Intermediate Algebra

MATH 083 Textbook

Third Edition (2020)

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Tyler Wallace
http://www.wallace.ccfaculty.org/book/book.html

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Factoring Polynomials

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Chapter 1

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Section 1.1: Greatest Common Factor

Objectives: Find the greatest common factor of a polynomial.
Factor the GCF from a polynomial.

The inverse of multiplying polynomials together is factoring polynomials. There are many benefits of factoring a polynomial. We use factored polynomials to help us solve equations, study behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials, it is very important to have strong factoring skills.

In this lesson, we will focus on factoring using the Greatest Common Factor or GCF of a polynomial. When multiplying monomials by polynomials, such as $4x^2(2x^2-3x+8)$, we distribute to get a product of $8x^4-12x^3+32x^2$. In this lesson, we will work backwards, starting with $8x^4-12x^3+32x^2$ and factoring to write as the product $4x^2(2x^2-3x+8)$.

DETERMINING THE GREATEST COMMON FACTOR

We will first introduce this idea by finding the GCF of several numbers. To find a GCF of several numbers, we look for the largest number that can divide each number without leaving a remainder.

Example 1. Determine the GCF of 15, 24, and 27.

$\frac{15}{3}=5, \frac{24}{3}=8, \frac{27}{3}=9$ Each of the numbers can be divided by 3

GCF = 3 Our Answer

When there are variables in our problem, we can first find the GCF of the numbers as in Example 1 above. Then we take any variables that are in common to all terms. The variable part of the GCF uses the *smallest* power of each variable that appears in all terms. This idea is shown in the next example.

Example 2. Determine the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

$\frac{24}{6}=4, \frac{18}{6}=3, \frac{12}{6}=2$ Each number can be divided by 6

Use the lowest exponent for each common variable; each term contains x^2y.
Note that z is not part of the GCF because the term $18x^2y^4$ does not contain the variable z.

GCF = $6x^2y$ Our Answer

FACTORING THE GREATEST COMMON FACTOR

Now we will learn to factor the GCF from a polynomial with two or more terms. Remember that factoring is the inverse process of multiplying. In particular, factoring the GCF reverses the distributive property of multiplication.

To factor the GCF from a polynomial, we first identify the GCF of all the terms. The GCF is the factor that goes in front of the parentheses. Then we divide each term of the given polynomial by the GCF. For the second factor, enclose the quotients within the parentheses. In the final answer, the GCF is outside the parentheses and the remaining quotients are enclosed within the parentheses.

Example 3. Factor using the GCF.

$4x^2-20x-16$ GCF of $4x^2$, $-20x$, and 16 is 4; divide each term by 4

$$\frac{4x^2}{4}=x^2, \frac{-20x}{4}=-5x, \frac{-16}{4}=-4$$

The quotients are the terms left inside the parentheses; keep the GCF outside the parentheses

$=4(x^2-5x-4)$ Our Answer

With factoring, we can always check our answers by multiplying (distributing); the resulting product should be the original expression.

Example 4. Factor using the GCF.

$25x^4-15x^3+20x^2$ GCF of $25x^4$, $-15x^3$, and $20x^2$ is $5x^2$; divide each term by $5x^2$

$$\frac{25x^4}{5x^2}=5x^2, \frac{-15x^3}{5x^2}=-3x, \frac{20x^2}{5x^2}=4$$

These quotients are the terms left inside the parentheses; keep the GCF outside the parentheses

$=5x^2(5x^2-3x+4)$ Our Answer

Example 5. Factor using the GCF.

$3x^3y^2z+5x^4y^3z^5-4xy^4$ GCF of $3x^3y^2z$, $5x^4y^3z^5$, and $-4xy^4$ is xy^2; divide each term by xy^2

$$\frac{3x^3y^2z}{xy^2}=3x^2z, \frac{5x^4y^3z^5}{xy^2}=5x^3yz^5, \frac{-4xy^4}{xy^2}=-4y^2$$

These quotients are the terms left inside the parentheses; keep the GCF outside the parentheses

$=xy^2(3x^2z+5x^3yz^5-4y^2)$ Our Answer

Example 6. Factor using the GCF.

$21x^3+14x^2+7x$ GCF of $21x^3$, $14x^2$, and $7x$ is $7x$;
divide each term by $7x$

$$\frac{21x^3}{7x}=3x^2, \frac{14x^2}{7x}=2x, \frac{7x}{7x}=1$$

The factors are the GCF and the result of the division;
These quotients are the terms left inside the parentheses;
keep the GCF outside the parentheses.

$=7x(3x^2+2x+1)$ Our Answer

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was also one of the terms, so dividing resulting in a quotient of 1. Factoring will never make terms disappear. Anything divided by itself is 1; be sure not to forget to put the 1 into the solution.

Often the second line is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parentheses containing the remaining factors as shown in the following example.

Example 7. Factor using the GCF.

$18a^4b^3-27a^3b^3+9a^2b^3$ GCF is $9a^2b^3$, divide each term by $9a^2b^3$

$=9a^2b^3(2a^2-3a+1)$ Our Answer

Again, in the previous problem when dividing $9a^2b^3$ by itself, the result is 1. Be very careful that each term is accounted for in your final solution.

GREATEST COMMON FACTOR EQUAL TO 1

Sometimes an expression has a GCF of 1. If there is no common factor other than 1, the polynomial expression cannot be factored using the GCF. This is shown in the following example.

Example 8. Factor using the GCF.

$8ab-17c+49$ GCF is 1 because there are no other factors in common to all 3 terms

cannot be factored using the GCF Our Answer

FACTORING THE *NEGATIVE* OF THE GCF

If the first term of a polynomial has a negative coefficient, always make the GCF negative in order to make the first term inside the parentheses have a positive coefficient. See Example 9 on the next page.

Example 9. Factor using the GCF.

$-12x^5y^2+6x^4y^4-8x^3y^5$	GCF of $\left(-12x^5y^2\right)$, $\left(6x^4y^4\right)$, and $\left(-8x^3y^5\right)$ is $(-2x^3y^2)$; because the first term is negative; divide each term by $(-2x^3y^2)$ $\frac{-12x^5y^2}{-2x^3y^2}=6x^2, \frac{6x^4y^4}{-2x^3y^2}=-3xy^2, \frac{-8x^3y^5}{-2x^3y^2}=4y^3$ The results are what is left inside the parentheses
$=-2x^3y^2(6x^2-3xy^2+4y^3)$	Our Answer

We will always begin factoring by looking for a Greatest Common Factor and factoring it out if there is one. In the rest of this chapter, we will learn other factoring techniques that might be used to write a polynomial as a product of prime polynomials.

Practice Exercises
Section 1.1: Greatest Common Factor

Factor using the GCF.
If the GCF is 1, state that the polynomial "cannot be factored using the GCF".

1) $15x+20$

2) $12-8x$

3) $9x-9$

4) $3x^2+5x$

5) $10x^3-18x$

6) $7ab-35a^2b$

7) $9+8x^2$

8) $4x^3y^2+8x^3$

9) $24x^2y^5-18x^3y^2$

10) $-3a^2b+6a^3b^2$

11) $5x^3-7$

12) $-32n^9+32n^6+40n^5$

13) $20x^4-30x+30$

14) $21p^6+30p^2+27$

15) $28m^4+40m^3+8$

16) $-10x^4+20x^2+12x$

17) $30b^9+5ab-15a^2$

18) $27y^7+12xy^2+9y^2$

19) $-48a^2b^2-56a^3b-56a^5b$

20) $30m^6+15mn^2-25$

21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$

22) $3p+12q-15q^2r^2$

23) $50x^2y+10y^2+70xz^2$

24) $30x^5y^4z^3+50y^4z^5-10xy^4z^3$

25) $30pqr-5pq+5q$

26) $28b+14b^2+35b^3+7b^5$

27) $-18n^5+3n^3-21n+3$

28) $30a^8+6a^5+27a^3+21a^2$

29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$

30) $-24x^6-4x^4+12x^3+4x^2$

31) $-32mn^8+4m^6n+12mn^4+16mn$

32) $-10y^7+6y^{10}-4xy^{10}-8xy^8$

ANSWERS to Practice Exercises
Section 1.1: Greatest Common Factor

1) $5(3x+4)$

2) $4(3-2x)$

3) $9(x-1)$

4) $x(3x+5)$

5) $2x(5x^2-9)$

6) $7ab(1-5a)$

7) cannot be factored using the GCF

8) $4x^3(y^2+2)$

9) $6x^2y^2(4y^3-3x)$

10) $-3a^2b(1-2ab)$

11) cannot be factored using the GCF

12) $-8n^5(4n^4-4n-5)$

13) $10(2x^4-3x+3)$

14) $3(7p^6+10p^2+9)$

15) $4(7m^4+10m^3+2)$

16) $-2x(5x^3-10x-6)$

17) $5(6b^9+ab-3a^2)$

18) $3y^2(9y^5+4x+3)$

19) $-8a^2b(6b+7a+7a^3)$

20) $5(6m^6+3mn^2-5)$

21) $5x^3y^2z(4x^5z+3x^2+7y)$

22) $3(p+4q-5q^2r^2)$

23) $10(5x^2y+y^2+7xz^2)$

24) $10y^4z^3(3x^5+5z^2-x)$

25) $5q(6pr-p+1)$

26) $7b(4+2b+5b^2+b^4)$

27) $-3(6n^5-n^3+7n-1)$

28) $3a^2(10a^6+2a^3+9a+7)$

29) $-10x^{11}(4+2x-5x^2+5x^3)$

30) $-4x^2(6x^4+x^2-3x-1)$

31) $-4mn(8n^7-m^5-3n^3-4)$

32) $-2y^7(5-3y^3+2xy^3+4xy)$

Section 1.2: Factoring by Grouping

Objective: Factor polynomials with four terms by grouping.

Whenever possible, we will always do when factoring a polynomial is factor out the greatest common factor (GCF). This GCF is often a monomial. For example, the GCF of $5xy+10xz$ is the monomial $5x$, so we would factor as $5x(y+2z)$. However, a GCF does not have to be a monomial; it could be a polynomial. To see this, consider the following two examples.

Example 1. Factor completely.

$3ax-7bx$	Both terms have x in common, factor it out
$=x(3a-7b)$	Our Answer

Now we have a similar problem, but instead of the monomial x, we have the binomial $(2a+5b)$ as the GCF.

Example 2. Factor completely.

$3a(2a+5b)-7b(2a+5b)$	Both terms have $(2a+5b)$ in common, factor it out
$=(2a+5b)(3a-7b)$	Our Answer

In the same way we factored out the GCF of x, we can factor out the GCF which is a binomial, $(2a+5b)$.

FACTORING BY GROUPING

When a polynomial has a GCF of 1, it still may be factorable. Additional factoring strategies will be needed.

When a polynomial has **four** terms, we will attempt to factor it using a strategy called **grouping**.

Remember, factoring is the reverse of multiplying, so first we will look at a multiplication problem and then try to reverse the process.

Example 3. Multiply.

$(2a+3)(5b+2)$	Distribute $(2a+3)$ to each term in the second parentheses
$=5b(2a+3)+2(2a+3)$	Distribute each monomial
$=10ab+15b+4a+6$	Our Answer

The product has four terms. We arrived at this answer by looking at the two parts, $5b(2a+3)$ and $2(2a+3)$.

When we are factoring by grouping, we split the expression into two groups: the first two terms and the last two terms. Then we can factor the GCF out of each group of two terms. When we do this, our hope is what remains in the parentheses will match in both the left group and the right group. If they match, we can pull this matching binomial GCF out front, putting the rest in parentheses and the expression will be factored.

The next example is the same problem worked backwards, factoring instead of multiplying.

Example 4. Factor completely.

$10ab+15b+4a+6$	Split expression into two groups
$=\boxed{10ab+15b}\boxed{+4a+6}$	Factor the GCF from each group of two terms
$=\boxed{5b(2a+3)}\boxed{+2(2a+3)}$	$(2a+3)$ is common to both terms; Factor out this binomial GCF
$=(2a+3)(5b+2)$	Our Answer

Example 5. Factor completely.

$6x^3-15x^2+2x-5$	Split expression into two groups
$=\boxed{6x^3-15x^2}\boxed{+2x-5}$	Factor the GCF from each group of two terms
$=\boxed{3x^2(2x-5)}\boxed{+1(2x-5)}$	$(2x-5)$ is common to both terms; Factor out this binomial GCF
$=(2x-5)(3x^2+1)$	Our Answer

The key for grouping to work is after the GCF is factored out of the left and right groups, the two binomials must match exactly. If there is any difference between the two binomials, we either have to do some adjusting or it can't be factored using the grouping method.

Sometimes, we will need to factor the negative of the GCF of a grouping to be sure the remaining binomials match.

Example 6. Factor completely.

$6x^2+9xy-14x-21y$	Split expression into two groups
$=\boxed{6x^2+9xy}\boxed{-14x-21y}$	Factor the GCF from each group of two terms
$=\boxed{3x(2x+3y)}\boxed{-7(2x+3y)}$	$(2x+3y)$ is common to both terms; Factor out this binomial GCF
$=(2x+3y)(3x-7)$	Our Answer

Example 7. Factor completely.

$5xy-8x-10y+16$	Split expression into two groups
$=\boxed{5xy-8x}\boxed{-10y+16}$	Factor the GCF from each group of two terms
$=\boxed{x(5y-8)}\boxed{-2(5y-8)}$	$(5y-8)$ is common to both terms; Factor out this binomial GCF
$=(5y-8)(x-2)$	Our Answer

Example 8. Factor completely.

$12ab-14a-6b+7$	Split expression into two groups
$=\boxed{12ab-14a}\boxed{-6b+7}$	Factor the GCF from each group of two terms
$=\boxed{2a(6b-7)}\boxed{-1(6b-7)}$	$(6b-7)$ is common to both terms; Factor out this binomial GCF
$=(6b-7)(2a-1)$	Our Answer

Example 9. Factor completely.

$4a^2+6ab-14ab^2-21b^3$	Split expression into two groups
$=\boxed{4a^2+6ab}\boxed{-14ab^2-21b^3}$	Factor the GCF from each group of two terms
$=\boxed{2a(2a+3b)}\boxed{-7b^2(2a+3b)}$	$(2a+3b)$ is common to both terms; Factor out this binomial GCF
$=(2a+3b)(2a-7b^2)$	Our Answer

Example 10. Factor completely.

$8xy-12y-10x+15$	Split expression into two groups
$=\boxed{8xy-12y}\boxed{-10x+15}$	Factor the GCF from each group of two terms
$=\boxed{4y(2x-3)}\boxed{-5(2x-3)}$	$(2x-3)$ is common to both terms; Factor out this binomial GCF
$=(2x-3)(4y-5)$	Our Answer

Sometimes the terms in the expression must be rearranged in order for factoring by grouping to work.

Example 11. Factor completely.

$6xy+4+3x+8y$	Split expression into two groups
$= \boxed{6xy+4}\boxed{+3x+8y}$	Factor the GCF from each group of two terms
$= \boxed{2(3xy+2)}\boxed{+1(3x+8y)}$	The remaining factors are not the same; rearrange the terms.
$6xy+3x+8y+4$	Split expression into two groups
$= \boxed{6xy+3x}\boxed{+8y+4}$	Factor the GCF from each group of two terms
$= \boxed{3x(2y+1)}\boxed{+4(2y+1)}$	$(2y+1)$ is common to both terms; Factor out this binomial GCF
$= (2y+1)(3x+4)$	Our Answer

Practice Exercises
Section 1.2: Factoring by Grouping

Factor completely.

1) $x^3+3x^2+4x+12$
2) $x^3-3x^2+6x-18$
3) $x^3-5x^2-2x+10$
4) x^3+x^2-3x-3
5) $40r^3-8r^2-25r+5$
6) $35x^3-10x^2-56x+16$
7) $3n^3-2n^2-9n+6$
8) $14v^3+10v^2-7v-5$
9) $15b^3+21b^2-35b-49$
10) $6x^3-48x^2+5x-40$
11) $3x^3+15x^2+2x+10$
12) $35x^3-28x^2-20x+16$
13) $7n^3+21n^2-5n-15$
14) $7xy-49x+5y-35$
15) $42r^3-49r^2+18r-21$
16) $32xy+40x^2+12y+15x$
17) $15ab-6a+5b^3-2b^2$
18) $16xy-56x+2y-7$
19) $3mn-8m+15n-40$
20) $5mn+2m-25n-10$
21) $40xy+35x-8y^2-7y$
22) $32uv-20u+24v-15$
23) $10xy+30+25x+12y$
24) $24xy+25y^2-20x-30y^3$
25) $3uv+14u-6u^2-7v$
26) $56ab+14-49a-16b$
27) $2xy-8x^2+7y^3-28xy^2$
28) $28p^3+21p^2+20p+15$
29) $16xy-3x-6x^2+8y$
30) $8xy+56x-y-7$

ANSWERS to Practice Exercises
Section 1.2: Factoring by Grouping

1) $(x+3)(x^2+4)$

2) $(x-3)(x^2+6)$

3) $(x-5)(x^2-2)$

4) $(x+1)(x^2-3)$

5) $(5r-1)(8r^2-5)$

6) $(7x-2)(5x^2-8)$

7) $(3n-2)(n^2-3)$

8) $(7v+5)(2v^2-1)$

9) $(5b+7)(3b^2-7)$

10) $(x-8)(6x^2+5)$

11) $(x+5)(3x^2+2)$

12) $(5x-4)(7x^2-4)$

13) $(n+3)(7n^2-5)$

14) $(y-7)(7x+5)$

15) $(6r-7)(7r^2+3)$

16) $(4y+5x)(8x+3)$

17) $(5b-2)(3a+b^2)$

18) $(2y-7)(8x+1)$

19) $(3n-8)(m+5)$

20) $(5n+2)(m-5)$

21) $(8y+7)(5x-y)$

22) $(8v-5)(4u+3)$

23) $(2y+5)(5x+6)$

24) $(6y-5)(4x-5y^2)$

25) $(v-2u)(3u-7)$

26) $(8b-7)(7a-2)$

27) $(y-4x)(2x+7y^2)$

28) $(4p+3)(7p^2+5)$

29) $(8y-3x)(2x+1)$

30) $(y+7)(8x-1)$

Section 1.3: Factor Trinomials Whose Leading Coefficient is 1

Objective: Factor trinomials when the leading coefficient is 1.

We will now learn a strategy for factoring trinomials (polynomials with three terms). In this section, we will focus on trinomials of the form x^2+bx+c. In this case, the leading coefficient (the coefficient of the first term) is 1.

Since factoring is the reverse of multiplication, we will start with a multiplication problem and look at how we can reverse the process.

Example 1. Multiply and simplify.

$(x+6)(x-4)$	Distribute $(x+6)$ to each term in the second set of parentheses
$=x(x+6)-4(x+6)$	Distribute each monomial through each se of parentheses
$=x^2+6x-4x-24$	Combine like terms
$=x^2+2x-24$	Our Answer

Notice that if you reverse the last three steps, the process is factoring by grouping! The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This process is shown in the following example, which is this same problem worked backwards:

Example 2. Factor completely.

$x^2+2x-24$	Replace the middle term $+2x$ with $+6x-4x$
$=x^2+6x-4x-24$	Split expression into two pairs of terms; factor the GCF from each pair
$=x(x+6)-4(x+6)$	$(x+6)$ is common to both terms*:* factor this binomial GCF
$=(x+6)(x-4)$	Our Answer

The key to making these problems work is in the way we split the middle term. Why did we choose $+6x-4x$ and not $+5x-3x$? To find the correct way to split the middle term, we will use what is called the *ac method*. The *ac method* works by finding a pair of numbers that multiply to obtain the last number in the trinomial **and** also add up to the coefficient of the middle term of the trinomial. In the previous example, the numbers must multiply to -24 and add to $+2$. The only numbers that can do this are 6 and -4. Notice that $6\cdot(-4)=-24$ and $6+(-4)=2$.

FACTORING TRINOMIALS OF THE FORM x^2+bx+c

Example 3. Factor completely.

$x^2+9x+18$	Find factors that multiply to 18 and add to 9: Use 6 and 3 Replace $9x$ with $6x+3x$
$=x^2+6x+3x+18$	Factor by grouping
$=x(x+6)+3(x+6)$	
$=(x+6)(x+3)$	Our Answer

Example 4. Factor completely.

x^2-4x+3	Find factors that multiply to 3 and add to -4: Use -3 and -1 Replace $-4x$ with $-3x+(-1x)=-3x-1x$
$=x^2-3x-x+3$	Factor by grouping
$=x(x-3)-1\,(x-3)$	
$=(x-3)(x-1)$	Our Answer

Example 5. Factor completely.

