Section 3.2: Negative Exponents

Objective: Simplify expressions with negative exponents using the properties of exponents.

There are a few special exponent properties that deal with exponents that are not positive. The first is considered in the following example, which is written out in two different ways.

Example 1. Simplify.

$\frac{a^3}{a^3}$ Use quotient rule; subtract exponents

a^0 Our Solution; now consider the problem in the second way

$\frac{a^3}{a^3}$ Rewrite exponents; use repeated multiplication

$\frac{aaa}{aaa}$ Reduce out all the a's

$\frac{1}{1}=1$ Our Solution; combine the two solutions, get:

$a^0=1$ Our Final Solution

This final result is an important property known as the zero power rule of exponents.

Zero Power Rule of Exponents: $a^0=1$

Any non-zero number or expression raised to the zero power will always be 1. This is illustrated in the following example.

Example 2. Simplify.

$(3x^2)^0$ Zero power rule

1 Our Solution

Here we are assuming that x is not 0. If $x=0$, then $(3\cdot 0^2)^0$ would equal 0, not 1. Another property we will consider here deals with negative exponents. Again we will solve the following example in two ways.

Example 3. Simplify and write the answer using only positive exponents.

$\frac{a^3}{a^5}$ Use quotient rule; subtract exponents

a^{-2} Our Solution; solve this problem another way

$$\frac{a^3}{a^5}$$ Rewrite exponents; use repeated multiplication

$$\frac{aaa}{aaaaa}$$ Reduce three a's out of top and bottom

$$\frac{1}{aa}$$ Simplify to exponents

$$\frac{1}{a^2}$$ Our Solution; combine the two solutions; get:

$$a^{-2} = \frac{1}{a^2}$$ Our Final Solution

This example illustrates an important property of exponents. Negative exponents yield the reciprocal of the base. Once we take the reciprocal, the exponent is now positive. Also, it is important to note that a negative exponent does not mean that the expression is negative; only that we need the reciprocal of the base. The rules of negative exponents follow.

Rules of Negative Exponents:

$$\boldsymbol{a^{-m} = \frac{1}{a^m}}$$

$$\boldsymbol{\frac{1}{a^{-m}} = a^m}$$

$$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$$

Negative exponents can be combined in several different ways. As a general rule, if we think of our expression as a fraction, negative exponents in the numerator must be moved to the denominator; likewise, negative exponents in the denominator need to be moved to the numerator. When the base with exponent moves, the exponent is now positive. This is illustrated in the following example.

Example 4. Simplify and write the answer using only positive exponents.

$$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2}$$ Move negative exponents on b, d, and e;
Exponents become positive

$$\frac{a^3cde^4}{2b^2f^2}$$ Our Solution

As we simplified our fraction, we took special care to move the bases that had a negative exponent, but the expression itself did not become negative because of those exponents. Also, it is important to remember that exponents only effect the base they are attached to. The 2 in the denominator of the above example does not have an exponent on it, so it does not move with the d.

We now have the following nine properties of exponents. It is important that we are very familiar with all of them. Note that when simplifying expressions involving exponents, the final form is usually written using only positive exponents.

Properties of Exponents

$a^m a^n = a^{m+n}$	$(ab)^m = a^m b^m$	$a^{-m} = \dfrac{1}{a^m}$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\dfrac{1}{a^{-m}} = a^m$
$\left(a^m\right)^n = a^{mn}$	$a^0 = 1$	$\left(\dfrac{a}{b}\right)^{-m} = \dfrac{b^m}{a^m}$

World View Note: Nicolas Chuquet, the French mathematician of the 15th century, wrote $12^{1\bar{m}}$ to indicate $12x^{-1}$. This was the first known use of the negative exponent.

Simplifying with negative exponents is much the same as simplifying with positive exponents. It is the advice of the author to keep the negative exponents until the end of the problem and then move them around to their correct location (numerator or denominator). As we do this, it is important to be very careful of rules for adding, subtracting, and multiplying with negatives. This is illustrated in the following examples.

Example 5. Simplify and write the answer using only positive exponents.