$x^2-8x-20$	Find factors that multiply to -20 and add to -8: Use -10 and 2 Replace $-8x$ with $-10x+2x$
$=x^2-10x+2x-20$	Factor by grouping
$=x(x-10)+2(x-10)$	
$=(x-10)(x+2)$	Our Answer

FACTORING TRINOMIALS IN TWO VARIABLES

Often we are asked to factor trinomials in two variables. The *ac method* works in much the same way: Find a pair of terms that multiplies to obtain the last term in the trinomial **and** also adds up to the the middle term of the trinomial.

Example 6. Factor completely.

$a^2-9ab+14b^2$	Find factors that multiply to 14 and add to -9: Use -7 and -2 Replace $-9ab$ with $-7ab-2ab$
$=a^2-7ab-2ab+14b^2$	Factor by grouping
$=a(a-7b)-2b(a-7b)$	
$=(a-7b)(a-2b)$	Our Answer

As the past few examples illustrate, it is very important to find the correct pair of terms we will use to replace the middle term. Consider the following example, done **incorrectly**, where we mistakenly found two factors that multiply to 6 instead of -6:

Warning!

x^2+5x-6	Find factors that multiply to 6 and add to 5:
	Use 2 and 3
	Replace $5x$ with $2x+3x$
$=x^2+2x+3x-6$	Factor by grouping
$=x(x+2)+3(x-2)$	
???	Binomials do not match!

Because we did not use the negative sign with the 6 to find our pair of terms, the binomials did not match and grouping was not able to work at the end. The problem is done correctly below by choosing the correct pair of terms:

Example 7. Factor completely.

x^2+5x-6	Find factors that multiply to -6 and add to 5:
	Use 6 and -1
	Replace $5x$ with $6x-1x$
$=x^2+6x-1x-6$	Factor by grouping
$=x(x+6)-1(x+6)$	
$=(x+6)(x-1)$	Our Answer

FACTORING SHORTCUT

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1; our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it, however, when the leading coefficient of the trinomial is 1. In all of the problems we have factored in this lesson, the leading coefficient is 1. If this is the case, then we can use this shortcut. This process is shown in the next few examples.

Example 8. Factor completely.

$x^2-7x-18$	Find factors that multiply to -18 and add to -7:
	Use -9 and 2
	Write the binomial factors using -9 and 2
$=(x-9)(x+2)$	Our Answer

Example 9. Factor completely.

$m^2 - mn - 30n^2$	Find factors that multiply to -30 and add to -1:
	Use 5 and -6
	Write the binomial factors using 5 and -6
	Don't forget the second variable
$= (m+5n)(m-6n)$	Our Answer

It is possible to have an expression that does not factor. If there is no combination of numbers that multiply and add up to the correct numbers, then we cannot factor the polynomial and we say the polynomial is *prime*. This is shown in the following example.

Example 10. Factor completely.

$x^2 + 2x + 6$	Find factors that multiply to 6 and add to 2:
	$1\cdot 6$, $2\cdot 3$, $(-1)\cdot(-6)$, and $(-2)\cdot(-3)$ are the only ways to multiply to 6 but none of these pairs adds to 2
prime	Our Answer

FACTORING USING MORE THAN ONE STRATEGY

When factoring any polynomial, it is important to first factor any GCF (other than 1). If all of the terms in a polynomial have a common factor other than 1, we will want to first factor out the GCF before attempting any other method. The next three examples illustrate this technique.

Example 11. Factor completely.

$3x^2 - 24x + 45$	GCF of all three terms is 3*;* divide each term by 3
$= 3(x^2 - 8x + 15)$	Find factors that multiply to 15 and add to -8:
	Use -5 and -3
	Write the binomial factors using -5 and -3
$= 3(x-5)(x-3)$	Our Answer

Example 12. Factor completely.

$4x^2y - 8xy - 32y$	GCF of all three terms is $4y$*;* divide each terms by $4y$
$= 4y(x^2 - 2x - 8)$	Find factors that multiply to -8 and add to -2:
	Use -4 and 2
	Write the binomial factors using -4 and 2
$= 4y(x-4)(x+2)$	Our Answer

Example 13. Factor completely.

$7a^4b^2 + 28a^3b^2 - 35a^2b^2$	GCF of all three terms is $7a^2b^2$; divide each term by $7a^2b^2$
$= 7a^2b^2(a^2 + 4a - 5)$	Find factors that multiply to -5 and to 4: Use -1 and 5
	Write the binomial factors using -1 and 5
$= 7a^2b^2(a-1)(a+5)$	Our Answer

Again it is important to comment on the shortcut of jumping right to the factors. This shortcut only works if the leading coefficient is 1. In the next lesson, we will look at how this process changes slightly when we have a number other than 1 as the leading coefficient.

Practice Exercises
Section 1.3: Factoring Trinomials Whose Leading Coefficient is 1

Factor completely.

1) $x^2+12x+32$

2) $x^2+13x+40$

3) $x^2-7x+10$

4) x^2-9x+8

5) x^2+x-30

6) x^2+x-72

7) $x^2-6x-27$

8) $x^2-9x-10$

9) $y^2-17y+70$

10) $x^2+3x-18$

11) $p^2+17p+72$

12) $x^2+3x-70$

13) $p^2+15p+54$

14) $p^2+7p-30$

15) c^2-4c+9

16) $m^2-15mn+50n^2$

17) $u^2-10uv+21v^2$

18) $m^2-3mn-40n^2$

19) $m^2+2mn-8n^2$

20) $x^2+10xy+16y^2$

21) $x^2-11xy+18y^2$

22) $u^2-10uv-24v^2$

23) $x^2+xy-12y^2$

24) $x^2+14xy+45y^2$

25) $x^2+4xy-12y^2$

26) $4x^2+52x+168$

27) $5a^2+60a+100$

28) $7w^2+35w-63$

29) $6a^2+24a-192$

30) $-5v^2-20v+25$

31) $6x^2+18xy+12y^2$

32) $5m^2+30mn-80n^2$

33) $6x^2+96xy+378y^2$

34) $-n^2+15n-56$

35) $-16t^2+48t+64$

ANSWERS to Practice Exercises
Section 1.3: Factoring Trinomials Whose Leading Coefficient is 1

1) $(x+8)(x+4)$

2) $(x+8)(x+5)$

3) $(x-2)(x-5)$

4) $(x-1)(x-8)$

5) $(x-5)(x+6)$

6) $(x+9)(x-8)$

7) $(x-9)(x+3)$

8) $(x-10)(x+1)$

9) $(y-10)(y-7)$

10) $(x+6)(x-3)$

11) $(p+9)(p+8)$

12) $(x+10)(x-7)$

13) $(p+6)(p+9)$

14) $(p+10)(p-3)$

15) prime

16) $(m-5n)(m-10n)$

17) $(u-7v)(u-3v)$

18) $(m+5n)(m-8n)$

19) $(m+4n)(m-2n)$

20) $(x+8y)(x+2y)$

21) $(x-9y)(x-2y)$

22) $(u-12v)(u+2v)$

23) $(x-3y)(x+4y)$

24) $(x+5y)(x+9y)$

25) $(x+6y)(x-2y)$

26) $4(x+7)(x+6)$

27) $5(a+10)(a+2)$

28) $7(w^2+5w-9)$

29) $6(a-4)(a+8)$

30) $-5(v-1)(v+5)$

31) $6(x+2y)(x+y)$

32) $5(m-2n)(m+8n)$

33) $6(x+9y)(x+7y)$

34) $-(n-8)(n-7)$

35) $-16(t-4)(t+1)$

Section 1.4: Factor Trinomials Whose Leading Coefficient is not 1

Objective: Factor trinomials using the *ac* method when the leading coefficient of the polynomial is not 1.

When factoring trinomials, we use the *ac method* to split the middle term and then factor by grouping. The *ac method* gets its name from the general trinomial expression, ax^2+bx+c, where a, b, and c are the numbers (coefficients) in front of x^2 and x terms, and the constant at the end, respectively.

The ac *method* is named ac because we multiply $a \cdot c$ to find out what we want the two numbers to multiply to. In the previous lesson, we always multiplied to just c because there was no number written in front of x^2. This meant the leading coefficient was 1 and we were multiplying $1 \cdot c$, which is c. Now we will have a number other than 1 as the leading coefficient; so, we will be looking for two numbers that multiply to $a \cdot c$ and that add to b.

FACTORING TRINOMIALS OF THE FORM ax^2+bx+c WHERE $a \neq 1$

Example 1. Factor completely.

$3x^2+11x+6$	Multiply $a \cdot c$: $(3)(6)=18$
	Find factors that multiply to 18 and add to 11:
	Use 9 and 2
	Replace $11x$ with $9x+2x$
$=3x^2+9x+2x+6$	Factor by grouping
$=3x(x+3)+2(x+3)$	
$=(x+3)(3x+2)$	Our Answer

When $a=1$, we are able to use the shortcut discussed in the previous section, using the numbers that split the middle term to write the binomial factors. Example 1 above illustrates that the shortcut does not work when $a \neq 1$. **We must go through *all* the steps of grouping in order to factor the expression.**

Example 2. Factor completely.

$8x^2-2x-15$	Multiply $a \cdot c$: $(8)(-15)=-120$
	Find factors that multiply to -120 and add to -2:
	Use -12 and 10
	Replace $-2x$ with $-12x+10x$
$=8x^2-12x+10x-15$	Factor by grouping
$=4x(2x-3)+5(2x-3)$	
$=(2x-3)(4x+5)$	Our Answer

Example 3. Factor completely.

$10x^2 - 27x + 5$	Multiply $a \cdot c$: $(10)(5) = 50$
	Find factors that multiply to 50 and add to -27:
	Use -25 and -2
	Replace $-27x$ with $-25x - 2x$
$= 10x^2 - 25x - 2x + 5$	Factor by grouping
$= 5x(2x-5) - 1(2x-5)$	
$= (2x-5)(5x-1)$	Our Answer

The same process works with polynomials in two variables.

Example 4. Factor completely.

$4x^2 - xy - 5y^2$	Multiply $a \cdot c$: $(4)(-5) = -20$
	Find factors that multiply to -20 and add to -1:
	Use 4 and -5
	Replace $-xy$ with $4xy - 5xy$
$= 4x^2 + 4xy - 5xy - 5y^2$	Factor by grouping
$= 4x(x+y) - 5y(x+y)$	
$= (x+y)(4x-5y)$	Our Answer

FACTORING USING MORE THAN ONE STRATEGY

As always, when factoring we will first look for a GCF before using any other method, including the *ac method*. Factoring out the GCF first also has the added bonus of making the numbers smaller so the *ac method* becomes easier.

Example 5. Factor completely.

$18x^3 + 33x^2 - 30x$	$\text{GCF} = 3x$; factor from each term
$= 3x[6x^2 + 11x - 10]$	Multiply $a \cdot c$: $(6)(-10) = -60$
	Find factors that multiply to -60 and add to 11:
	Use 15 and -4
	Replace $11x$ with $15x - 4x$
$= 3x[6x^2 + 15x - 4x - 10]$	Factor by grouping
$= 3x[3x(2x+5) - 2(2x+5)]$	
$= 3x(2x+5)(3x-2)$	Our Answer

As was the case with trinomials when $a = 1$, not all trinomials can be factored. If there is no combination of numbers that multiplies and adds up to the correct numbers, then we cannot factor the polynomial and we say the polynomial is *prime*.

Example 6. Factor completely.

$3x^2 + 2x - 7$	Multiply $a \cdot c$: $(3)(-7) = -21$ $-3(7)$, $-7(3)$, $-1(21)$, and $-21(1)$ are the only ways to multiply to -21, but none of these pairs sums to 2
prime	Our Answer

Practice Exercises
Section 1.4: Factoring Trinomials Whose Leading Coefficient is not 1

Factor completely.

1) $5x^2+13x+6$
2) $2x^2-5x+2$
3) $3r^2-4r-4$
4) $4r^2+3r-7$
5) $2x^2-x+3$
6) $4k^2-17k+4$
7) $2b^2-b-3$
8) $6p^2+11p-7$
9) $4r^2+r-3$
10) $2x^2+19x+35$
11) $3x^2-17x+20$
12) $5n^2-4n-20$
13) $7x^2-48x+36$
14) $7n^2-44n+12$
15) $-7x^2-15x-2$
16) $7v^2-24v-16$
17) $5a^2-13a-28$
18) $7x^2+29x-30$
19) $5k^2-26k+24$
20) $-3r^2-16r-21$
21) $3u^2+13uv-10v^2$
22) $3x^2+17xy+10y^2$
23) $7x^2-2xy-5y^2$
24) $5x^2+28xy-49y^2$
25) $5u^2+31uv-28v^2$
26) $6x^2-39x-21$
27) $-10a^2+54a+36$
28) $21k^2-87k-90$
29) $-21n^2-45n+54$
30) $14x^2-60x+16$
31) $6x^2+29x+20$
32) $4x^2+9xy+2y^2$
33) $4m^2+6mn+6n^2$
34) $4m^2-9mn-9n^2$
35) $4x^2-6xy+30y^2$
36) $4x^2+13xy+3y^2$
37) $18u^2-3uv-36v^2$
38) $12x^2+62xy+70y^2$
39) $16x^2+60xy+36y^2$
40) $24x^2-52xy+8y^2$

ANSWERS to Practice Exercises
Section 1.4: Factoring Trinomials Whose Leading Coefficient is not 1

1) $(5x+3)(x+2)$
2) $(2x-1)(x-2)$
3) $(3r+2)(r-2)$
4) $(r-1)(4r+7)$
5) prime
6) $(k-4)(4k-1)$
7) $(2b-3)(b+1)$
8) $(3p+7)(2p-1)$
9) $(r+1)(4r-3)$
10) $(2x+5)(x+7)$
11) $(3x-5)(x-4)$
12) prime
13) $(7x-6)(x-6)$
14) $(7n-2)(n-6)$
15) $-(7x+1)(x+2)$
16) $(7v+4)(v-4)$
17) $(5a+7)(a-4)$
18) $(7x-6)(x+5)$
19) $(5k-6)(k-4)$
20) $-(3r+7)(r+3)$
21) $(3u-2v)(u+5v)$
22) $(3x+2y)(x+5y)$
23) $(7x+5y)(x-y)$
24) $(5x-7y)(x+7y)$
25) $(5u-4v)(u+7v)$
26) $3(2x+1)(x-7)$
27) $-2(5a+3)(a-6)$
28) $3(7k+6)(k-5)$
29) $-3(7n-6)(n+3)$
30) $2(7x-2)(x-4)$
31) $(x+4)(6x+5)$
32) $(x+2y)(4x+y)$
33) $2(2m^2+3mn+3n^2)$
34) $(m-3n)(4m+3n)$
35) $2(2x^2-3xy+15y^2)$
36) $(x+3y)(4x+y)$
37) $3(3u+4v)(2u-3v)$
38) $2(2x+7y)(3x+5y)$
39) $4(x+3y)(4x+3y)$
40) $4(x-2y)(6x-y)$

Section 1.5: Factoring Special Products

Objective: Identify and factor special products including a difference of two perfect squares, perfect square trinomials, and sum and difference of two perfect cubes.

When factoring there are a few special products that, if we can recognize them, help us factor polynomials.

DIFFERENCE OF TWO PERFECT SQUARES

When multiplying special products, we found that a sum of a binomial and a difference of a binomial could multiply to a difference of two perfect squares. Here, we will use this special product to help us factor.

Difference of Two Perfect Squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Example 1. Factor completely.

$x^2 - 16$	Express each term as the square of a monomial
$= (x)^2 - (4)^2$	Apply the difference of two perfect squares formula: Here, $a = x$ and $b = 4$
$= (x+4)(x-4)$	Our Answer

Example 2. Factor completely.

$36 - y^2$	Express each term as the square of a monomial
$= (6)^2 - (y)^2$	Apply the difference of two perfect squares formula: Here, $a = 6$ and $b = y$
$= (6+y)(6-y)$	Our Answer

Example 3. Factor completely.

$9a^2 - 25b^2$	Express each term as the square of a monomial
$= (3a)^2 - (5b)^2$	Apply the difference of two perfect squares formula: Here, $a = 3a$ and $b = 5b$
$= (3a+5b)(3a-5b)$	Our Answer

PERFECT SQUARE TRINOMIAL

Another special case involves the perfect square trinomial. We had a shortcut for squaring a binomial, which can be reversed to help us factor a perfect square trinomial.

Perfect Square Trinomial:

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

If we do not recognize a perfect square trinomial at first glance, we use the *ac method*. If we get two of the same numbers, we know we have a perfect square trinomial. Then we can factor using the square roots of the first and last terms, and the sign from the middle term.

Example 4. Factor completely.

x^2-6x+9	Multiply to 9, sum to -6
	Numbers are -3 and -3, the same; a perfect square trinomial
	Use square roots from first and last terms and sign from middle term
$=(x-3)^2$	Our Answer

Example 5. Factor completely.

$4x^2+20xy+25y^2$	Multiply to 100, sum to 20
	Numbers are 10 and 10, the same; perfect square trinomial
	Use square roots from first and last terms and sign from middle term
$=(2x+5y)^2$	Our Answer

SUM OR DIFFERENCE OF TWO PERFECT CUBES

Another special case involves the sum or difference of two perfect cubes. The sum and the difference of two perfect cubes have very similar factoring formulas:

Sum of Two Perfect Cubes:

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Difference of Two Perfect Cubes:

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Start by expressing each term as the cube of a monomial. Use these results to determine the factored form of the expression. Comparing the formulas, you may notice that the only difference is the **signs** between the terms. One way to keep these two formulas straight is to think of **SOAP**.

> **S** stands for **S**ame sign as the original polynomial. If we have a sum of two perfect cubes, we add first; if we have a difference of two perfect cubes we subtract first.
> **O** stands for **O**pposite sign. If we have a sum, then subtraction is the second sign; a difference has addition for the second sign.
> **AP** stands for **A**lways **P**ositive. The last term for both formulas has an addition sign.

The following examples demonstrate factoring the sum or difference of two perfect cubes.

Example 6. Factor completely.

$m^3 - 27$	Express each term as the cube of a monomial
$= (m)^3 - (3)^3$	Apply the difference of two perfect cubes formula $(m \quad 3)(m^2 \quad 3m \quad 9)$; Use SOAP to fill in signs
$= (m-3)\,(m^2+3m+9)$	Our Answer

Example 7. Factor completely.

$125p^3 + 8r^3$	Express each term as the cube of a monomial
$= (5p)^3 + (2r)^3$	Apply the sum of two perfect cubes formula $(5p \quad 2r)\,(25p^2 \quad 10r \quad 4r^2)$; Use SOAP to fill in signs
$= (5p+2r)\,(25p^2 - 10pr + 4r^2)$	Our Answer

The previous example illustrates an important point. When we fill in the trinomial's first and last terms, we square the monomials $5p$ and $2r$. So, our squared terms in the second set of parentheses are $5p \cdot 5p = 25p^2$ and $2r \cdot 2r = 4r^2$. Notice that when done correctly, both the number and the variable are squared. Sometimes students forget to square both the number and the variable.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial, further. As a general rule, this factor will always be *prime* (unless there is a GCF that should have been factored before applying the appropriate perfect cubes rule).

SUMMARY OF FACTORING SPECIAL PRODUCTS

The following table summarizes all of the methods that we can use to factor special products:

FACTORING SPECIAL PRODUCTS

Difference of Squares:	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares:	prime
Perfect Square Trinomial:	$a^2 + 2ab + b^2 = (a+b)^2$
	$a^2 - 2ab + b^2 = (a-b)^2$
Sum of Cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

FACTORING USING MORE THAN ONE STRATEGY

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This process is shown in the following examples.

Example 8. Factor completely.

$72x^2 - 2$	GCF is 2; factor from each term
$= 2\,(36x^2 - 1)$	Difference of two perfect squares: $36x^2 = (6x)^2$ and $1 = (1)^2$
$= 2\,(6x+1)\,(6x-1)$	Our Answer

Example 9. Factor completely.

$48x^2y - 24xy + 3y$	GCF is $3y$; factor from each term
$= 3y\,(16x^2 - 8x + 1)$	Multiply to 16, sum to -8 Numbers are -4 and -4, the same; perfect square trinomial Use square roots from first and last terms and sign from middle term
$= 3y\,(4x-1)^2$	Our Answer

Example 10. Factor completely.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$; factor from each term
$= 2ab^2(64a^3 + 27b^3)$	Sum of two perfect cubes: $64a^3 = (4a)^3$ and $27b^3 = (3b)^3$
$= 2ab^2(4a+3b)\,(16a^2 - 12ab + 9b^2)$	Our Answer

Practice Exercises
Section 1.5: Factoring Special Products

Factor completely.

1) $x^2 - 49$
2) $x^2 - 9$
3) $v^2 - 25$
4) $1 - x^2$
5) $p^2 - 4$
6) $4v^2 - 1$
7) $64x^2 - 9y^2$
8) $9a^2 - 1$
9) $9x^2 + 1$
10) $3x^2 - 27$
11) $5n^2 - 20$
12) $16x^2 - 36$
13) $125x^2 + 45y^2$
14) $98a^2 - 50b^2$
15) $4m^2 + 64n^2$
16) $a^2 - 2a + 1$
17) $k^2 + 4k + 4$
18) $x^2 + 6x + 9$
19) $n^2 - 8n + 16$
20) $x^2 - 6x + 9$
21) $k^2 - 4k + 4$
22) $25p^2 - 10p + 1$
23) $x^2 + 2x + 1$
24) $25a^2 + 30ab + 9b^2$
25) $x^2 + 8xy + 16y^2$
26) $4a^2 - 20ab + 25b^2$
27) $49x^2 + 36y^2$
28) $8x^2 - 24xy + 18y^2$
29) $20x^2 + 20xy + 5y^2$
30) $x^3 - 8$
31) $x^3 + 64$
32) $x^3 - 64$
33) $x^3 + 8$
34) $216 - u^3$
35) $125x^3 - 216$
36) $125a^3 - 64$
37) $64x^3 - 27$
38) $64x^3 + 27y^3$
39) $32m^3 - 108n^3$
40) $54x^3 + 250y^3$

ANSWERS to Practice Exercises
Section 1.5: Factoring Special Products

1) $(x+7)(x-7)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(1+x)(1-x)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(8x+3y)(8x-3y)$
8) $(3a+1)(3a-1)$
9) prime
10) $3(x+3)(x-3)$
11) $5(n+2)(n-2)$
12) $4(2x+3)(2x-3)$
13) $5(25x^2+9y^2)$
14) $2(7a+5b)(7a-5b)$
15) $4(m^2+16n^2)$
16) $(a-1)^2$
17) $(k+2)^2$
18) $(x+3)^2$
19) $(n-4)^2$
20) $(x-3)^2$
21) $(k-2)^2$
22) $(5p-1)^2$
23) $(x+1)^2$
24) $(5a+3b)^2$
25) $(x+4y)^2$
26) $(2a-5b)^2$
27) prime
28) $2(2x-3y)^2$
29) $5(2x+y)^2$
30) $(x-2)(x^2+2x+4)$
31) $(x+4)(x^2-4x+16)$
32) $(x-4)(x^2+4x+16)$
33) $(x+2)(x^2-2x+4)$
34) $(6-u)(36+6u+u^2)$
35) $(5x-6)(25x^2+30x+36)$
36) $(5a-4)(25a^2+20a+16)$
37) $(4x-3)(16x^2+12x+9)$
38) $(4x+3y)(16x^2-12xy+9y^2)$
39) $4(2m-3n)(4m^2+6mn+9n^2)$
40) $2(3x+5y)(9x^2-15xy+25y^2)$

Section 1.6: Factoring Strategy

Objective: Identify and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which strategy to use and when. Here, we will organize all the different factoring methods we have seen.

A large part of deciding how to factor a polynomial is based the number of terms in the polynomial.

For all problem types, we will always try to factor out the GCF first.

FACTORING STRATEGY

1. If there is a common factor other than 1, factor the **Greatest Common Factor (GCF)**. Always look for the GCF first!

2. Count the number of terms. Select a method to try based on the number of terms:

   - **2 terms:** Try one of the special methods:
     Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
     Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
     Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

   - **3 terms:** Try the ***ac method***. Find two numbers that multiply to $a \cdot c$ and sum to b. Split the middle term and then continue to factor by grouping.
     If the trinomial is recognized as a perfect square trinomial, factor using one of these forms:
     $$a^2 + 2ab + b^2 = (a+b)^2 \quad \textbf{or} \quad a^2 - 2ab + b^2 = (a-b)^2$$

   - **4 terms:** Try factoring by **grouping**.

3. Check if the polynomial has been factored **completely**.

We will use the above strategy to factor each of the following examples. Here, the emphasis will be on which strategy to use rather than the steps used in that method.

Example 1. Factor completely.

$x^2 - 23x + 42$	GCF = 1, so no need to divide by GCF
	Three terms: try *ac method*
	Find factors to multiply to 42 and add to -23:
	Use -2 and -21
$= (x-2)(x-21)$	Our Answer

Example 2. Factor completely.

$z^2 + 6z - 9$	GCF = 1, so no need to divide by GCF
	Three terms: try *ac method*
	Find factors to multiply to -9 and add to 6
	Try $(-1)(9)$, $(1)(-9)$, $(3)(-3)$; none sum to 6
prime	Our Answer

Example 3. Factor completely.

$4x^2 + 56xy + 196y^2$	GCF = 4, first factor GCF from each term
$= 4(x^2 + 14xy + 49y^2)$	Three terms inside parentheses*:* try *ac method*
	Find factors to multiply to 49 and add to 14
	7 and 7; a perfect square trinomial
$= 4(x+7y)^2$	Our Answer

Example 4. Factor completely.

$5x^2y + 15xy - 35x^2 - 105x$	GCF = $5x$, first factor GCF from each term
$= 5x(xy + 3y - 7x - 21)$	Four terms inside parentheses*:* try grouping
$= 5x[y(x+3) - 7(x+3)]$	
$= 5x(x+3)(y-7)$	Our Answer

Example 5. Factor completely.

$100x^2 - 400$	GCF = 100, first factor GCF from both terms
$= 100(x^2 - 4)$	Two terms inside parentheses*:* try difference of two perfect squares
$= 100(x+2)(x-2)$	Our Answer

Example 6. Factor completely.

$108x^3y^2 - 39x^2y^2 + 3xy^2$	GCF = $3xy^2$, first factor GCF from each term
$= 3xy^2(36x^2 - 13x + 1)$	Three terms inside parentheses: try *ac method*
	Find factors to multiply to 36 and add to -13"
	Use -9 and -4

$= 3xy^2(36x^2 - 9x - 4x + 1)$	Factor by grouping
$= 3xy^2[9x(4x-1) - 1(4x-1)]$	
$= 3xy^2(4x-1)(9x-1)$	Our Answer

Example 7. Factor completely.

$5 + 625y^3$	GCF first, factor out 5 from each term
$= 5(1+125y^3)$	Two terms inside parentheses: try sum of two perfect cubes
$= 5(1+5y)(1-5y+25y^2)$	Our Answer

It is important to be comfortable and confident not just with using all the factoring methods, but also with deciding on which method to use. Your practice with these problems is very important!

Practice Exercises
Section 1.6: Factoring Strategy

Factor completely.

1) $24az-18ah+60yz-45hy$
2) $2x^2-11x+15$
3) $5u^2-9uv+4v^2$
4) $16x^2+48xy+36y^2$
5) $-2x^3+128y^3$
6) $20uv-60u^3-5xv+15xu^2$
7) $5n^3+7n^2-6n$
8) $2x^3+5x^2y+3y^2x$
9) $54u^3-16$
10) $54-128x^3$
11) n^2-n
12) $5x^2-22x-15$
13) $x^2-4xy+3y^2$
14) $45u^2-150uv+125v^2$
15) $64x^2+49y^2$
16) x^3-27y^3
17) m^2-4n^2
18) $12ab-18a+6bn-9n$
19) $36b^2c-16dx-24b^2d+24cx$
20) $3m^3-6m^2n-24mn^2$
21) $128+54x^3$
22) $64m^3+27n^3$
23) $2x^3+6x^2y-20xy^2$
24) $3ac+15ad^2+cx^2+5d^2x^2$
25) n^3+7n^2+10n
26) $64m^3-n^3$
27) $27x^3-64$
28) $16a^2-9b^2$
29) $5x^2+2x$
30) $2x^2-10x+12$
31) $-3k^3+27k^2-60k$
32) $75x^2-12y^2$
33) $mn-12x+3m-4nx$
34) $2k^2+k-10$
35) $16x^2-8xy+y^2$
36) v^2+v
37) $27m^2-48n^2$
38) x^3+4x^2
39) $9x^3+21x^2y-60xy^2$
40) $9n^3-3n^2$
41) $2m^2+6mn-20n^2$
42) $2u^2v^2-11uv^3+15v^4$
43) $5x^2-6x+7$
44) $9x^2-25y^2$
45) $2x^2-2x+14$
46) x^2-100

ANSWERS to Practice Exercises
Section 1.6: Factoring Strategy

1) $3(2a+5y)(4z-3h)$
2) $(2x-5)(x-3)$
3) $(5u-4v)(u-v)$
4) $4(2x+3y)^2$
5) $-2(x-4y)(x^2+4xy+16y^2)$
6) $5(4u-x)(v-3u^2)$
7) $n(5n-3)(n+2)$
8) $x(2x+3y)(x+y)$
9) $2(3u-2)(9u^2+6u+4)$
10) $2(3-4x)(9+12x+16x^2)$
11) $n(n-1)$
12) $(5x+3)(x-5)$
13) $(x-3y)(x-y)$
14) $5(3u-5v)^2$
15) prime
16) $(x-3y)(x^2+3xy+9y^2)$
17) $(m+2n)(m-2n)$
18) $3(2a+n)(2b-3)$
19) $4(3b^2+2x)(3c-2d)$
20) $3m(m+2n)(m-4n)$
21) $2(4+3x)(16-12x+9x^2)$
22) $(4m+3n)(16m^2-12mn+9n^2)$
23) $2x(x+5y)(x-2y)$
24) $(3a+x^2)(c+5d^2)$
25) $n(n+2)(n+5)$
26) $(4m-n)(16m^2+4mn+n^2)$
27) $(3x-4)(9x^2+12x+16)$
28) $(4a+3b)(4a-3b)$
29) $x(5x+2)$
30) $2(x-2)(x-3)$
31) $-3k(k-5)(k-4)$
32) $3(5x+2y)(5x-2y)$
33) $(m-4x)(n+3)$
34) $(2k+5)(k-2)$
35) $(4x-y)^2$
36) $v(v+1)$
37) $3(3m+4n)(3m-4n)$
38) $x^2(x+4)$
39) $3x(3x-5y)(x+4y)$
40) $3n^2(3n-1)$
41) $2(m-2n)(m+5n)$
42) $v^2(2u-5v)(u-3v)$
43) prime
44) $(3x+5y)(3x-5y)$
45) $2(x^2-x+7)$
46) $(x+10)(x-10)$

Section 1.7: Solving Equations by Factoring

Objective: Solve equations by factoring and using the zero product rule.

When solving linear equations such as $2x-5=21$, we can solve for the variable directly by adding 5 and dividing by 2 to get 13. However, when we have x^2 (or a higher power of x), we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the variable will require us to use a property known as the zero product rule.

Zero Product Rule:

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The zero product rule tells us that if two factors are multiplied together and the result is zero, then one of the factors must be zero.

USING THE ZERO PRODUCT RULE TO SOLVE EQUATIONS

We can use the zero product rule to help us solve equations having zero on one side and a factored expression on the other side as in the following example.

Example 1. Solve the equation.

$$(2x-3)(5x+1)=0 \quad \text{Set each factor equal to zero}$$

$$2x-3=0 \quad \text{or} \quad 5x+1=0 \quad \text{Solve each equation}$$

$$\underline{\;+3\;\;+3\;} \qquad \underline{\;-1\;\;-1\;}$$

$$\frac{2x}{2}=\frac{3}{2} \quad \text{or} \quad \frac{5x}{5}=\frac{-1}{5}$$

$$x=\frac{3}{2} \quad \text{or} \quad -\frac{1}{5} \quad \text{Our Solutions}$$

For the zero product rule to work, we must have factors to set equal to zero. This means if the expression is not already factored, we will need to factor it first, if possible.

Example 2. Solve the equation.

$$4x^2+x-3=0$$

Factor using the *ac* method: Find factors to multiply to -12 and add to 1

Use -3 and 4

$$4x^2-3x+4x-3=0 \quad \text{Factor by grouping}$$

$$x(4x-3)+1(4x-3)=0$$

$(4x-3)(x+1)=0$	Set each factor equal to zero
$4x-3=0$ or $x+1=0$	Solve each equation
$\underline{\quad +3 \ +3}$ $\qquad$ $\underline{\quad -1 \ -1}$	
$\dfrac{4x}{4}=\dfrac{3}{4}$ or $x=-1$	
$x=\dfrac{3}{4}$ or -1	Our Solutions

Another important part of the zero product rule is that before we factor, one side of the equation must be zero. If one side of the equation is not zero, we must move terms around so that one side of the equation is zero. Generally, we like the coefficient of the x^2 term to be positive.

Example 3. Solve the equation.

$x^2=8x-15$	Set one side equal to 0, adding $-8x$ and 15 to both sides of the equation
$\underline{-8x+15 \quad -8x+15}$	
$x^2-8x+15=0$	Factor using the *ac method*: Find factors to multiply to 15 and add to -8 Use -5 and -3
$(x-5)(x-3)=0$	Set each factor equal to zero
$x-5=0$ or $x-3=0$	Solve each equation
$\underline{\ +5 \ +5}$ $\qquad$ $\underline{\ +3 \ +3}$	
$x=5$ or $x=3$	
$x=5$ or $x=3$	Our Solutions

Example 4. Solve the equation.

$(x-7)(x+3)=-9$	Not equal to zero; multiply first using FOIL
$x^2-7x+3x-21=-9$	Combine like terms
$x^2-4x-21=-9$	Set one side equal to 0 by adding 9 to both sides of the equation
$\underline{+9=+9}$	
$x^2-4x-12=0$	Factor using the *ac method*: Find factors to multiply to -12 and add to -4 Use 6 and -2
$(x-6)(x+2)=0$	Set each factor equal to zero
$x-6=0$ or $x+2=0$	Solve each equation
$\underline{\ +6 \ +6}$ $\qquad$ $\underline{\ -2 \ -2}$	
$x=6$ or $x=-2$	Our Solutions

Example 5. Solve the equation.

$$\begin{array}{rcll} 3x^2+4x-5 &=& 7x^2+4x-14 & \text{Set one side of the equation equal to 0} \\ \underline{-3x^2-4x+5} & & \underline{-3x^2-4x+5} & \\ 0 &=& 4x^2-9 & \text{Factor using the difference of two perfect squares} \\ 0 &=& (2x+3)\,(2x-3) & \text{Set each factor equal to zero} \end{array}$$

$$\begin{array}{ccc l} 2x+3=0 & \text{or} & 2x-3=0 & \text{Solve each equation} \\ \underline{\;-3\quad -3\;} & & \underline{\;+3\quad +3\;} & \\ \dfrac{2x}{2}=\dfrac{-3}{2} & \text{or} & \dfrac{2x}{2}=\dfrac{3}{2} & \\ x=-\dfrac{3}{2} & \text{or} & x=\dfrac{3}{2} & \text{Our Solutions} \end{array}$$

Most quadratic equations (equations where the highest exponent of the variable is 2) have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 6. Solve the equation.

$$\begin{array}{rl} 4x^2 = 12x-9 & \text{Set one side of the equation equal to 0} \\ \underline{-12x+9} \quad \underline{-12x+9} & \\ 4x^2-12x+9=0 & \text{Factor using the } \textit{ac method}\text{: Find factors to multiply to} \\ & \text{36 and add to } -12 \\ & \quad \text{Use } -6 \text{ and } -6 \text{ (the same); a perfect square trinomial} \\ (2x-3)^2=0 & \text{Since } (2x-3)^2=(2x-3)(2x-3)\text{, both factors are} \\ & \text{identical. Set the factor } (2x-3) \text{ equal to zero.} \\ 2x-3=0 & \text{Solve the equation} \\ \underline{\;+3\quad +3\;} & \\ \dfrac{2x}{2}=\dfrac{3}{2} & \\ x=\dfrac{3}{2} & \text{Our Solution} \end{array}$$

As always, it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This might give us more than just two solutions. The next few examples illustrate this.

Example 7. Solve the equation.

$$\begin{array}{rl} 4x^2 = 8x & \text{Set equal to 0 by subtracting } 8x \text{ from both sides} \\ \underline{-8x \quad -8x} & \\ 4x^2-8x=0 & \text{Factor the GCF of } 4x \end{array}$$

$$4x(x-2)=0$$ Set each factor equal to zero

$$\frac{4x}{4}=\frac{0}{4} \quad \text{or} \quad x-2=0$$ Solve each equation

$$x=0 \quad \text{or} \quad \underline{\;+2 \;\; +2\;} \quad x=2$$

$$x=0 \quad \text{or} \quad x=2$$ Our Solutions

Example 8. Solve the equation.

$$2x^3-14x^2+24x=0$$ Factor the GCF of $2x$

$$2x(x^2-7x+12)=0$$ Factor with *ac method*: Find factors to multiply to 12 and add to -7

Use -3 and -4

$$2x(x-3)(x-4)=0$$ Set each factor equal to zero

$$\frac{2x}{2}=\frac{0}{2} \quad \text{or} \quad x-3=0 \quad \text{or} \quad x-4=0$$ Solve each equation

$$\underline{\;+3 \;\; +3\;} \qquad \underline{\;+4 \;\; +4\;}$$

$$x=0 \qquad x=3 \qquad x=4$$

$$x=0,\ 3, \text{or } 4$$ Our Solutions

Example 9. Solve the equation.

$$6x^2+21x-27=0$$ Factor the GCF of 3

$$3(2x^2+7x-9)=0$$ Factor with *ac method*: Find factors to multiply to -18 and add to 7

Use 9 and -2

$$3(2x^2+9x-2x-9)=0$$ Factor by grouping

$$3[x(2x+9)-1(2x+9)]=0$$

$$3(2x+9)(x-1)=0$$ Set each factor equal to zero

$$3=0 \quad \text{or} \quad 2x+9=0 \quad \text{or} \quad x-1=0$$ Solve each equation

$$\underline{\;-9=-9\;} \qquad \underline{\;+1 \;\; +1\;}$$

$$3\neq 0 \quad \text{or} \quad \frac{2x}{2}=\frac{-9}{2} \quad \text{or} \quad x=1$$

$$x=-\frac{9}{2}$$

$$x=-\frac{9}{2} \quad \text{or} \quad x=1$$ Our Solutions

In the previous example, the GCF did not contain a variable. When we set this factor equal to zero, we get a false statement. No solution comes from this factor. When the GCF has no variables, we may skip setting the GCF equal to zero.

Just as not all polynomials can be factored, not all equations can be solved by factoring. If an equation cannot be solved by factoring, we will have to use another method. These other methods are saved for another lesson.

APPLICATIONS OF SOLVING EQUATIONS

In science, we often use a mathematical model to describe a physical situation. To answer questions about the situation, we may need to set up and solve an equation. In this section, we will be able to solve the equations by factoring.

Example 10. Bob is on the balcony of his apartment, which is 80 feet above the ground. He tosses a ball vertically upward. The ball's height above the ground as it travels is modeled by the equation $h=-16t^2+64t+80$ where t is the time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

$h=-16t^2+64t+80$ The time that has passed, t, is unknown; when the ball hits the ground, its height h is zero

$0=-16t^2+64t+80$ Set the quadratic equation equal to zero and solve by factoring

$0=-16(t^2-4t-5)$ Factor the GCF of -16

$0=-16(t-5)(t+1)$ Factor with *ac method:* Find factors to multiply to -5 and add to -4
Use -5 and 1

$-16=0$ or $t-5=0$ or $t+1=0$ Set each factor equal to zero

$$\begin{array}{rcrcr} & & t-5=0 & & t+1=0 \\ & & \underline{+5\ \ +5} & & \underline{-1\ \ -1} \\ -16\neq 0 & \text{or} & t=5 & \text{or} & t=-1 \end{array}$$

$t=5$ or $t=-1$ Time cannot be negative; so $t=-1$ extraneous (not considered to be a solution to the equation in this context.

The ball hits the ground after 5 seconds. Our Solution

Practice Exercises
Section 1.7: Solving Equations by Factoring

Solve each equation by factoring.