$\dfrac{4x^{-5}y^{-3}\cdot 3x^3y^{-2}}{6x^{-5}y^3}$ Simplify numerator with product rule; add exponents

$\dfrac{12x^{-2}y^{-5}}{6x^{-5}y^3}$ Use quotient rule; subtract exponents; be careful with negatives
$(-2)-(-5)=(-2)+5=3$
$(-5)-3=(-5)+(-3)=-8$

$2x^3y^{-8}$ Move negative exponent to denominator; exponent becomes positive

$\dfrac{2x^3}{y^8}$ Our Solution

Example 6. Simplify and write the answer using only positive exponents.

$\dfrac{(3ab^3)^{-2}ab^{-3}}{2a^{-4}b^0}$ In numerator, use power rule with -2; multiply exponents
In denominator, $b^0=1$

$\dfrac{3^{-2}a^{-2}b^{-6}ab^{-3}}{2a^{-4}}$ In numerator, use product rule; add exponents

$\dfrac{3^{-2}a^{-1}b^{-9}}{2a^{-4}}$ Use quotient rule; subtract exponents; be careful with negatives
$(-1)-(-4)=(-1)+4=3$

$$\frac{3^{-2}a^3b^{-9}}{2}$$ Move 3 and b to denominator because of negative exponents; Exponents become positive

$$\frac{a^3}{3^2 2b^9}$$ Evaluate $3^2*2=9*2=18$

$$\frac{a^3}{18b^9}$$ Our Solution

In the previous example it is important to point out that when we simplified 3^{-2}, we moved the three to the denominator and the exponent became positive. We did not make the number negative! Negative exponents never make the bases negative; they simply mean we have to take the reciprocal of the base. One final example with negative exponents is given here.

Example 7. Simplify and write the answer using only positive exponents.

$$\left(\frac{3x^{-2}y^5z^3\cdot 6x^{-6}y^{-2}z^{-3}}{9(x^2y^{-2})^{-3}}\right)^{-3}$$ In numerator, use product rule; adding exponents
In denominator, use power rule, multiplying exponents

$$\left(\frac{18x^{-8}y^3z^0}{9x^{-6}y^6}\right)^{-3}$$ Use quotient rule to subtract exponents, be careful with negatives:
$(-8)-(-6)=(-8)+6=-2$
$3-6=3+(-6)=-3$

$$(2x^{-2}y^{-3}z^0)^{-3}$$ Parentheses are done; use power rule with -3

$$2^{-3}x^6y^9z^0$$ Move 2 with negative exponent down and $z^0=1$; exponent of 2 becomes positive

$$\frac{x^6y^9}{2^3}$$ Evaluate 2^3

$$\frac{x^6y^9}{8}$$ Our Solution

3.2 Practice

Simplify each expression. Your answer should contain only positive exponents.

1) $2x^4y^{-2}\cdot(2xy^3)^4$
2) $2a^{-2}b^{-3}\cdot(2a^0b^4)^4$
3) $(a^4b^{-3})^3\cdot 2a^3b^{-2}$
4) $2x^3y^2\cdot(2x^3)^0$
5) $(2x^2y^2)^4\cdot x^{-4}$
6) $(m^0n^3\cdot 2m^{-3}n^{-3})^0$
7) $(x^3y^4)^3\cdot x^{-4}y^4$
8) $2m^{-1}n^{-3}\cdot(2m^{-1}n^{-3})^4$
9) $\dfrac{2x^{-3}y^2}{3x^{-3}y^3\cdot 3x^0}$
10) $\dfrac{3y^3}{3yx^3\cdot 2x^4y^{-3}}$
11) $\dfrac{4xy^{-3}\cdot x^{-4}y^0}{4y^{-1}}$
12) $\dfrac{3x^3y^2}{4y^{-2}\cdot 3x^{-2}y^{-4}}$
13) $\dfrac{u^2v^{-1}}{2u^0v^4\cdot 2uv}$
14) $\dfrac{2xy^2\cdot 4x^3y^{-4}}{4x^{-4}y^{-4}\cdot 4x}$
15) $\dfrac{u^2}{4u^0v^3\cdot 3v^2}$
16) $\dfrac{2x^{-2}y^2}{4yx^2}$
17) $\dfrac{2y}{(x^0y^2)^4}$
18) $\dfrac{(a^4)^4}{2b}$
19) $\left(\dfrac{2a^2b^3}{a^{-1}}\right)^4$