1) $(x-1)(x+4)=0$
2) $0=(2x+5)(x-7)$
3) $x^2-4=0$
4) $2x^2-18x=0$
5) $6x^2-150=0$
6) $p^2+4p-32=0$
7) $2n^2+10n-28=0$
8) $m^2-m-30=0$
9) $7x^3+26x^2+15x=0$
10) $-16t^2+24t+16=0$
11) $x^2-4x-8=-8$
12) $x^2-5x-1=-5$
13) $a^3-6a^2+6a=-2a$
14) $7x^2+17x-20=-8$
15) $4n^2-13n+8=5$
16) $7r^2+84=-49r$
17) $x^2-6x=-9$
18) $7n^2-28n=0$
19) $3v^2+7v=40$
20) $6b^2=5+7b$
21) $35x^2+120x=-45$
22) $3n^2+3n=6$
23) $k^2+24k+89=6k+8$
24) $a^2+7a-9=-3+6a$
25) $9x^2-46+7x=7x+8x^2+3$
26) $x^2+10x+30=6$
27) $2m^2+19m+40=-2m$
28) $5n^2+41n+40=-2$
29) $24x^2+11x-80=3x$
30) $121w^2+8w-7=8w-6$

Solve each application.

31) A ball is tossed vertically upward from a building which is 96 feet above the ground. The ball's height above the ground as it travels is modeled by the equation $h=-16t^2+16t+96$ where t is the time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

32) An explosion causes debris to fly vertically upward with an initial speed of 80 feet per second. The height of the debris above the ground is modeled by the equation $h=-16t^2+80t$ where t is the time (in seconds) after the explosion and h is the height of the debris (in feet) at any particular time. How long does it take for the debris to hit the ground?

ANSWERS to Practice Exercises
Section 1.7: Solving Equations by Factoring

1) $1, -4$
2) $-\frac{5}{2}, 7$
3) $-2, 2$
4) $0, 9$
5) $-5, 5$
6) $4, -8$
7) $2, -7$
8) $-5, 6$
9) $-\frac{5}{7}, -3, 0$
10) $-\frac{1}{2}, 2$
11) $4, 0$
12) $1, 4$
13) $0, 4, 2$
14) $\frac{4}{7}, -3$
15) $\frac{1}{4}, 3$
16) $-4, -3$
17) 3
18) $4, 0$
19) $\frac{8}{3}, -5$
20) $-\frac{1}{2}, \frac{5}{3}$
21) $-\frac{3}{7}, -3$
22) $-2, 1$
23) -9
24) $2, -3$
25) $-7, 7$
26) $-4, -6$
27) $-\frac{5}{2}, -8$
28) $-\frac{6}{5}, -7$
29) $\frac{5}{3}, -2$
30) $-\frac{1}{11}, \frac{1}{11}$

31) The ball hits the ground after 3 seconds.

32) The debris hits the ground after 5 seconds.

Review: Chapter 1

Factor completely.

1) $20x^3 + 15x^2 + 35x$
2) $w^3 + 3w^2 - 2w - 6$
3) $z^2 + 8z - 20$
4) $40z^{18} + 5z^{12} + 10z^6$
5) $125x^3 + 64y^3$
6) $80x^2 - 45y^2$
7) $25x^2 + 9$
8) $10k^2 - k - 3$
9) $4y^2 - 23y - 35$
10) $4x^2 - 12x + 9$
11) $8x^3 - 1$
12) $-2x^3 + 8x^2 - 6x$
13) $45w^2 + 6w - 3$

Solve.

14) $(2x-3)(x+2) = 0$
15) $m^2 - 3m = 0$
16) $p^2 = -6p + 27$
17) $x^2 - 3x + 1 = 5$
18) $a^3 - a^2 - 6a = -4a$
19) $25x^2 = 16$
20) A ball is tossed vertically upward from a building which is 96 feet above the ground. The ball's height above the ground as it travels is modeled by the equation $h = -16t^2 + 64t + 80$ where t is the time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time. How long does it take for the ball to hit the ground?

ANSWERS to Review: Chapter 1

1) $5x(4x^2+3x+7)$
2) $(w+3)(w^2-2)$
3) $(z+10)(z-2)$
4) $5z^6(8z^{12}+z^6+2)$
5) $(5x+4y)(25x^2-20xy+16y^2)$
6) $5(4x+3y)(4x-3y)$
7) prime
8) $(5k-3)(2k+1)$
9) $(4y+5)(y-7)$
10) $(2x-3)^2$
11) $(2x-1)(4x^2+2x+1)$
12) $-2x(x-1)(x-3)$
13) $3(5w-1)(3w+1)$
14) $-2, \dfrac{3}{2}$
15) $0, 3$
16) $-9, 3$
17) $-1, 4$
18) $-1, 0, 2$
19) $-\dfrac{4}{5}, \dfrac{4}{5}$
20) The ball hits the ground after 5 seconds.

CHAPTER 2

Rational Expressions and Equations

Objectives
Chapter 2

- Reduce rational expressions by dividing out common factors.
- Multiply rational expressions.
- Divide rational expressions.
- Identify the least common denominator and build equivalent fractions using that common denominator.
- Add and subtract rational expressions with like and different denominators.
- Solve proportions using the cross product.
- Use proportions to solve application problems.
- Solve rational equations by identifying and multiplying by the least common denominator.
- Solve application problems by creating a rational equation to model the problem.
- Model and solve direct, inverse, joint and combined variation problems.

Section 2.1: Reduce Rational Expressions

Objective: Reduce rational expressions by dividing out common factors.

A **rational expression** is a quotient of polynomials. Examples of rational expressions include:

$$\frac{x^2-x-12}{x^2-9x+20} \quad \text{and} \quad \frac{3}{x-2} \quad \text{and} \quad \frac{a-b}{b-a} \quad \text{and} \quad \frac{3}{2}$$

DETERMINING EXCLUDED VALUES FOR A RATIONAL EXPRESSION

It is important to remember that the denominator of a fraction cannot have a value of zero. A rational expression is undefined when its denominator equals 0. W must exclude all value(s) of the variable that make a denominator zero. We must determine the value(s) that will give us zero in the denominator, and then exclude those values as possible values of the variable.

Example 1. State the excluded value(s).

$\dfrac{4x}{x+2}$	Find values of x for which denominator is 0 Set the denominator equal to 0
$x+2=\ 0$	Solve the equation
$\underline{-2\ \ -2}$	
$x=-2$	
Excluded value: $x=-2$	Our Answer

We can evaluate the rational expression for any other value of x except for -2.

Example 2. State the excluded value(s).

$\dfrac{x^2-1}{3x^2+5x}$	Find values of x for which denominator is 0 Set the denominator equal to 0
$3x^2+5x=0$	Factor the left side
$x(3x+5)=0$	Set each factor equal to zero
$x=0$ or $3x+5=0$	Solve each equation
$\underline{-5\ \ -5}$	
$\dfrac{3x}{3}=-\dfrac{5}{3}$	
$x=-\dfrac{5}{3}$	Second equation solved
Excluded values: $x=0$ and $-\dfrac{5}{3}$	Our Answer

EVALUATING A RATIONAL EXPRESSION

We evaluate rational expressions by substituting the given value for the variable and then simplifying using the order of operations.

Example 3. Evaluate the expression for the given value of the variable.

$\dfrac{x^2-4}{x^2+6x+8}$ when $x=-6$	Substitute -6 in for each variable
$\dfrac{(-6)^2-4}{(-6)^2+6(-6)+8}$	Exponents first
$=\dfrac{36-4}{36+6\,(-6)+8}$	Multiply
$=\dfrac{36-4}{36-36+8}$	Add and subtract
$=\dfrac{32}{8}$	Reduce
$=4$	Our Answer

SIMPLIFYING A RATIONAL EXPRESSION

Just as we reduced the fraction in the previous example, often a rational expression can be reduced. When we reduce, we divide out common factors. A rational expression is reduced, or simplified, if the numerator and denominator have no common factors other than 1 or -1.

If the rational expression only has monomials, we can reduce the coefficients, and divide out common factors of the variables.

Example 4. Simplify the expression.

$\dfrac{15x^4y^2}{25x^2y^6}$	Reduce 15 and 25 by dividing out the common factor of 5; Subtract the exponents: $x^{4-2}=x^2$ and $y^{2-6}=y^{-4}$; Negative exponents move to denominator because $y^{-4}=\dfrac{1}{y^4}$
$=\dfrac{3x^2}{5y^4}$	Our Answer

If the rational expression has more than one term in either the numerator or denominator, we need to first factor the numerator or denominator and then divide out common factors.

Example 5. Simplify the expression.

$\dfrac{28}{8x^2-16}$	Denominator has a common factor of 8; factor.
$=\dfrac{28}{8(x^2-2)}$	Reduce by dividing 28 and 8 by 4.
$=\dfrac{7}{2(x^2-2)}$	Our Answer

Example 6. Simplify the expression.

$\dfrac{9x-3}{18x-6}$	Numerator has a common factor of 3; denominator has a common factor of 6; factor both the numerator and denominator
$=\dfrac{3(3x-1)}{6(3x-1)}$	Divide out common factor of $(3x-1)$, and divide both 3 and 6 by 3
$=\dfrac{1}{2}$	Our Answer

Example 7. Simplify the expression.

$\dfrac{x^2-25}{x^2+8x+15}$	Factor both the numerator and the denominator; numerator is the difference of two squares; denominator can be factored using the *ac method*;
$=\dfrac{(x+5)\,(x-5)}{(x+5)\,(x+3)}$	Divide out common factor of $(x+5)$
$=\dfrac{x-5}{x+3}$	Our Answer

It is important to remember that we cannot divide *terms*, only *factors*. This means if there are any $+$ or $-$ signs between the parts we want to reduce, we cannot reduce. In the previous example, we had the solution $\dfrac{x-5}{x+3}$ we cannot divide out the x s because they are terms (separated by $+$ or $-$), not factors (separated by multiplication).

FACTORS THAT ARE OPPOSITES

Sometimes, factors in the numerator and denominator are *opposites*. To simplify, factor -1 from the numerator or denominator.

Example 8. Simplify the expression.

$\dfrac{5-x}{x-5}$	Rewrite the numerator
$=\dfrac{-x+5}{x-5}$	Factor -1 from the numerator
$=\dfrac{-1(x-5)}{x-5}$	Divide out common factor of $(x-5)$
$=-1$	Our Answer

Example 9. Simplify the expression.

$\dfrac{7-x}{x^2-49}$	Rewrite the numerator
$=\dfrac{-x+7}{x^2-49}$	Factor -1 from the numerator; Factor denominator as difference of two squares
$=\dfrac{-1\,(x-7)}{(x+7)\,(x-7)}$	Divide out common factor of $(x-7)$
$=-\dfrac{1}{x+7}$	Our Answer

Practice Exercises
Section 2.1: Reduce Rational Expressions

Evaluate.

1) $\dfrac{4v+2}{6}$ when $v=4$

2) $\dfrac{b-3}{3b-9}$ when $b=-2$

3) $\dfrac{x-3}{x^2-4x+3}$ when $x=-4$

4) $\dfrac{a+2}{a^2+3a+2}$ when $a=-1$

5) $\dfrac{b+2}{b^2+4b+4}$ when $b=0$

6) $\dfrac{n^2-n-6}{n-3}$ when $n=4$

State the excluded value(s).

7) $\dfrac{3k^2+30k}{k+10}$

8) $\dfrac{27p}{18p^2-36p}$

9) $\dfrac{15n^2}{10n+25}$

10) $\dfrac{x+10}{8x^2+80x}$

11) $\dfrac{10m^2+8m}{10m}$

12) $\dfrac{10x+16}{6x+20}$

13) $\dfrac{r^2+3r+2}{5r+10}$

14) $\dfrac{6n^2-21n}{6n^2+3n}$

15) $\dfrac{b^2+12b+32}{b^2+4b-32}$

16) $\dfrac{10v^2-30v}{35v^2-5v}$

Simplify each expression.

17) $\dfrac{21x^2}{18x}$

18) $\dfrac{24a}{40a^2}$

19) $\dfrac{32x^3}{8x^4}$

20) $\dfrac{90x^2}{20x}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.1 (continued)

Simplify each expression.

21) $\dfrac{18m-24}{60}$

22) $\dfrac{20}{4p+2}$

23) $\dfrac{x+3}{3+x}$

24) $\dfrac{5-x}{x-5}$

25) $\dfrac{9-n}{9n-81}$

26) $\dfrac{x+7}{7-x}$

27) $\dfrac{x+1}{x^2+8x+7}$

28) $\dfrac{28m+12}{36}$

29) $\dfrac{32x^2}{28x^2+28x}$

30) $\dfrac{49r+56}{56r}$

31) $\dfrac{n^2+4n-12}{n^2-7n+10}$

32) $\dfrac{b^2+14b+48}{b^2+15b+56}$

33) $\dfrac{9v+54}{v^2-4v-60}$

34) $\dfrac{30x-90}{50x+40}$

35) $\dfrac{12x^2-42x}{30x^2-42x}$

36) $\dfrac{k^2-12k+32}{k^2-64}$

37) $\dfrac{6a-10}{10a+4}$

38) $\dfrac{9p+18}{p^2+4p+4}$

39) $\dfrac{2n^2+19n-10}{9n+90}$

40) $\dfrac{3x^2-29x+40}{5x^2-30x-80}$

41) $\dfrac{3-x}{5x^2-45}$

42) $\dfrac{9r^2+63r}{5r^2+40r+35}$

43) $\dfrac{2x^2-10x+8}{3x^2-7x+4}$

44) $\dfrac{50b-80}{50b+20}$

45) $\dfrac{7n^2-32n+16}{4n-16}$

46) $\dfrac{35v+35}{21v+7}$

47) $\dfrac{n^2-2n+1}{6n+6}$

48) $\dfrac{56x-48}{24x^2+56x+32}$

49) $\dfrac{7a^2-26a-45}{6a^2-34a+20}$

50) $\dfrac{4k^3-2k^2-2k}{9k^3-18k^2+9k}$

ANSWERS to Practice Exercises
Section 2.1: Reduce Rational Expressions

1) 3

2) $\frac{1}{3}$

3) $-\frac{1}{5}$

4) undefined

5) $\frac{1}{2}$

6) 6

---

7) -10

8) $0, 2$

9) $-\frac{5}{2}$

10) $0, -10$

11) 0

12) $-\frac{10}{3}$

13) -2

14) $0, -\frac{1}{2}$

15) $-8, 4$

16) $0, \frac{1}{7}$

---

17) $\frac{7x}{6}$

18) $\frac{3}{5a}$

19) $\frac{4}{x}$

20) $\frac{9x}{2}$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 2.1 (continued)

21) $\frac{3m-4}{10}$

22) $\frac{10}{2p+1}$

23) 1

24) -1

25) $-\frac{1}{9}$

26) cannot be simplified

27) $\frac{1}{x+7}$

28) $\frac{7m+3}{9}$

29) $\frac{8x}{7(x+1)}$

30) $\frac{7r+8}{8r}$

31) $\frac{n+6}{n-5}$

32) $\frac{b+6}{b+7}$

33) $\frac{9}{v-10}$

34) $\frac{3(x-3)}{5x+4}$

35) $\frac{2x-7}{5x-7}$

36) $\frac{k-4}{k+8}$

37) $\frac{3a-5}{5a+2}$

38) $\frac{9}{p+2}$

39) $\frac{2n-1}{9}$

40) $\frac{3x-5}{5(x+2)}$

41) $-\frac{1}{5(x+3)}$

42) $\frac{9r}{5(r+1)}$

43) $\frac{2(x-4)}{3x-4}$

44) $\frac{5b-8}{5b+2}$

45) $\frac{7n-4}{4}$

46) $\frac{5(v+1)}{3v+1}$

47) $\frac{(n-1)^2}{6(n+1)}$

48) $\frac{7x-6}{(3x+4)(x+1)}$

49) $\frac{7a+9}{2(3a-2)}$

50) $\frac{2(2k+1)}{9(k-1)}$

Section 2.2: Multiply and Divide Rational Expressions

Objectives: Multiply rational expressions.
Divide rational expressions.

When, multiplying and dividing rational expressions, we will use the same process as we do when multiplying and dividing fractions. Always be sure the answer is written in simplest form.

MULTIPLYING RATIONAL EXPRESSIONS

Example 1. Multiply, expressing the resulting fraction in its lowest terms.

$\frac{15}{49}\cdot\frac{14}{45}$ First, reduce by dividing out the common factors from numerator and denominator (15 and 7)

$=\frac{1}{7}\cdot\frac{2}{3}$ Multiply the numerators together and the denominators together

$=\frac{2}{21}$ Our Answer

When multiplying rational expressions, we first divide the numerators and denominators by any common factors. Then we multiply the remaining factors straight across.

Example 2. Multiply.

$\frac{25x^2}{9y^8}\cdot\frac{24y^4}{55x^7}$ Reduce coefficients by dividing out the common factors from the numerator and the denominator (3 and 5)

Reduce the variable terms by subtracting exponents
$(x^{2-7})=x^{-5}$; $(y^{4-8})=y^{-4}$

Negative exponent (x^{-5}) moves to the denominator as $\frac{1}{x^5}$;

Likewise, (y^{-4}) *t*o the denominator as $\frac{1}{y^4}$

$=\frac{5}{3y^4}\cdot\frac{8}{11x^5}$ Multiply across (numerators together; denominators together)

$=\frac{40}{33x^5y^4}$ Our Answer

If the rational expression in either the numerator or the denominator is factorable, it must be factored first. That way, any common factors can be divided out before multiplying.

Example 3. Multiply.

$\dfrac{x^2-9}{x^2+x-20}\cdot\dfrac{x^2-8x+16}{3x+9}$	Factor each numerator and denominator
$=\dfrac{(x+3)(x-3)}{(x-4)(x+5)}\cdot\dfrac{(x-4)(x-4)}{3(x+3)}$	Divide out the common factors $(x+3)$ and $(x-4)$
$=\dfrac{x-3}{x+5}\cdot\dfrac{x-4}{3}$	Multiply across
$=\dfrac{(x-3)(x-4)}{3(x+5)}$	Our Answer

DIVIDING RATIONAL EXPRESSIONS

When dividing rational expressions, we change the division problem into an equivalent multiplication problem. Multiply the first expression by the reciprocal of the divisor. In other words, keep the first expression, change the operation from division to multiplication, and "flip" the second expression. Then, multiply as shown in the examples above, writing the answer in simplest form.

Example 4. Divide.

$\dfrac{a^4b^2}{a}\div\dfrac{b^4}{4}$	Multiply the first expression by the reciprocal of the second expression
$=\dfrac{a^4b^2}{a}\cdot\dfrac{4}{b^4}$	Subtract exponents; negative exponents move to denominator
$=\dfrac{a^3}{1}\cdot\dfrac{4}{b^2}$	Multiply across (numerators together; denominators together)
$=\dfrac{4a^3}{b^2}$	Our Answer

Example 5. Divide.

$\dfrac{x^2-x-12}{x^2-2x-8} \div \dfrac{5x^2+15x}{x^2+x-2}$	Multiply by the reciprocal
$= \dfrac{x^2-x-12}{x^2-2x-8} \cdot \dfrac{x^2+x-2}{5x^2+15x}$	Factor each numerator and denominator
$= \dfrac{(x-4)\,(x+3)}{(x+2)\,(x-4)} \cdot \dfrac{(x+2)\,(x-1)}{5x\,(x+3)}$	Divide out the common factors $(x-4)$, $(x+3)$, and $(x+2)$
$= \dfrac{1}{1} \cdot \dfrac{x-1}{5x}$	Multiply across (numerators together; denominators together)
$= \dfrac{x-1}{5x}$	Our Answer

The example below contains both multiplication and division. To perform these operations, we change division to multiplication by the reciprocal of the divisor, factor wherever possible, reduce if possible, and then multiply the remaining factors.

Example 6. Multiply and divide as indicated.

$\dfrac{a^2+7a+10}{a^2+6a+5} \cdot \dfrac{a+1}{4a+8} \div \dfrac{(a-1)}{(a+2)}$	Factor each numerator and denominator
$= \dfrac{(a+5)\,(a+2)}{(a+5)\,(a+1)} \cdot \dfrac{(a+1)}{4(a+2)} \div \dfrac{(a-1)}{(a+2)}$	Multiply by the reciprocal of last fraction
$= \dfrac{(a+5)\,(a+2)}{(a+5)\,(a+1)} \cdot \dfrac{(a+1)}{4(a+2)} \cdot \dfrac{(a+2)}{(a-1)}$	Divide out the common factors $(a+5)$, $(a+2)$, and $(a+1)$
$= \dfrac{a+2}{4(a-1)}$	Our Answer

Practice Exercises
Section 2.2: Multiply and Divide Rational Expressions

Multiply.

1) $\frac{8x^2}{9}\cdot\frac{9}{2}$

2) $\frac{9n}{2n}\cdot\frac{7}{5n}$

3) $\frac{5x^2}{4}\cdot\frac{6}{5}$

4) $\frac{7(m-6)}{m-6}\cdot\frac{5m(7m-5)}{7(7m-5)}$

5) $\frac{6x(x+4)}{x-3}\cdot\frac{(x-3)(x-6)}{6x(x-6)}$

6) $\frac{25n+25}{5}\cdot\frac{4}{30n+30}$

7) $\frac{v-1}{4}\cdot\frac{4}{v^2-11v+10}$

8) $\frac{x^2-6x-7}{x+5}\cdot\frac{x+5}{x-7}$

Divide.

9) $\frac{8x}{3x}\div\frac{4}{7}$

10) $\frac{9m}{5m^2}\div\frac{7}{2}$

11) $\frac{10p}{5}\div\frac{8}{10}$

12) $\frac{7}{10(n+3)}\div\frac{n-2}{(n+3)(n-2)}$

13) $\frac{7r}{7r(r+10)}\div\frac{r-6}{(r-6)^2}$

14) $\frac{9}{b^2-b-12}\div\frac{b-5}{b^2-b-12}$

15) $\frac{x-10}{35x+21}\div\frac{7}{35x+21}$

16) $\frac{8k}{24k^2-40k}\div\frac{1}{15k-25}$

Perform the indicated operation.

17) $\frac{1}{a-6}\cdot\frac{8a+80}{8}$

18) $\frac{p-8}{p^2-12p+32}\div\frac{1}{p-10}$

19) $(n-8)\cdot\frac{6}{10n-80}$

20) $\frac{x^2-7x+10}{x-2}\cdot\frac{x+10}{x^2-x-20}$

21) $\frac{4m+36}{m+9}\cdot\frac{m-5}{5m^2}$

22) $\frac{2r}{r+6}\div\frac{2r}{7r+42}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.2 (continued)

Perform the indicated operation.

23) $\dfrac{3x-6}{12x-24}\cdot(x+3)$

24) $\dfrac{2n^2-12n-54}{n+7}\div(2n+6)$

25) $\dfrac{b+2}{40b^2-24b}(5b-3)$

26) $\dfrac{21v^2+16v-16}{3v+4}\div\dfrac{35v-20}{v-9}$

27) $\dfrac{n-7}{6n-12}\cdot\dfrac{12-6n}{n^2-13n+42}$

28) $\dfrac{x^2+11x+24}{6x^3+18x^2}\cdot\dfrac{6x^3+6x^2}{x^2+5x-24}$

29) $\dfrac{27a+36}{9a+63}\div\dfrac{6a+8}{2}$

30) $\dfrac{k-7}{k^2-k-12}\cdot\dfrac{7k^2-28k}{8k^2-56k}$

31) $\dfrac{x^2-12x+32}{x^2-6x-16}\cdot\dfrac{7x^2+14x}{7x^2+21x}$

32) $\dfrac{9x^3+54x^2}{x^2+5x-14}\cdot\dfrac{x^2+5x-14}{10x^2}$

33) $(10m^2+100m)\cdot\dfrac{18m^3-36m^2}{20m^2-40m}$

34) $\dfrac{n-7}{n^2-2n-35}\div\dfrac{9n+54}{10n+50}$

35) $\dfrac{7p^2+25p+12}{6p+48}\cdot\dfrac{3p-8}{21p^2-44p-32}$

36) $\dfrac{7x^2-66x+80}{49x^2+7x-72}\div\dfrac{7x^2+39x-70}{49x^2+7x-72}$

37) $\dfrac{10b^2}{30b+20}\cdot\dfrac{30b+20}{2b^2+10b}$

38) $\dfrac{35n^2-12n-32}{49n^2-91n+40}\cdot\dfrac{7n^2+16n-15}{5n+4}$

39) $\dfrac{7r^2-53r-24}{7r+2}\div\dfrac{49r+21}{49r+14}$

40) $\dfrac{12x+24}{10x^2+34x+28}\cdot\dfrac{15x+21}{5}$

41) $\dfrac{x^2-1}{2x-4}\cdot\dfrac{x^2-4}{x^2-x-2}\div\dfrac{x^2+x-2}{3x-6}$

42) $\dfrac{a^3+b^3}{a^2+3ab+2b^2}\cdot\dfrac{3a-6b}{3a^2-3ab+3b^2}\div\dfrac{a^2-4b^2}{a+2b}$

ANSWERS to Practice Exercises
Section 2.2: Multiply and Divide Rational Expressions

1) $4x^2$

2) $\frac{63}{10n}$

3) $\frac{3x^2}{2}$

4) $5m$

5) $x+4$

6) $\frac{2}{3}$

7) $\frac{1}{v-10}$

8) $x+1$

---

9) $\frac{14}{3}$

10) $\frac{18}{35m}$

11) $\frac{5p}{2}$

12) $\frac{7}{10}$

13) $\frac{r-6}{r+10}$

14) $\frac{9}{b-5}$

15) $\frac{x-10}{7}$

16) 5

---

17) $\frac{a+10}{a-6}$

18) $\frac{p-10}{p-4}$

19) $\frac{3}{5}$

20) $\frac{x+10}{x+4}$

21) $\frac{4(m-5)}{5m^2}$

22) 7

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 2.2 (continued)

23) $\frac{x+3}{4}$

24) $\frac{n-9}{n+7}$

25) $\frac{b+2}{8b}$

26) $\frac{v-9}{5}$

27) $-\frac{1}{n-6}$

28) $\frac{x+1}{x-3}$

29) $\frac{1}{a+7}$

30) $\frac{7}{8(k+3)}$

31) $\frac{x-4}{x+3}$

32) $\frac{9(x+6)}{10}$

33) $9m^2(m+10)$

34) $\frac{10}{9(n+6)}$

35) $\frac{p+3}{6(p+8)}$

36) $\frac{x-8}{x+7}$

37) $\frac{5b}{b+5}$

38) $n+3$

39) $r-8$

40) $\frac{18}{5}$

41) $\frac{3}{2}$

42) $\frac{1}{a+2b}$

Section 2.3: Least Common Denominator

Objective: Identify the least common denominator and build equivalent fractions using that common denominator.

In the previous section, we performed the operations of multiplication and division of rational expressions. We also need to learn to perform addition and subtraction of rational expressions.

In this section, we will focus on finding the LCD of two or more rational expressions. We will then express each rational expression as an equivalent one with the LCD as the denominator. We will hold off on actually adding or subtracting until the next section.

FINDING THE LEAST COMMON DENOMINATOR (LCD)

The process used to find the LCD of rational expressions is based on the process used to find the LCD of integers.

Example 1. Find the Least Common Denominator.

Find the LCD of $\frac{1}{8}$ and $\frac{5}{6}$	Consider multiples of the larger denominator: $8, 16, 24, \ldots$ 24 is the first multiple of 8 that is also divisible by 6
LCD is 24	Our Answer

When finding the LCD of several monomials, we first find the LCD of the coefficients. The variable part of the LCD uses the *highest* exponent of each unique variable.

Example 2. Find the Least Common Denominator.

Find the LCD of $\frac{1}{4x^2y^5}$ and $\frac{7}{6x^4y^3z^6}$	First find the LCD of coefficients 4 and 6: 12 is the LCD of 4 and 6 because it is the smallest number that both 4 and 6 divide into without a remainder Then find the LCD of variables: Use the highest exponent for each variable: x^4, y^5, and z^6
LCD is $12x^4y^5z^6$	Our Answer

The same idea is used when finding the LCD of polynomials that have more than one term. First factor each polynomial and then identify all the factors to be used (attaching highest exponent if necessary).

Example 3. Find the Least Common Denominator.

Find the LCD of $\dfrac{4}{x^2+2x-3}$ and $\dfrac{9}{x^2-x-12}$

$x^2+2x-3=(x-1)(x+3)$	Factor each polynomial
$x^2-x-12=(x-4)(x+3)$	
$(x-1)(x+3)$ and $(x-4)(x+3)$	LCD uses all unique factors with the highest exponent; Notice $(x+3)$ is common to both expressions; we only include it once
LCD is $(x-1)(x+3)(x-4)$	Our Answer

Notice we only used the factor (x+3) once in our LCD because it only appears as a factor once in either polynomial.

We will need to repeat a factor or use an exponent on a factor if there are multiple like factors associated with one of the polynomials in completely factored form.

Example 4. Find the Least Common Denominator.

Find the LCD of $\dfrac{3x}{x^2-10x+25}$ and $\dfrac{1}{x^2-14x+45}$

$x^2-10x+25=(x-5)(x-5)=(x-5)^2$	Factor each polynomial
$x^2-14x+45=(x-5)(x-9)$	
$(x-5)^2$ and $(x-5)(x-9)$	LCD uses all unique factors with the highest exponent
LCD is $(x-5)^2(x-9)$	Our Answer

The previous example could have also been done by factoring the first polynomial to $(x-5)(x-5)$ and not expressing it as $(x-5)^2$. We still would have included $(x-5)$ twice in the LCD because it showed up twice in one of the polynomials. However, expressing the factors using exponents allows us to use the same pattern (including the highest exponent in the LCD) that we previously used with monomials.

BUILDING EQUIVALENT EXPRESSIONS WITH THE LCD AS DENOMINATOR

Once we know the LCD, our goal will be to build up fractions so that they have matching denominators. Whenever we alter the denominator of a fraction by multiplying to get the LCD, we must multiply by the same factor in the numerator of that fraction in order to keep the fraction equivalent to its original value. We can build up a fraction to an equivalent one with a specified denominator by multiplying the numerator and denominator by any factors that are part of the LCD, but not part of the original denominator.

Example 5. Build the rational expression to an equivalent expression with the specified denominator. State the value of the missing numerator.

$\dfrac{5a}{3a^2b} = \dfrac{?}{6a^5b^3}$ Identify what factors we need to match denominators:

The missing factor is $2a^3b^2$ because $3 \cdot 2 = 6$ and we need three more factors of a and two more factors of b

$\dfrac{5a}{3a^2b}\left(\dfrac{2a^3b^2}{2a^3b^2}\right)$ Multiply both numerator and denominator by missing factor

$= \dfrac{10a^4b^2}{6a^5b^3}$ Find the value of the numerator

$10a^4b^2$ Our Answer

Example 6. Build the rational expression to an equivalent expression with the specified denominator. State the value of the missing numerator.

$\dfrac{x-2}{x+4} = \dfrac{?}{x^2+7x+12}$ Factor to identify factors we need to match denominators:

$(x+4)\,(x+3)$ The missing factor is $(x+3)$

$\dfrac{x-2}{x+4}\left(\dfrac{x+3}{x+3}\right)$ Multiply numerator and denominator by missing factor

$= \dfrac{(x-2)\,(x+3)}{(x+4)\,(x+3)}$ Multiply (FOIL) the numerator

$= \dfrac{x^2+x-6}{(x+4)\,(x+3)}$ Find the value of the numerator

x^2+x-6 Our Answer

As the above example illustrates, we multiply out the numerators, but keep the denominators in factored form. The reason for multiplying out only the numerators is that in order to add and subtract rational expressions, we need to combine like terms in the numerators. However, once the like terms have been added/subtracted in the numerator, we factor so that we can see if the expression can be reduced. Since we reduce factors, both the numerator and the denominator should be in completely factored format, and common factors can be divided out. We will discuss this process in detail in the next section.

In the previous examples, we were given a specified denominator and asked to build up to an equivalent expression with that given denominator. Now we will consider more than one rational expression at a time. First, we will identify the Least Common Denominator of the given expressions. Then, we will build up each rational expression to an equivalent one with that LCD as the denominator. In this section, we are not adding and subtracting fractions, just building them up to a common denominator.

Example 7. Build up each rational expression so they have a common denominator.

$\frac{5a}{4b^3c}$ and $\frac{3c}{6a^2b}$	First, identify the LCD; in this case the least common denominator is $12a^2b^3c$
	Determine what factors each fraction's denominator is missing:
	First is missing $3a^2$ and second is missing $2b^2c$
$\frac{5a}{4b^3c}\left(\frac{3a^2}{3a^2}\right)$ and $\frac{3c}{6a^2b}\left(\frac{2b^2c}{2b^2c}\right)$	Multiply each corresponding fraction's numerator and denominator by that denominator's missing factors to get equivalent fractions
$\frac{15a^3}{12a^2b^3c}$ and $\frac{6b^2c^2}{12a^2b^3c}$	Our Answers

Example 8. Build up each rational expression so they have a common denominator.

$\frac{5x}{x^2-5x-6}$ and $\frac{x-2}{x^2+4x+3}$	Factor denominators to find LCD
$(x-6)\,(x+1)$ $\quad$ $(x+1)\,(x+3)$	Use factors to find LCD:
	LCD is $(x-6)\,(x+1)\,(x+3)$
	Determine what factors each fraction's denominator is missing:
	First: $(x+3)$ Second: $(x-6)$

$$\frac{5x}{(x-6)\,(x+1)}\left(\frac{x+3}{x+3}\right) \text{ and } \frac{x-2}{(x+1)\,(x+3)}\left(\frac{x-6}{x-6}\right)$$

Multiply each corresponding fraction's numerator and denominator by that denominator's missing factors to get equivalent fractions

$$\frac{5x^2+15x}{(x-6)\,(x+1)\,(x+3)} \text{ and } \frac{x^2-8x+12}{(x-6)\,(x+1)\,(x+3)}$$

Our Answers

Practice Exercises
Section 2.3: Least Common Denominator

Build the rational expression to an equivalent expression with the specified denominator. State the value of the missing numerator.

1) $\frac{3}{8} = \frac{?}{48}$

2) $\frac{a}{5} = \frac{?}{5a}$

3) $\frac{a}{x} = \frac{?}{xy}$

4) $\frac{5}{2x^2} = \frac{?}{8x^3y}$

5) $\frac{2}{3a^3b^2c} = \frac{?}{9a^5b^2c^4}$

6) $\frac{4}{3a^5b^2c^4} = \frac{?}{9a^5b^2c^4}$

7) $\frac{2}{x+4} = \frac{?}{x^2-16}$

8) $\frac{x+1}{x-3} = \frac{?}{x^2-6x+9}$

9) $\frac{x-4}{x+2} = \frac{?}{x^2+5x+6}$

10) $\frac{x-6}{x+3} = \frac{?}{x^2-2x-15}$

Find the Least Common Denominator.

11) $\frac{1}{2a^3}, \frac{7}{6a^4b^2}, \frac{3}{4a^3b^5}$

12) $\frac{4}{5x^2y}, \frac{9}{25x^3y^5z}$

13) $\frac{5}{x^2-3x}, \frac{1}{x-3}, \frac{2}{x}$

14) $\frac{7}{4x-8}, \frac{3}{x-2}, \frac{1}{4}$

15) $\frac{2}{x+2}, \frac{8}{x-4}$

16) $\frac{5}{x}, \frac{1}{x-7}, \frac{9x}{x+1}$

17) $\frac{4}{x^2-25}, \frac{6}{x+5}$

18) $\frac{7x}{x^2-9}, \frac{3}{x^2-6x+9}$

19) $\frac{10}{x^2+3x+2}, \frac{1}{x^2+5x+6}$

20) $\frac{2}{x^2-7x+10}, \frac{20}{x^2-2x-15}, \frac{3x^2}{x^2+x-6}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.3 (continued)

Find the LCD and build up each rational expression so they have a common denominator.

21) $\dfrac{3a}{5b^2}, \dfrac{2}{10a^3b}$

22) $\dfrac{3x}{x-4}, \dfrac{2}{x+2}$

23) $\dfrac{x+2}{x-3}, \dfrac{x-3}{x+2}$

24) $\dfrac{5}{x^2-6x}, \dfrac{2}{x}, \dfrac{-3}{x-6}$

25) $\dfrac{x}{x^2-16}, \dfrac{3x}{x^2-8x+16}$

26) $\dfrac{5x+1}{x^2-3x-10}, \dfrac{4}{x-5}$

27) $\dfrac{x+1}{x^2-36}, \dfrac{2x+3}{x^2+12x+36}$

28) $\dfrac{3x+1}{x^2-x-12}, \dfrac{2x}{x^2+4x+3}$

29) $\dfrac{4x}{x^2-x-6}, \dfrac{x+2}{x-3}$

30) $\dfrac{3x}{x^2-6x+8}, \dfrac{x-2}{x^2+x-20}, \dfrac{5}{x^2+3x-10}$

ANSWERS to Practice Exercises
Section 2.3: Least Common Denominator

1) 18
2) a^2
3) ay
4) $20xy$
5) $6a^2c^3$
6) 12
7) $2x-8$
8) x^2-2x-3
9) x^2-x-12
10) $x^2-11x+30$

---

11) $12a^4b^5$
12) $25x^3y^5z$
13) $x(x-3)$
14) $4(x-2)$
15) $(x+2)(x-4)$
16) $x(x-7)(x+1)$
17) $(x+5)(x-5)$
18) $(x-3)^2(x+3)$
19) $(x+1)(x+2)(x+3)$
20) $(x-2)(x-5)(x+3)$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 2.3 (continued)

21) $\dfrac{6a^4}{10a^3b^2}, \dfrac{2b}{10a^3b^2}$

22) $\dfrac{3x^2+6x}{(x-4)(x+2)}, \dfrac{2x-8}{(x-4)(x+2)}$

23) $\dfrac{x^2+4x+4}{(x-3)(x+2)}, \dfrac{x^2-6x+9}{(x-3)(x+2)}$

24) $\dfrac{5}{x(x-6)}, \dfrac{2x-12}{x(x-6)}, \dfrac{-3x}{x(x-6)}$

25) $\dfrac{x^2-4x}{(x-4)^2(x+4)}, \dfrac{3x^2+12x}{(x-4)^2(x+4)}$

26) $\dfrac{5x+1}{(x-5)(x+2)}, \dfrac{4x+8}{(x-5)(x+2)}$

27) $\dfrac{x^2+7x+6}{(x-6)(x+6)^2}, \dfrac{2x^2-9x-18}{(x-6)(x+6)^2}$

28) $\dfrac{3x^2+4x+1}{(x-4)(x+3)(x+1)}, \dfrac{2x^2-8x}{(x-4)(x+3)(x+1)}$

29) $\dfrac{4x}{(x-3)(x+2)}, \dfrac{x^2+4x+4}{(x-3)(x+2)}$

30) $\dfrac{3x^2+15x}{(x-4)(x-2)(x+5)}, \dfrac{x^2-4x+4}{(x-4)(x-2)(x+5)}, \dfrac{5x-20}{(x-4)(x-2)(x+5)}$

Section 2.4: Add and Subtract Rational Expressions

Objective: Add and subtract rational expressions with like and different denominators.

You will recall that when adding fractions with a common denominator, we add the numerators and keep the denominator. This same process is used with rational expressions. Remember to reduce your sum or difference, if possible, to obtain your final answer.

ADDING AND SUBTRACTING RATIONAL EXPRESSIONS WITH THE *SAME* DENOMINATOR

Example 1. Add the rational expressions, and simplify if possible.

$$\frac{x-4}{x^2-2x-8}+\frac{x+8}{x^2-2x-8}$$ Same denominator; add numerators; combine like terms

$$=\frac{2x+4}{x^2-2x-8}$$ Factor numerator and denominator

$$=\frac{2(x+2)}{(x+2)(x-4)}$$ Divide out common factor of $(x+2)$ to reduce the fraction to its lowest terms

$$=\frac{2}{x-4}$$ Our Answer

Subtraction with common denominators follows the same pattern. However, subtraction can cause problems if we are not careful to avoid sign errors. Consequently, we will first distribute the subtraction sign to every term in the numerator of the fraction that follows the subtraction sign. Then we can treat it like an addition problem.

Example 2. Subtract the rational expressions, and simplify if possible.

$$\frac{6x-12}{3x-6}-\frac{15x-6}{3x-6}$$ Add the opposite of the second fraction (distribute the subtraction to each term in the second fraction)

$$=\frac{6x-12}{3x-6}+\frac{-15x+6}{3x-6}$$ Add the numerators; combine like terms

$$=\frac{-9x-6}{3x-6}$$ Factor numerator and denominator

$$= \frac{-3(3x+2)}{3(x-2)}$$ Divide out common factor of 3

$$= -\frac{(3x+2)}{x-2}$$ Our Answer

ADDING AND SUBTRACTING RATIONAL EXPRESSIONS WITH *DIFFERENT* DENOMINATORS

When the denominators of the rational expressions are not like, we find their least common denominator (LCD). Then we build up each fraction to an equivalent one with that LCD as the denominator. The following example shows this process with fractions.

Example 3. Add the fractions, and simplify if possible.

$$\frac{5}{6}+\frac{1}{4}$$ The LCD is 12. Build up, multiply 6 by 2 and 4 by 3 to get the common denominator;

$$=\frac{5}{6}\left(\frac{2}{2}\right)+\frac{1}{4}\left(\frac{3}{3}\right)$$ Multiply first fraction by $\left(\frac{2}{2}\right)$ and second by $\left(\frac{3}{3}\right)$

$$=\frac{10}{12}+\frac{3}{12}$$ Add the numerators

$$=\frac{13}{12}$$ Our Answer

The same process is used with rational expressions containing variables.