20) $\left(\frac{2y^{-4}}{x^2}\right)^{-2}$

21) $\frac{2nm^4}{(2m^2n^2)^4}$

22) $\frac{2y^2}{(x^4y^0)^{-4}}$

23) $\frac{(2mn)^4}{m^0n^{-2}}$

24) $\frac{2x^{-3}}{(x^4y^{-3})^{-1}}$

25) $\frac{y^3 \cdot x^{-3}y^2}{(x^4y^2)^3}$

26) $\frac{2x^{-2}y^0 \cdot 2xy^4}{(xy^0)^{-1}}$

27) $\frac{2u^{-2}v^3 \cdot (2uv^4)^{-1}}{2u^{-4}v^0}$

28) $\frac{2yx^2 \cdot x^{-2}}{(2x^0y^4)^{-1}}$

29) $\left(\frac{2x^0 \cdot y^4}{y^4}\right)^3$

30) $\frac{u^{-3}v^{-4}}{2v(2u^{-3}v^4)^0}$

31) $\frac{y(2x^4y^2)^2}{2x^4y^0}$

32) $\frac{b^{-1}}{(2a^4b^0)^0 \cdot 2a^{-3}b^2}$

33) $\frac{2yzx^2}{2x^4y^4z^{-2} \cdot (zy^2)^4}$

34) $\frac{2b^4c^{-2} \cdot (2b^3c^2)^{-4}}{a^{-2}b^4}$

35) $\frac{2kh^0 \cdot 2h^{-3}k^0}{(2kj^3)^2}$

36) $\left(\frac{(2x^{-3}y^0z^{-1})^3 \cdot x^{-3}y^2}{2x^3}\right)^{-2}$

37) $\dfrac{(cb^3)^2 \cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3}$

38) $\dfrac{2q^4 \cdot m^2p^2q^4}{(2m^{-4}p^2)^3}$

39) $\dfrac{(yx^{-4}z^2)^{-1}}{z^3 \cdot x^2y^3z^{-1}}$

40) $\dfrac{2mpn^{-3}}{(m^0n^{-4}p^2)^3 \cdot 2n^2p^0}$

3.2 Answers

1) $32x^8y^{10}$
2) $\frac{32b^{13}}{a^2}$
3) $\frac{2a^{15}}{b^{11}}$
4) $2x^3y^2$
5) $16x^4y^8$
6) 1
7) $y^{16}x^5$
8) $\frac{32}{m^5n^{15}}$
9) $\frac{2}{9y}$
10) $\frac{y^5}{2x^7}$
11) $\frac{1}{y^2x^3}$
12) $\frac{y^8x^5}{4}$
13) $\frac{u}{4v^6}$
14) $\frac{x^7y^2}{2}$
15) $\frac{u^2}{12v^5}$
16) $\frac{y}{2x^4}$
17) $\frac{2}{y^7}$
18) $\frac{a^{16}}{2b}$
19) $16a^{12}b^{12}$
20) $\frac{y^8x^4}{4}$

21) $\frac{1}{8m^4n^7}$

22) $2x^{16}y^2$

23) $16n^6m^4$

24) $\frac{2x}{y^3}$

25) $\frac{1}{x^{15}y}$

26) $4y^4$

27) $\frac{u}{2v}$

28) $4y^5$

29) 8

30) $\frac{1}{2u^3v^5}$

31) $2y^5x^4$

32) $\frac{a^3}{2b^3}$

33) $\frac{1}{x^2y^{11}z}$

34) $\frac{a^2}{8c^{10}b^{12}}$

35) $\frac{1}{h^3kj^6}$

36) $\frac{x^{30}z^6}{16y^4}$

37) $\frac{2b^{14}}{a^{12}c^7}$

38) $\frac{m^{14}q^8}{4p^4}$

39) $\frac{x^2}{y^4z^4}$

40) $\frac{mn^7}{p^5}$