Example 4. Add the rational expressions, and simplify if possible.

$$\frac{7a}{3a^2b}+\frac{4b}{6ab^4}$$ The LCD is $6a^2b^4$. Build up each expression:

$(3a^2b)(2b^3)=6a^2b^4$ and $(6ab^4)(a)=6a^2b^4$

$$=\frac{7a}{3a^2b}\left(\frac{2b^3}{2b^3}\right)+\frac{4b}{6ab^4}\left(\frac{a}{a}\right)$$ Multiply first fraction by $\left(\frac{2b^3}{2b^3}\right)$ and second by $\left(\frac{a}{a}\right)$

$=\dfrac{14ab^3}{6a^2b^4}+\dfrac{4ab}{6a^2b^4}$ Add the numerators, there are no like terms to combine

$=\dfrac{14ab^3+4ab}{6a^2b^4}$ Factor the numerator

$=\dfrac{2ab(7b^2+2)}{6a^2b^4}$ Reduce, dividing out common factors of 2, a, and b

$=\dfrac{7b^2+2}{3ab^3}$ Our Answer

The same process is used for subtraction; we will simply include the first step of changing the subtraction to addition of the opposite value.

Example 5. Subtract the rational expressions, and simplify if possible.

$\dfrac{4}{5a}-\dfrac{7b}{4a^2}$ Change subtraction to addition of the opposite

$=\dfrac{4}{5a}+\dfrac{-7b}{4a^2}$ The LCD is $20a^2$. Build up each expression:

$(5a)(4a)=20a^2$ and $(4a^2)(5)=20a^2$

$=\dfrac{4}{5a}\left(\dfrac{4a}{4a}\right)+\dfrac{-7b}{4a^2}\left(\dfrac{5}{5}\right)$ Multiply first fraction by $\left(\dfrac{4a}{4a}\right)$ and second by $\left(\dfrac{5}{5}\right)$

$=\dfrac{16a}{20a^2}+\dfrac{-35b}{20a^2}$ Add numerators; there are no like terms to combine

$=\dfrac{16a-35b}{20a^2}$ Factor the numerator, if possible. In this case, the numerator is prime. This is our answer.

If our denominators have more than one term, then we will need to factor first to find the LCD. Next, we build up each fraction using the factors that are missing from each denominator.

Example 6. Add the rational expressions, and simplify if possible.

$\dfrac{6}{8a+4}+\dfrac{3a}{8}$ Factor denominators to find LCD

$=\dfrac{6}{4(2a+1)}+\dfrac{3a}{8}$ The LCD is $8(2a+1)$. Build up each expression:

$4(2a+1)\cdot 2=8(2a+1)$ and

$(8)(2a+1)=8(2a+1)$

$=\dfrac{6}{4(2a+1)}\left(\dfrac{2}{2}\right)+\dfrac{3a}{8}\left(\dfrac{2a+1}{2a+1}\right)$ Multiply first fraction by $\left(\dfrac{2}{2}\right)$, second by $\left(\dfrac{2a+1}{2a+1}\right)$

$=\dfrac{12}{8(2a+1)}+\dfrac{6a^2+3a}{8(2a+1)}$ Add numerators

$=\dfrac{6a^2+3a+12}{8(2a+1)}$ Factor the numerator

$=\dfrac{3(2a^2+a+4)}{8(2a+1)}$ Cannot be simplified

$=\dfrac{3(2a^2+a+4)}{8(2a+1)}$ Our Answer

Example 7. Add the rational expressions, and simplify if possible.

$\dfrac{3}{5y+20}+\dfrac{y+2}{6y^2+23y-4}$ Factor denominators to find LCD

$=\dfrac{3}{5(y+4)}+\dfrac{y+2}{(6y-1)(y+4)}$ The LCD is $5(y+4)\,(6y-1)$. Build up each expression

$=\dfrac{3}{5(y+4)}+\dfrac{y+2}{(6y-1)(y+4)}$ Multiply first fraction by $\left(\dfrac{6y-1}{6y-1}\right)$ and second by $\left(\dfrac{5}{5}\right)$

$=\dfrac{3}{5(y+4)}\cdot\left(\dfrac{6y-1}{6y-1}\right)+\dfrac{y+2}{(6y-1)(y+4)}\cdot\left(\dfrac{5}{5}\right)$ Multiply the numerators and denominators

$=\dfrac{18y-3}{5(y+4)(6y-1)}+\dfrac{5y+10}{5(6y-1)(y+4)}$ Add numerators, combine like terms

$=\dfrac{23y+7}{5(y+4)(6y-1)}$ Factor the numerator, if possible. In this case, the numerator is prime. This is our answer.

Whenever you encounter a subtraction problem, remember to rewrite the problem using addition of the opposite.

Example 8. Subtract the rational expressions, and simplify if possible.

$\dfrac{x+1}{x-4}-\dfrac{x+1}{x^2-7x+12}$	Add the opposite: Distribute the subtraction to each term in the numerator of the second fraction
$=\dfrac{x+1}{x-4}+\dfrac{-x-1}{x^2-7x+12}$	Factor denominators to find LCD
$=\dfrac{x+1}{x-4}+\dfrac{-x-1}{(x-3)(x-4)}$	The LCD is $(x-4)(x-3)$. Build up each expression
$=\dfrac{x+1}{x-4}\left(\dfrac{x-3}{x-3}\right)+\dfrac{-x-1}{(x-3)(x-4)}$	Only the first fraction needs to be multiplied by $(x-3)$
$=\dfrac{x^2-2x-3}{(x-3)(x-4)}+\dfrac{-x-1}{(x-3)(x-4)}$	Add the numerators, combine like terms
$=\dfrac{x^2-3x-4}{(x-3)(x-4)}$	Factor the numerator
$=\dfrac{(x-4)(x+1)}{(x-3)(x-4)}$	Divide out the common factor of $(x-4)$
$=\dfrac{x+1}{x-3}$	Our Answer

Practice Exercises
Section 2.4: Add and Subtract Rational Expressions

Add or subtract, expressing the result in its lowest terms.

1) $\dfrac{2}{a+3}+\dfrac{4}{a+3}$

2) $\dfrac{x^2}{x-2}-\dfrac{6x-8}{x-2}$

3) $\dfrac{t^2+4t}{t-1}+\dfrac{2t-7}{t-1}$

4) $\dfrac{a^2+3a}{a^2+5a-6}-\dfrac{4}{a^2+5a-6}$

5) $\dfrac{2x^2+3}{x^2-6x+5}-\dfrac{x^2-5x+9}{x^2-6x+5}$

6) $\dfrac{3}{x}+\dfrac{4}{x^2}$

7) $\dfrac{5}{6r}-\dfrac{5}{8r}$

8) $\dfrac{7}{xy^2}+\dfrac{3}{x^2y}$

9) $\dfrac{8}{9t^3}+\dfrac{5}{6t^2}$

10) $\dfrac{x+5}{8}+\dfrac{x-3}{12}$

11) $\dfrac{a+2}{2}-\dfrac{a-4}{4}$

12) $\dfrac{2a-1}{3a^2}+\dfrac{5a+1}{9a}$

13) $\dfrac{x-1}{4x}-\dfrac{2x+3}{x}$

14) $\dfrac{2c-d}{c^2d}-\dfrac{c+d}{cd^2}$

15) $\dfrac{5x+3y}{2x^2y}-\dfrac{3x+4y}{xy^2}$

16) $\dfrac{2}{x-1}+\dfrac{2}{x+1}$

17) $\dfrac{2z}{z-1}-\dfrac{3z}{z+1}$

18) $\dfrac{2}{x-5}+\dfrac{3}{4x}$

19) $\dfrac{8}{x^2-4}-\dfrac{3}{x+2}$

20) $\dfrac{4x}{x^2-25}+\dfrac{x}{x+5}$

21) $\dfrac{t}{t-3}-\dfrac{5}{4t-12}$

22) $\dfrac{2}{x+3}+\dfrac{4}{(x+3)^2}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.4 (continued)

23) $\frac{2}{5x^2+5x}-\frac{4}{3x+3}$

24) $\frac{3a}{4a-20}+\frac{9a}{6a-30}$

25) $\frac{t}{y-t}-\frac{y}{y+t}$

26) $\frac{x}{x-5}+\frac{x-5}{x}$

27) $\frac{x}{x^2+5x+6}-\frac{2}{x^2+3x+2}$

28) $\frac{2x}{x^2-1}-\frac{3}{x^2+5x+4}$

29) $\frac{x}{x^2+15x+56}-\frac{7}{x^2+13x+42}$

30) $\frac{2x}{x^2-9}+\frac{5}{x^2+x-6}$

31) $\frac{5x}{x^2-x-6}-\frac{18}{x^2-9}$

32) $\frac{4x}{x^2-2x-3}-\frac{3}{x^2-5x+6}$

33) $\frac{2x}{x^2-1}-\frac{4}{x^2+2x-3}$

34) $\frac{x-1}{x^2+3x+2}+\frac{x+5}{x^2+4x+3}$

35) $\frac{x+1}{x^2-2x-35}+\frac{x+6}{x^2+7x+10}$

36) $\frac{3x+2}{3x+6}+\frac{x}{4-x^2}$

37) $\frac{4-a^2}{a^2-9}-\frac{a-2}{3-a}$

38) $\frac{4y}{y^2-1}-\frac{2}{y}-\frac{2}{y+1}$

39) $\frac{2z}{1-2z}+\frac{3z}{2z+1}-\frac{3}{4z^2-1}$

40) $\frac{2r}{r^2-s^2}+\frac{1}{r+s}-\frac{1}{r-s}$

41) $\frac{2x-3}{x^2+3x+2}+\frac{3x-1}{x^2+5x+6}$

42) $\frac{x+2}{x^2-4x+3}+\frac{4x+5}{x^2+4x-5}$

43) $\frac{2x+7}{x^2-2x-3}-\frac{3x-2}{x^2+6x+5}$

44) $\frac{3x-8}{x^2+6x+8}+\frac{2x-3}{x^2+3x+2}$

ANSWERS to Practice Exercises
Section 2.4: Add and Subtract Rational Expressions

1) $\dfrac{6}{a+3}$

2) $x-4$

3) $t+7$

4) $\dfrac{a+4}{a+6}$

5) $\dfrac{x+6}{x-5}$

6) $\dfrac{3x+4}{x^2}$

7) $\dfrac{5}{24r}$

8) $\dfrac{7x+3y}{x^2y^2}$

9) $\dfrac{15t+16}{18t^3}$

10) $\dfrac{5x+9}{24}$

11) $\dfrac{a+8}{4}$

12) $\dfrac{5a^2+7a-3}{9a^2}$

13) $\dfrac{-7x-13}{4x}$

14) $\dfrac{-c^2+cd-d^2}{c^2d^2}$

15) $\dfrac{3y^2-3xy-6x^2}{2x^2y^2}$

16) $\dfrac{4x}{(x+1)(x-1)}$

17) $\dfrac{-z^2+5z}{(z+1)(z-1)}$

18) $\dfrac{11x-15}{4x(x-5)}$

19) $\dfrac{14-3x}{(x+2)(x-2)}$

20) $\dfrac{x^2-x}{(x+5)(x-5)}$

21) $\dfrac{4t-5}{4(t-3)}$

22) $\dfrac{2x+10}{(x+3)^2}$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 2.4 (continued)

23) $\dfrac{6-20x}{15x(x+1)}$

24) $\dfrac{9a}{4(a-5)}$

25) $\dfrac{t^2+2ty-y^2}{(y+t)(y-t)}$

26) $\dfrac{2x^2-10x+25}{x(x-5)}$

27) $\dfrac{x-3}{(x+3)(x+1)}$

28) $\dfrac{2x+3}{(x-1)(x+4)}$

29) $\dfrac{x-8}{(x+8)(x+6)}$

30) $\dfrac{2x-5}{(x-3)(x-2)}$

31) $\dfrac{5x+12}{(x+3)(x+2)}$

32) $\dfrac{4x+1}{(x+1)(x-2)}$

33) $\dfrac{2x+4}{(x+3)(x+1)}$

34) $\dfrac{2x+7}{(x+3)(x+2)}$

35) $\dfrac{2x-8}{(x-7)(x+2)}$

36) $\dfrac{-3x^2+7x+4}{3(x+2)(2-x)}$

37) $\dfrac{a-2}{(a+3)(a-3)}$

38) $\dfrac{2}{y(y-1)}$

39) $\dfrac{z-3}{2z-1}$

40) $\dfrac{2}{r+s}$

41) $\dfrac{5(x-1)}{(x+1)(x+3)}$

42) $\dfrac{5x+5}{(x+5)(x-3)}$

43) $\dfrac{-(x-29)}{(x-3)(x+5)}$

44) $\dfrac{5x-10}{(x+4)(x+1)}$

Section 2.5: Proportions

Objectives: Solve proportions using the cross product.
Use proportions to solve application problems.

When two fractions are equal, they are said to be in proportion. This definition can be generalized for two equal rational expressions.

The following principle is true for any proportion and will be useful when solving proportions.

CROSS PRODUCT:

If $\frac{a}{b} = \frac{c}{d}$ **, then** $ad = bc$**.**

SOLVING A PROPORTION USING THE CROSS PRODUCT

To solve a proportion $\frac{a}{b} = \frac{c}{d}$, set the cross products ad and bc equal and solve the resulting equation.

Example 1. Solve the proportion for x.

$\frac{20}{6} = \frac{x}{9}$ Set the cross products equal

$(20)(9) = 6x$ Multiply

$\frac{180}{6} = \frac{6x}{6}$ Divide both sides by 6

$30 = x$ Our Solution

If the proportion has more than one term in either the numerator or denominator, we distribute when calculating the cross product.

Example 2. Solve the proportion for x.

$$\frac{x+3}{4}=\frac{2}{5}$$ Set the cross products equal

$$5(x+3)=(4)(2)$$ Multiply and distribute

$$5x+15=8$$ Solve

$$\underline{\quad -15 \quad -15}$$ Subtract 15 from both sides

$$\frac{5x}{5}=\frac{-7}{5}$$ Divide both sides by 5

$$x=-\frac{7}{5}$$ Our Solution

Example 3. Solve the proportion for x.

$$\frac{4}{x}=\frac{6}{3x+2}$$ Set the cross products equal

$$4(3x+2)=6x$$ Distribute

$$12x+8=6x$$ Move variables to one side of the equation

$$\underline{-12x \quad -12x}$$ Subtract $12x$ from both sides

$$\frac{8}{-6}=\frac{-6x}{-6}$$ Divide both sides by -6, simplify fraction

$$-\frac{4}{3}=x$$ Our Solution

Example 4. Solve the proportion for x.

$$\frac{2x-3}{7x+4}=\frac{2}{5}$$ Set the cross products equal

$$5(2x-3)=2(7x+4)$$ Distribute

$10x-15=14x+8$	Move variables to one side of the equation
$-10x \quad -10x$	Subtract $10x$ from both sides
$-15=4x+8$	
$-8 \quad -8$	Subtract 8 from both sides
$\frac{-23}{4}=\frac{4x}{4}$	Divide both sides by 4
$-\frac{23}{4}=x$	Our Solution

When solving a proportion, we may end up with a quadratic equation to solve. In this section, we will solve the quadratic equations in the same way we solved quadratics previously - by factoring. Other methods, such as completing the square or utilizing the quadratic formula, will be discussed in a later chapter. As before, we will generally end up with two solutions.

Example 5. Solve the proportion for k.

$\frac{k+3}{3}=\frac{8}{k-2}$	Set the cross products equal
$(k+3)(k-2)=(8)(3)$	FOIL and multiply
$k^2+k-6=24$	Set the equation equal to zero
$-24 \quad -24$	Subtract 24 from both sides
$k^2+k-30=0$	Factor completely
$(k+6)(k-5)=0$	Set each factor equal to zero
$k+6=0$ or $k-5=0$	Solve each equation
$-6 \ -6$ $+5 \ +5$	Add or subtract
$k=-6$ or $k=5$	Our Solutions

SOLVING APPLICATION PROBLEMS USING PROPORTIONS

Proportions are very useful in that they can be used in many different types of applications. We can use them to compare different quantities and make conclusions about how quantities are related.

Proportions can be used in situations where multiplying one variable by a value k results in the other variable also being multiplied by k. For example, suppose that a person gets an hourly wage. If that person works twice as long they would make twice as much money. A contrasting example is someone who has a normal work week of 40 hours and will get an overtime bonus for working extra hours. Working twice as much would get that person more than twice as much money. Another example of proportional reasoning is if something has a set price, then buying twice as much would cost twice as much. A contrasting example is if there is a "buy 12, get one free" deal. Buying twice as much then would not always cost twice as much. In the following examples, assume that the variables involved are indeed proportional.

As we set up these problems using proportions, it is important to stay organized. For example, if we are comparing dogs and cats and the number of dogs is in the numerator of the first fraction, then the numerator of the second fraction must also refer to the number of dogs. This consistency of the numerator and denominator is essential in setting up proportions.

Example 6. Solve.

A six foot tall man casts a shadow that is 3.5 feet long. If the shadow of a flag pole is 8 feet long, how tall is the flag pole? Round the answer to the tenths place.

$\frac{\text{shadow}}{\text{height}}$ We will put shadows in numerator; heights in denominator

The man has a shadow of 3.5 feet and a height of 6 feet: Write $\frac{3.5}{6}$

The flag pole has a shadow of 8 feet, but the height is unknown: Write $\frac{8}{x}$

Set up the proportion

$\frac{3.5}{6} = \frac{8}{x}$ Set the cross products equal

$3.5x = (8)(6)$ Multiply

$\frac{3.5x}{3.5} = \frac{48}{3.5}$ Divide both sides by 3.5 and round to the tenths place.

$x \approx 13.7$ feet Our Solution

Example 7. Solve.

In a basketball game, the home team was down by 9 points at the end of the game. They only scored 6 points for every 7 points the visiting team scored. What was the final score of the game?

$\frac{\text{home}}{\text{visitor}}$ We will put the home team in numerator, visitors in denominator

The solution is continued on the next page.

$\dfrac{\text{home}}{\text{visitor}}$	Visitor's score is unknown so label as x; the home team scored 9 points less than the visitors or $x-9$: Write $\dfrac{x-9}{x}$ Home team scored 6 points for every 7 points the visiting team scored: Write $\dfrac{6}{7}$ Set up the proportion
$\dfrac{x-9}{x}=\dfrac{6}{7}$	Set the cross products equal
$7(x-9)=6x$	Distribute
$7x-63=6x$ $\underline{-7x \quad -7x}$	Move variables to one side Subtract $7x$ from both sides
$\dfrac{-63}{-1}=-\dfrac{-x}{-1}$	Divide both sides by -1
$63=x$	We used x for the visiting team's score.
$63-9=54$	Subtract 9 to get the home team's score
63 to 54	Our Solution — The visiting team scored 63 points and the home team scored 54 points

Practice Exercises
Section 2.5: Proportions

Solve each proportion.

1) $\frac{10}{a}=\frac{6}{8}$

2) $\frac{7}{9}=\frac{n}{6}$

3) $\frac{7}{6}=\frac{2}{k}$

4) $\frac{8}{x}=\frac{4}{8}$

5) $\frac{6}{x}=\frac{8}{2}$

6) $\frac{n-10}{8}=\frac{9}{3}$

7) $\frac{m-1}{5}=\frac{8}{2}$

8) $\frac{8}{5}=\frac{3}{x-8}$

9) $\frac{2}{9}=\frac{10}{p-4}$

10) $\frac{9}{n+2}=\frac{3}{9}$

11) $\frac{b-10}{7}=\frac{b}{4}$

12) $\frac{9}{4}=\frac{r}{r-4}$

13) $\frac{x}{5}=\frac{x+2}{9}$

14) $\frac{n}{8}=\frac{n-4}{3}$

15) $\frac{3}{10}=\frac{a}{a+2}$

16) $\frac{x+1}{9}=\frac{x+2}{2}$

17) $\frac{v-5}{v+6}=\frac{4}{9}$

18) $\frac{n+8}{10}=\frac{n-9}{4}$

19) $\frac{7}{x-1}=\frac{4}{x-6}$

20) $\frac{k+5}{k-6}=\frac{8}{5}$

21) $\frac{x+5}{5}=\frac{6}{x-2}$

22) $\frac{4}{x-3}=\frac{x+5}{5}$

23) $\frac{m+3}{4}=\frac{11}{m-4}$

24) $\frac{x-5}{8}=\frac{4}{x-1}$

25) $\frac{2}{p+4}=\frac{p+5}{3}$

26) $\frac{5}{n+1}=\frac{n-4}{10}$

27) $\frac{n+4}{3}=\frac{-3}{n-2}$

28) $\frac{1}{n+3}=\frac{n+2}{2}$

29) $\frac{3}{x+4}=\frac{x+2}{5}$

30) $\frac{x-5}{4}=\frac{-3}{x+3}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.5 (continued)

Answer each question. Round your answer to the nearest tenth. Round dollar amounts to the nearest cent.

31) The currency in Western Samoa is the Tala. The exchange rate is approximately $0.70 to 1 Tala. At this rate, how many dollars would you get if you exchanged 13.3 Tala?

32) If you can buy one plantain for $0.49 then how many plantains can you buy with $7.84?

33) Kali reduced the size of a painting to a height of 1.3 inches. What is the new width if it was originally 5.2 inches tall and 10 inches wide?

34) A model train has a scale of 1.2 inches : 2.9 feet. If the model train is 5 inches tall, how tall is the real train?

35) A bird bath that is 5.3 feet tall casts a shadow that is 25.4 feet long. Find the length of the shadow that an 8.2 feet adult elephant casts.

36) Victoria and Georgetown are 36.2 miles from each other. How far apart would the cities be on a map that has a scale of 0.9 inches : 10.5 miles?

37) The Vikings led the Timberwolves by 19 points at the half. If the Vikings scored 3 points for every 2 points the Timberwolves scored, what was the half time score?

38) Sarah worked 10 more hours than Josh. If Sarah worked 7 hours for every 2 hours Josh worked, how long did they each work?

39) Computer Services Inc. charges $8 more for a repair than Low Cost Computer Repair. If the ratio of the costs is 3 : 6, what will it cost for the repair at Low Cost Computer Repair?

40) Kelsey's commute is 15 minutes longer than Christina's. If Christina drives 12 minutes for every 17 minutes Kelsey drives, how long is each commute?

ANSWERS to Practice Exercises
Section 2.5: Proportions

1) $a = \frac{40}{3}$

2) $n = \frac{14}{3}$

3) $k = \frac{12}{7}$

4) $x = 16$

5) $x = \frac{3}{2}$

6) $n = 34$

7) $m = 21$

8) $x = \frac{79}{8}$

9) $p = 49$

10) $n = 25$

11) $b = -\frac{40}{3}$

12) $r = \frac{36}{5}$

13) $x = \frac{5}{2}$

14) $n = \frac{32}{5}$

15) $a = \frac{6}{7}$

16) $v = -\frac{16}{7}$

17) $v = \frac{69}{5}$

18) $n = \frac{61}{3}$

19) $x = \frac{38}{3}$

20) $k = \frac{73}{3}$

21) $x = -8, 5$

22) $x = -7, 5$

23) $m = -7, 8$

24) $x = -3, 9$

25) $p = -7, -2$

26) $n = -6, 9$

27) $n = -1$

28) $x = -4, -1$

29) $x = -7, 1$

30) $x = -1, 3$

ANSWERS to Practice Exercises: Section 2.5 (continued)

31) $\$9.31$

32) 16

33) 2.5 inches

34) 12.1 feet

35) 39.3 feet

36) 3.1 inches

37) T: 38, V: 57

38) J: 4 hours, S: 14 hours

39) $\$8$

40) C: 36 minutes, K: 51 minutes

Section 2.6: Solving Rational Equations

Objective: Solve rational equations by identifying and multiplying by the least common denominator.

When solving equations that are made up of rational expressions, we will use the same strategy we use to solve linear equations with fractions. As shown in the example below, we clear the fractions by multiplying both sides of the equation by the least common denominator.

Example 1. Solve the equation.

$$\frac{2}{3}x-\frac{5}{6}=\frac{3}{4}$$ Multiply each term by the LCD 12

$$\frac{2(12)}{3}x-\frac{5(12)}{6}=\frac{3(12)}{4}$$ Reduce fractions, which clears the denominators $\frac{2(12)}{3}x=8x;\quad -\frac{5(12)}{6}=-10;\quad \frac{3(12)}{4}=9$

$$8x-10=9$$ Solve for x

$$\underline{\quad +10\quad +10}$$ First, add 10 to both sides

$$\frac{8x}{8}=\frac{19}{8}$$ Then; divide both sides by 8

$$x=\frac{19}{8}$$ Our Solution

To solve rational equations, we will use the same process of multiplying by the LCD to clear the fractions. We will also have to be aware of values of the variable that must be excluded. Recall that whenever its denominator is zero, a fraction is undefined; so, any solution that would make the denominator's value zero is not a part of the solution set. For this reason, we will always check our proposed solutions to make sure they are not excluded values.

SOLVING RATIONAL EQUATIONS

1. Determine the excluded value(s), the value(s) of the variable making a denominator zero.
2. Multiply both sides of the equation by the LCD to clear the fractions.
3. Solve the resulting equation.
4. Reject any proposed solution(s) that is an excluded value.

Example 2. Solve the equation.

$$\frac{5x+5}{x+2}+3x=\frac{x^2}{x+2}$$

Find values of x for which denominators are 0:
$x+2=0$ when $x=-2$
So $x=-2$ is an excluded value

Multiply each term by the LCD $(x+2)$

$$\frac{(5x+5)(x+2)}{x+2}+3x(x+2)=\frac{x^2(x+2)}{x+2}$$

Reduce fractions

$$5x+5+3x(x+2)=x^2$$

Distribute

$$5x+5+3x^2+6x=x^2$$

Combine like terms

$$\begin{array}{l} 3x^2+11x+5=x^2 \\ \underline{-x^2 \qquad\qquad\quad -x^2} \end{array}$$

Make equation equal zero by subtracting x^2 from both sides

$$2x^2+11x+5=0$$

Factor completely

$$(2x+1)(x+5)=0$$

Set each factor equal to zero

$$\begin{array}{ccc} 2x+1=0 & \text{or} & x+5=0 \\ \underline{\quad -1 \quad -1} & & \underline{\quad -5 \quad -5} \\ \dfrac{2x}{2}=-\dfrac{1}{2} & \text{or} & x=-5 \end{array}$$

Solve each equation

$$x=-\frac{1}{2} \text{ or } -5$$

Proposed solutions do not match the excluded value $x=-2$ so these are our solutions

The LCD can contain several factors in these problems. Even as the LCD gets more complex, the process for solving a rational equation is the same.

Example 3. Solve the equation.

$$\frac{x}{x+2}+\frac{1}{x+1}=\frac{5}{(x+1)(x+2)}$$

Find values of x for which denominators are 0:
$x+2=0$ when $x=-2$
$x+1=0$ when $x=-1$
So $x=-2$ and $x=-1$ are excluded values

Use factors to find the LCD $(x+1)(x+2)$

Multiply each term by the LCD $(x+1)(x+2)$

$$\frac{x(x+1)(x+2)}{x+2}+\frac{1(x+1)(x+2)}{x+1}=\frac{5(x+1)(x+2)}{(x+1)(x+2)}$$ Reduce fractions

$$x(x+1)+1(x+2)=5$$ Distribute

$$x^2+x+x+2=5$$ Combine like terms

$$\begin{array}{r} x^2+2x+2=5 \\ -5 \quad -5 \\ \hline \end{array}$$ Make equation equal zero by subtracting 5 from both sides

$$x^2+2x-3=0$$ Factor completely

$$(x+3)(x-1)=0$$ Set each factor equal to zero

$$\begin{array}{rcr} x+3=0 & \text{or} & x-1=0 \\ -3 \quad -3 & & +1 \quad +1 \\ \hline x=-3 & \text{or} & x=1 \end{array}$$ Solve each equation

$x=-3$ or $x=1$ Proposed solutions do not match the excluded values $x=-2$ and $x=-1$ so these are our solutions

In the previous example the denominators were already in factored form. More often, we will need to factor before finding the LCD.

Example 4. Solve the equation.

$$\frac{x}{x-1}-\frac{1}{x-2}=\frac{11}{x^2-3x+2}$$ Factor denominator

$$\frac{x}{x-1}-\frac{1}{x-2}=\frac{11}{(x-1)(x-2)}$$ Find values of x for which denominators are 0:

$x-1=0$ when $x=1$

$x-2=0$ when $x=2$

So $x=1$ and $x=2$ are excluded values

Use factors to find LCD: $(x-1)(x-2)$

Multiply each term by the LCD $(x-1)(x-2)$

$$\frac{x(x-1)(x-2)}{x-1}-\frac{1(x-1)(x-2)}{x-2}=\frac{11(x-1)(x-2)}{(x-1)(x-2)}$$ Reduce fractions

$$x(x-2)-1(x-1)=11$$ Distribute

$$x^2-2x-x+1=11$$ Combine like terms

$$\begin{array}{r} x^2-3x+1=11 \\ -11 \quad -11 \\ \hline x^2-3x-10=0 \end{array}$$ Make equation equal zero by subtracting 11 from both sides

Factor completely

$$(x-5)(x+2)=0$$ Set each factor equal to zero

$$\begin{array}{rcr} x-5=0 & \text{or} & x+2=0 \\ +5 \quad +5 & & -2 \quad -2 \\ \hline x=5 & \text{or} & x=-2 \end{array}$$ Solve each equation

$$x=5 \text{ or } -2$$ Proposed solutions do not match the excluded values $x=1$ and $x=2$ so these are our solutions

If we are subtracting a fraction with more than one term in the numerator, it may be easier to avoid future sign errors by first distributing the negative throughout the numerator in the subtrahend (rational expression immediately following the subtraction sign).

Example 5. Solve the equation.

$$\frac{x-2}{x-3}-\frac{x+2}{x+2}=\frac{5}{8}$$ Distribute the negative through numerator

$$\frac{x-2}{x-3}+\frac{-x-2}{x+2}=\frac{5}{8}$$ Find values of x for which denominators are 0:

$x-3=0$ when $x=3$

$x+2=0$ when $x=-2$

So $x=3$ and $x=-2$ are excluded values

Use factors to find LCD: $8(x-3)(x+2)$

Multiply each term by the LCD $8(x-3)(x+2)$

$$\frac{(x-2)8(x-3)(x+2)}{x-3}+\frac{(-x-2)8(x-3)(x+2)}{x+2}=\frac{5\cdot 8(x-3)(x+2)}{8}$$ Reduce fractions

$$8(x-2)(x+2)+8(-x-2)(x-3)=5(x-3)(x+2)$$ Multiply (FOIL)

$$8(x^2-4)+8(-x^2+x+6)=5(x^2-x-6)$$ Distribute

$$8x^2-32-8x^2+8x+48=5x^2-5x-30$$ Combine like terms

$$\begin{array}{rcl} 8x+16 &=& 5x^2-5x-30 \\ \underline{-8x-16} && \underline{-8x-16} \\ 0 &=& 5x^2-13x-46 \end{array}$$

Make equation equal zero by subtracting $8x$ and 16

Factor completely using *ac method*

$$0=(5x-23)(x+2)$$ Set each factor equal to zero

$$\begin{array}{ccc} 5x-23=0 & \text{or} & x+2=0 \\ \underline{+23 \quad +23} & & \underline{-2 \quad -2} \\ \frac{5x}{5}=\frac{23}{5} & \text{or} & x=-2 \end{array}$$ Solve each equation

$$x=\frac{23}{5}\text{ or }-2$$ The proposed solution $x=\frac{23}{5}$ *is* a solution because it is not an excluded value.

The proposed solution $x=-2$ is *not* a solution because it is an excluded value.

$$x=\frac{23}{5}$$ Our Solution

In the previous example, one of the proposed solutions was an excluded value because it made one of the denominators zero. When this happens, we exclude this result. The only solution(s) to the original rational equation are those that do not result with zero in the denominator.

In Examples 2 through 5 above, the equation that resulted after clearing the fractions was a quadratic equation. Notice in Example 6 below that the resulting equation is a linear equation.

Example 6. Solve the equation.

$\dfrac{2}{x-2}=\dfrac{x}{x-2}-3$	Find values of x for which denominators are 0: $x-2=0$ when $x=2$ So $x=2$ is an excluded value Use factors to find LCD: $(x-2)$ Multiply each term by the LCD $(x-2)$
$\dfrac{2(x-2)}{x-2}=\dfrac{x(x-2)}{x-2}-3(x-2)$	Reduce fractions
$2=x-3(x-2)$	Distribute
$2=x-3x+6$	Combine like terms
$2=-2x+6$ $\underline{-6 \qquad\quad -6}$	Solve for x by subtracting 6
$\dfrac{-4}{-2}=\dfrac{-2x}{-2}$ $2=x$	Divide by -2
No solution	The proposed solution $x=2$ is *not* a solution because it is an excluded value. Since there are no other proposed solutions, the original equation has no solution.

Practice Exercises
Section 2.6: Solving Rational Equations

Solve each equation:

1) $\frac{x}{2}-\frac{x}{3}=4$

2) $\frac{3}{x}+\frac{1}{2}=\frac{7}{x}$

3) $\frac{1}{3x}=\frac{5}{6}+\frac{3}{8x}$

4) $\frac{3}{x+8}=\frac{4}{6-x}$

5) $\frac{x+1}{x-1}=\frac{2x+3}{2x-5}$

6) $\frac{4}{x+3}+\frac{5}{x-1}=\frac{-7}{x^2+2x-3}$

7) $\frac{7}{x-4}-\frac{3}{x+4}=\frac{22}{x^2-16}$

8) $\frac{x+7}{x+3}=\frac{4}{x+3}$

9) $\frac{3m}{2m-5}-\frac{7}{3m+1}=\frac{3}{2}$

10) $\frac{4x}{2x-6}-\frac{4}{5x-15}=\frac{1}{2}$

11) $\frac{4-x}{1-x}=\frac{12}{3-x}$

12) $\frac{1}{x-5}+\frac{1}{x+5}=\frac{10}{x^2-25}$

13) $\frac{7}{y-3}-\frac{1}{2}=\frac{y-2}{y-4}$

14) $\frac{2}{3-x}-\frac{6}{8-x}=1$

15) $\frac{1}{x+2}-\frac{1}{2-x}=\frac{3x+8}{x^2-4}$

16) $\frac{x+2}{3x-1}-\frac{1}{x}=\frac{3x-3}{3x^2-x}$

17) $\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{5}{6}$

18) $\frac{x-1}{x-3}+\frac{x+2}{x+3}=\frac{3}{4}$

19) $\frac{3}{2x+1}+\frac{2x+1}{1-2x}=1-\frac{8x^2}{4x^2-1}$

20) $\frac{3x-5}{5x-5}+\frac{5x-1}{7x-7}-\frac{x-4}{1-x}=2$

21) $\frac{x-2}{x+3}-\frac{1}{x-2}=\frac{1}{x^2+x-6}$

22) $\frac{x-1}{x-2}+\frac{x+4}{2x+1}=\frac{1}{2x^2-3x-2}$

23) $\frac{3}{x+2}+\frac{x-1}{x+5}=\frac{5x+20}{6x+24}$

24) $\frac{x}{x+3}-\frac{4}{x-2}=\frac{-5x^2}{x^2+x-6}$

25) $\frac{x}{x-1}-\frac{2}{x+1}=\frac{4x^2}{x^2-1}$

26) $\frac{2x}{x+2}+\frac{2}{x-4}=\frac{3x}{x^2-2x-8}$

27) $\frac{2x}{x+1}-\frac{3}{x+5}=\frac{-8x^2}{x^2+6x+5}$

28) $\frac{x}{x+1}-\frac{3}{x+3}=\frac{-2x^2}{x^2+4x+3}$

29) $\frac{x-5}{x-9}+\frac{x+3}{x-3}=\frac{-4x^2}{x^2-12x+27}$

30) $\frac{x-3}{x+6}+\frac{x-2}{x-3}=\frac{x^2}{x^2+3x-18}$

ANSWERS to Practice Exercises
Section 2.6: Solving Rational Equations

1) 24
2) 8
3) $-\frac{1}{20}$
4) -2
5) $-\frac{1}{2}$
6) -2
7) $-\frac{9}{2}$
8) no solution
9) -5
10) $-\frac{7}{15}$
11) $-5, 0$
12) no solution
13) $\frac{16}{3}, 5$
14) $2, 13$
15) -8
16) 2
17) $-\frac{1}{5}, 5$
18) $-\frac{9}{5}, 1$
19) $\frac{3}{2}$
20) 10
21) $0, 5$
22) $-2, \frac{5}{3}$
23) $4, 7$
24) -1
25) $\frac{2}{3}$
26) $\frac{1}{2}$
27) $\frac{3}{10}$
28) 1
29) $-\frac{2}{3}$
30) -1

Section 2.7: Motion and Work Applications

Objective: Solve application problems by creating a rational equation to model the problem.

In this section, we will solve a variety of application problems using rational equations.

UNIFORM MOTION PROBLEMS

Solving motion problems where time traveled is the main focus usually involves rational equations. The distance formula, $d = rt$, is still utilized. However, since the focus is time, the formula is rewritten as $t = \frac{d}{r}$. If the times are known to be equal, we can use a proportion.

Example 1. Use a rational equation to solve.

In the time it takes for a car to travel 120 miles, a train can travel 180 miles. If the train's rate is 20 miles per hour faster than the car's rate, what is the average rate for each?

	distance	rate	time
Train	180	$r+20$	$\frac{180}{r+20}$
Car	120	r	$\frac{120}{r}$

We do not know rate, r, or time, t, traveled by either the train or the car. But we do know the distances traveled, and that they traveled for the same amount of time.
Since the times traveled by the train and car are the same, we set them equal to each other.

$$\frac{180}{r+20} = \frac{120}{r}$$ We have a proportion; Set the cross products equal

$$180r = (120)(r+20)$$ Distribute on the right side

$$\begin{aligned} 180r &= 120r + 2400 \\ -120r &\quad -120r \\ \hline \frac{60r}{60} &= \frac{2400}{60} \end{aligned}$$ Add $(-120r)$ to both sides

Divide both sides by 60

$$r = 40$$ Speed of the car

$$r + 20 = (40) + 20 = 60$$ Speed of the train

car: 40 mph; train: 60 mph Our Solutions

Another type of motion problem concerns a boat traveling in a river either with the current or against the current (or an airplane flying with the wind or against the wind). If a boat is traveling downstream, the current will push it, increasing the rate by the speed of the current. If a boat is traveling upstream, the current will pull against it, decreasing the rate by the speed of the current. Applications of these types are shown in the examples below.

Example 2. Use a rational equation to solve.

A man rows downstream for 30 miles, then turns around. He travels 20 miles upstream in the same amount of time. In still water, his boat averages 15 miles per hour. What is the speed of the water's current?

	distance	rate	time
downstream	30	$15+x$	$\frac{30}{15+x}$
upstream	20	$15-x$	$\frac{20}{15-x}$

Let x represent the speed of the water's current.

Downstream, the boat's rate is increased by the speed of the current.

Upstream, the boat's rate is decreased by the speed of the current.

We know the time traveled is the same in both directions.

We set the times equal to each other.

$$\frac{30}{15+x}=\frac{20}{15-x}$$ We have a proportion; Set the cross products equal

$$(30)(15-x)=(20)(15+x)$$ Simplify by distributing on both sides

$$\begin{array}{r} 450-30x=300+20x \\ +30x \quad +30x \\ \hline \end{array}$$ Add $(+30x)$ to both sides

$$\begin{array}{r} 450=300+50x \\ -300 \quad -300 \\ \hline \end{array}$$ Add (-300) to both sides

$$\frac{150}{50}=\frac{50x}{50}$$ Divide both sides by 50

$$x=3$$ This is the speed of the current

The speed of the water's current is 3 mph. Our Solution

Example 3. Use a rational equation to solve.

Two planes left an airport at the same time with the same average speed in still air. The first plane traveled to a remote island, a distance of 450 miles, with a tailwind. The other plane flew to a mountain resort, a distance of 250 miles, in the same amount of time with a head wind. The wind current's speed for both planes was 55 miles per hour. What would be each plane's average speed in still air?

	distance	rate	time
With a tailwind	450	$x+55$	$\frac{450}{x+55}$
With a head wind	250	$x-55$	$\frac{250}{x-55}$

Let x represent the speed of the planes in still air*:*

With a tailwind, the speed of the plane in still air is increased by the wind current's speed of 55 mph.

With a head wind, the speed of the plane in still air is decreased by the wind current's speed of 55 mph.

We know the time traveled is the same so set the times equal.

$$\frac{450}{x+55}=\frac{250}{x-55}$$

We have a proportion;
Set the cross products equal

$$450(x-55)=250(x+55)$$

Simplify by distributing on both sides

$$\begin{array}{r} 450x-24750=250x+13750 \\ \underline{-250x \qquad\qquad\quad -250x} \\ 200x-24750=13750 \end{array}$$

Add $(-250x)$ to both sides

$$\begin{array}{r} 200x-24750=13750 \\ \underline{+24750 \quad +24750} \\ \frac{200x}{200}=\frac{38500}{200} \end{array}$$

Add 24750 to both sides

Divide both sides by 200

$$x=192.5$$

This is each plane's average speed in the air.

192.5 mph Our Solution

WORK PROBLEMS

Work problems typically involve finding the time it takes for two individuals working together to complete a job. The first person can complete the job in A hours working alone, and the second person can complete that job in B hours working alone. Each person is performing a fractional part of the work. We let the time spent working together be denoted by x. The first person's portion of the work is $\frac{x}{A}$ and the second person's portion of the work is $\frac{x}{B}$. Working together one whole job is completed. Thus, we add the two parts together and set the equation equal to 1 to signify that the whole job is finished. We use the rational equation below for this type of problem.

WORK EQUATION:

$$\frac{x}{A}+\frac{x}{B}=1$$

x = the time spent working together
A = time spent by one person working alone
B = time spent by another person working alone

The equation above is used to solve the work problems included in this section.

Example 4. Use a rational equation to solve.

Pat can paint a room in 3 hours. Les can paint the same room in 6 hours. How long will it take them to do the job together?

Pat: 3 hours, Les: 6 hours	The time working together is unknown, or x Use the work equation to form the rational equation:
$\frac{x}{3}+\frac{x}{6}=1$	Multiply each term by LCD 6
$\frac{x(6)}{3}+\frac{x(6)}{6}=1(6)$	Reduce; clear fractions
$2x+x=6$	Combine like terms
$3x=6$	Divide by 3 on both sides of the equation
$\frac{3x}{3}=\frac{6}{3}$	
$x=2$	Our solution for x
The job takes them 2 hours working together.	Our Solution

Example 5. Use a rational equation to solve.

Adam can clean a room in 3 hours. His sister Maria can clean it in 12 hours. How long will it take them to do the job together?

Adam: 3 hours, Maria: 12 hours	The time working together is unknown, or x Use the work equation to form the rational equation:
$\frac{x}{3}+\frac{x}{12}=1$	Multiply each term by LCD, 12
$\frac{x(12)}{3}+\frac{x(12)}{12}=1(12)$	Reduce; clear fractions
$4x+x=12$	Combine like terms
$5x=12$	Divide by 5 on both sides of the equation
$\frac{5x}{5}=\frac{12}{5}$	
$x=\frac{12}{5}=2\frac{2}{5}$	Our solution for x
The job takes them $2\frac{2}{5}$ hours working together.	Our Solution

Practice Exercises
Section 2.7: Motion and Work Applications

Use a rational equation to solve.

1) A powerboat travels upstream for 50 miles in the same amount of time it travels 80 miles downstream. If the speed of the powerboat in still water is 65 mph, find the rate of the current.

2) Mary drove 840 miles to Texas in the same amount of time that Sue drove 770 miles to Louisiana. Mary was traveling 5 miles per hour faster than Sue. How fast was each traveling?

3) Steve runs uphill 3 miles in the time that it takes Mark to run 5 miles downhill. If Steve is traveling 4 miles per hour slower than Mark, how fast is each one running?

4) Alberta went on a kayaking trip. She traveled 2 miles upstream in the same amount of time that she spent kayaking 6 miles downstream. The water's current was 2 miles per hour. What was her average rate of travel in still water?

5) A car travels 240 miles in the same amount of time that a motorcyclist travels 360 miles. The car is traveling an average of 20 miles per hour slower. How fast is each traveling?

6) If Andre can do a piece of work alone in 6 days and Bonita can do it alone in 4 days, how long will it take the two working together to complete the job?

7) Cedric can do a piece of work in 4 days and Dominic can do it in half the time. How long will it take them to do the work together?

8) A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take both pipes together to fill the tank?

9) If Sam can do a certain job in 3 days, while it takes Fred 6 days to do the same job, how long will it take them, working together, to complete the job?

10) Tim can finish a certain job in 10 hours. It takes his wife JoAnn only 8 hours to do the same job. If they work together, how long will it take them to complete the job?

ANSWERS to Practice Exercises
Section 2.7: Motion and Work Applications

1) The rate of the current is 15 mph.

2) Mary was traveling 60 mph and Sue was traveling 55 mph.

3) Steve runs at 6 mph and Mark runs at 10 mph.

4) Alberta's average rate in still water is 4 mph.

5) The car is traveling 40 mph and the motorcycle is traveling 60 mph.

6) The job takes 2.4 days working together.

7) The job takes $1\frac{1}{3}$ days working together.

8) The tank is filled in 12 minutes.

9) The job takes 2 days working together.

10) The job takes $4\frac{4}{9}$ hours working together.

Section 2.8: Variation

Objective: Model and solve direct, inverse, joint and combined variation problems.

Variation problems are used to show the relationship between quantities. In this section, we will examine how quantities can vary directly, inversely and jointly, as well as explore what happens when more than one relationship must be considered within a problem.

Each type of variation problem will require that we first find the constant of variation k. Once that constant has been established, the relationship is defined and specific problems can be solved.

DIRECT VARIATION

When quantities vary directly, we say that "*y varies directly as x*" or that "*y varies directly in proportion to x*". Direct variation problems are modeled using linear equations in the form of $y = kx$.

DIRECT VARIATION
$y = kx$

We first find the value of the constant of variation k; then, we solve for a specific quantity.

Example 1. Solve the variation problem.

Given: y varies directly as x, and $y = 80$ when $x = 20$. Find y when $x = 65$.

$80 = k(20)$	Use the direct variation model $y = kx$; Substitute the given values for x and y, and solve for k.
$\frac{80}{20} = \frac{k(20)}{20}$	Divide both sides by 20.
$4 = k$	The constant of variation k is 4.
$y = 4(65)$	Substitute $k = 4$ and $x = 65$ into $y = kx$; multiply to find the value of y.
$y = 260$	Our Solution

Example 2. Solve the variation problem.

The dosage of a medication that is prescribed by a doctor varies directly as the weight of the patient. A doctor prescribes 2.5 milliliters (ml) of a medication for a 200-pound patient. How many milliliters of this medication can be prescribed for someone who weighs 220 pounds?

$2.5 = k(2000)$	The dosage y varies directly as the weight x of the patient. Use the direct variation model $y = kx$. Substitute the given values $y = 2.5$ ml and $x = 200$ pounds, and solve for k.
$\frac{2.5}{200} = \frac{k(200)}{200}$	Divide both sides by 200.
$0.0125 = k$	The constant of variation k is 0.0125.
$y = 0.0125(220)$	Substitute $k = 0.0125$ and $x = 220$ into $y = kx$; multiply to find the value of y.
$y = 2.75\,\text{ml}$	Our Solution

INVERSE VARIATION

When quantities vary inversely, we say that "y varies inversely as x" or that "y varies inversely in proportion to x". Inverse variation problems are modeled using rational equations in the form of $y = \frac{k}{x}$.

INVERSE VARIATION

$$y = \frac{k}{x}$$

Example 3. Solve the variation problem.

Given: y varies inversely as x, and $y = 95$ when $x = 3$. Find y when $x = 15$.

$95 = \frac{k}{3}$	Use the inverse variation model $y = \frac{k}{x}$. Substitute the given values for x and y, and solve for k.

$(95)(3) = \left(\frac{k}{3}\right)\left(\frac{3}{1}\right)$ Multiply both sides by 3.

$k = 285$ The constant of variation k is 285.

$y = \frac{285}{15}$ Substitute $k = 285$ and $x = 15$ into $y = \frac{k}{x}$; divide to find the value of y.

$y = 19$ Our Solution

Example 4. Solve the variation problem.

The frequency of a vibrating string varies inversely with its length. A vibrating string of length 40 inches has a frequency of 440 hertz. What is the frequency of a string of length 20 inches?

$440 = \frac{k}{40}$ The frequency y varies inversely as the length x. Use the inverse variation model. $y = \frac{k}{x}$.
Substitute the given values for x and y, and solve for k.

$(440)(40) = \left(\frac{k}{40}\right)\left(\frac{40}{1}\right)$ Multiply both sides by 40.

$17{,}600 = k$ The constant of variation k is $17{,}600$.

$y = \frac{17{,}600}{20}$ Substitute $k = 17{,}600$ and $x = 20$ into $y = \frac{k}{x}$; divide to find the value of y.

$y = 880$ hertz Our Solution

JOINT VARIATION

When quantities vary jointly, we say that "y varies directly as the product of two or more variables" or that "y varies directly in proportion to two or more variables". Joint variation problems are modeled using equations in the form of $y = kxz$.

JOINT VARIATION

$$y = kxz$$

Example 5. Solve the variation problem.

Given: y varies jointly as the square of x and z, and $y = 1334$ when $x = 8$ and $z = 3$. Find y when $x = 5$ and $z = 6$.

$1344 = k(8)^2(3)$	Use the joint variation model $y = kxz$. In this case, we have the square of x so use the form $y = kx^2z$. Substitute the given values for x, y, and z, and solve for k.
$1344 = k(64)(3)$	Use order of operations to simplify.
$1344 = k(192)$	Divide both sides by 192.
$k = 7$	The constant of variation k is 7.
$y = 7(5)^2(6)$	Substitute $k = 7$, $x = 5$, and $z = 6$ into $y = kx^2z$; simplify.
$y = 1050$	Our Solution

Example 6. Solve the variation problem.

The volume (V) of a closed box with fixed height varies jointly as the length (L) of its base and the width (W) of its base. A box whose base's length is 40 inches and width is 5 inches has a volume of 700 cubic inches. What is the volume of a box whose base's length is 35 inches and whose width is 4 inches?

$700 = k(40)(5)$	The volume of the box V varies jointly as the length L and width W of its base. Use the joint variation model written as $V = kLW$. Substitute the given values for V, L and W, and solve for k.
$700 = k(200)$	Multiply $(40)(50)$.

$\frac{700}{200} = \frac{k(200)}{200}$ Divide both sides by 200.

$3.5 = k$ The constant of variation k is 3.5.

$V = 3.5\,(35)\,(4)$ Substitute $k = 3.5$, $L = 35$, and $W = 4$ into $V = kLW$; multiply to find V.

$V = 490$ cubic inches Our Solution

COMBINED VARIATION

Sometimes direct, inverse, and/or joint variation occurs within the same situation. When quantities vary in more than one way, we include all relationships in one equation. The examples below illustrate combined variation.

Example 7. Solve the variation problem.

Given: a varies directly as b and inversely as c raised to the fourth power, and $a = 5$ when $b = 10$ and $c = 2$. Find a when $b = 7$ and $c = 1$.

$5 = \frac{k(10)}{2^4}$ Start with $a = kb$ as the variation model since a varies directly as b.

We must also divide by c^4 since a varies inversely as c^4. So the combined variation model is $a = \frac{kb}{c^4}$.

Substitute the given values for a, b and c, and solve for k.

$5 = \frac{k(10)}{16}$ Use order of operations to simplify:

$5(16) = \frac{k(10)}{16}(16)$ Multiply both sides by 16.

$80 = k(10)$ Divide both sides by 10.

$k = 8$ The constant of variation k is 8.

$a = \frac{(8)(7)}{1^4}$ Substitute $k = 8$, $b = 7$, and $c = 1$ into $a = \frac{kb}{c^4}$; evaluate to find a.

$a = 56$ Our Solution

Example 8. Solve the variation problem.

A person's intelligence quotient (IQ) varies directly as a person's mental age (m) and inversely as their chronological age (c). A person whose mental age is 35 and whose chronological age is 25 has an IQ of 140. What is the IQ of a 70 year old person whose mental age is 77?

$140 = \frac{k(35)}{25}$ Start with $IQ = km$ as the variation model since IQ varies directly as m. We must also divide by c since IQ varies inversely as c. So the variation equation is modeled by $IQ = \frac{km}{c}$. Substitute the given values for IQ, m and c, and solve for k.

$140(25) = \frac{k(35)}{25}(25)$ Multiply both sides by 25.

$\frac{3500}{35} = \frac{k(35)}{35}$ Divide both sides by 35.

$100 = k$ The constant of variation, k, is 100.

$IQ = \frac{100(77)}{70}$ Substitute $k = 100$, $m = 77$, and $c = 70$ into $IQ = \frac{km}{c}$; evaluate to find IQ.

$IQ = 110$ Our Solution

Practice Exercises
Section 2.8: Variation

Solve each variation problem.

1) y varies directly as x. When $x = 15, y = 9$. What is the value of y when $x = 6.3$?

2) y varies directly as x. When $x = 80, y = 65$. What is the value of y when $x = 72$?

3) y varies directly as x. When $x = 15, y = 540$. What is the value of y when $x = 9.75$?

4) y varies directly as x. When $x = 4.8, y = 25.2$. What is the value of y when $x = 7.4$?

5) y varies inversely as x. When $x = 65, y = 9$. What is the value of y when $x = 50$?

6) y varies inversely as x. When $x = 12, y = 5$. What is the value of y when $x = 4$?

7) y varies inversely as x. When $x = 9.2, y = 2.5$. What is the value of y when $x = 4.6$?

8) y varies inversely as x. When $x = 3.2, y = 70$. What is the value of y when $x = 8$?

9) z varies directly as x and inversely as the square of y. When $x = 3$ and $y = 4$, $z = 12$. What is the value of z when $x = 12$ and $y = 8$?

10) The force, F, needed to stretch a spring varies directly as a certain distance, x, where k is the spring constant measured in Newtons/cm. If a 24 Newton force stretches a certain spring by 20 cm, determining the force needed to stretch the spring 36 cm.

11) Jean wants to invest in property in order to build and sell houses. In the area where she would like to make her purchase, she has found that zoning regulations dictate that the amount of land is directly proportional to the number of houses that can be built on that land. She was able to determine that on 3.42 acres of land, 6 houses can be built. She wishes to build 10 houses. How much land does she need?

12) The time that it takes to fill a swimming pool with water varies directly as the depth of the pool. A swimming pool manufacturer claims that a backyard swimming pool that is 4 feet deep can be filled with water in 3 hours. If this is true, how long would it take to fill a pool that is 8 feet deep?

The Practice Exercises are continued on the next page.

Practice Exercises: Section 2.8 (continued)

13) There is a direct relationship between the number of hours spent working on a project and the grade a student receives for that project. Students who spend 2.5 hours on a project earn an average of 75 points on that project. How many points should Greg earn if he spends 3 hours on his project?

14) The frequency of a vibrating piano string varies inversely to its length. An 18-inch piano string vibrates at 336 cycles/second. What is the frequency of a vibrating 21-inch piano string?

15) Entrepreneurs know that number of items sold varies inversely with the price of their product. Alice produces natural chew toys for dogs. She knows that she can sell 250 toys per month when she charges $6 per toy. How many toys should she expect to sell if she charges $4 per toy?

16) The cost per person to rent an airport limousine for one ride is inversely proportional to the number of passengers in the vehicle. When five people rent the airport limousine, each person pays $50. How much would each person pay if eight people rent the airport limousine?

17) Kinetic energy varies jointly with mass and the square of velocity. If the mass is 20 kilograms and the velocity is 3 meters per second, the kinetic energy is 90 Joules. What is the kinetic energy if the mass is 10 kilograms and the velocity is 5 meters per second?

18) The volume of a pyramid is jointly proportional to the area of its base and to its height. A pyramid with base area 40 square meters and height 6 meters has a volume of 80 cubic meters. What is the volume of a pyramid with base area 30 square meters and height 8 meters?

ANSWERS to Practice Exercises
Section 2.8: Variation

1) 3.78

2) 58.5

3) 351

4) 38.85

5) 11.7

6) 15

7) 5

8) 28

9) 12

10) 43.2 Newtons

11) 5.7 acres

12) 6 hours

13) 90 points

14) 288 cycles/second

15) 375 toys

16) $31.25

17) 125 Joules

18) 80 cubic meters

Review: Chapter 2

Evaluate.

1) $\dfrac{n^2-4n+1}{n-5}$ when $n=1$

2) $\dfrac{b+1}{4b-16}$ when $b=4$

State the excluded value(s).

3) $\dfrac{5k^2+15k}{k+3}$

4) $\dfrac{13p}{25p^2-50p}$

5) $\dfrac{b^2+12b+20}{b^2-6b-16}$

Simplify each expression.

6) $\dfrac{48a}{60a^3}$

7) $\dfrac{x+5}{x^2+13x+40}$

8) $\dfrac{k^2-8k+7}{k^2-1}$

9) $\dfrac{2n^2+23n+30}{3n+30}$

10) $\dfrac{3x^2+8x+5}{6x^2-18x-24}$

11) $\dfrac{7-x}{2x^2-98}$

Multiply or divide, expressing the result in lowest terms.

12) $\dfrac{20x}{3x}\div\dfrac{10}{6}$

13) $\dfrac{x-4}{24x+21}\div\dfrac{4}{24x+21}$

14) $\dfrac{x^2-11x+18}{x+7}\cdot\dfrac{x+7}{x-9}$

15) $\dfrac{x^2+7x-30}{x-5}\cdot\dfrac{x+9}{x^2+6x-27}$

16) $\dfrac{6v^2-11v-35}{3v+5}\div\dfrac{4v^2-49}{4v+4}$

17) $\dfrac{4p^2+43p+63}{p^2+18p+77}\div\dfrac{2p^2-162}{p^2-2p-63}$

Add or subtract, expressing the result in its lowest terms.

18) $\frac{x^2}{x-3}-\frac{7x-12}{x-3}$

19) $\frac{t^2-12t}{t-3}+\frac{2t+21}{t-3}$

20) $\frac{a^2+4a}{a^2+11a+24}-\frac{32}{a^2+11a+24}$

21) $\frac{3x}{x^2-1}+\frac{x}{x+1}$

22) $\frac{t}{t-9}-\frac{1}{5t-45}$

23) $\frac{x}{x^2+6x+5}-\frac{3}{x^2+8x+7}$

24) $\frac{x-1}{x-2}-\frac{-4x+6}{x^2-6x+8}$

25) $\frac{x+1}{x^2+7x+10}+\frac{x+7}{x^2-25}$

Solve.

26) $\frac{m-6}{2}=\frac{4}{8}$

27) $\frac{x+1}{6}=\frac{x+8}{4}$

28) $\frac{x+4}{x}+\frac{1}{7}=\frac{1}{x}$

29) $\frac{-5}{x+3}+\frac{7}{x-12}=\frac{9}{x^2-9x-36}$

30) $\frac{1}{x-1}+\frac{x}{x-2}=\frac{4x-5}{x^2-3x+2}$

31) $\frac{-9x}{x^2-49}-\frac{10}{x-7}=\frac{1}{x+7}$

Answer each question. Round your answer to the nearest tenth. Round dollar amounts to the nearest cent.

32) A swimming pool can be filled by one hose in 12 hours and by another hose in 8 hours. How long will it take both hoses together to fill the pool?

33) When exchanging US Dollars (USD) for Philippine Peso (PHP) the number of Philippine Pesos received is directly proportional to the number of US Dollars to be exchanged. The exchange rate is approximately \$0.02 to 1 PHP. At this rate, how many dollars would you get if you exchanged 25,000 PHP?

ANSWERS to Review: Chapter 2

1) $\frac{1}{2}$

2) undefined

---

3) -3

4) 0, 2

5) -2, 8

---

6) $\frac{4}{5a^2}$

7) $\frac{1}{x+8}$

8) $\frac{k-7}{k+1}$

9) $\frac{2n+3}{3}$

10) $\frac{3x+5}{6(x-4)}$

11) $-\frac{1}{2(x+7)}$

---

12) 4

13) $\frac{x-4}{4}$

14) $x-2$

15) $\frac{x+10}{x-5}$

16) $\frac{4(v+1)}{2v+7}$

17) $\frac{4p+7}{2(p+11)}$

18) $x-4$

19) $t-7$

20) $\frac{a-4}{a+3}$

21) $\frac{x(x+2)}{(x+1)(x-1)}$

22) $\frac{5t-1}{5(t-9)}$

23) $\frac{x^2+4x-15}{(x+1)(x+5)(x+7)}$

24) $\frac{x+1}{x-4}$

25) $\frac{2x^2+5x+9}{(x-5)(x+2)(x+5)}$

---

26) $m=7$

27) $x=-22$

28) $x=-\frac{21}{8}$

29) $x=-36$

30) $x=3$

31) $x=-\frac{63}{20}$

---

32) 4.8 hours

33) $500.00

CHAPTER 3

Radical Expressions and Equations

Objectives
Chapter 3

- Simplify expressions with square roots.
- Simplify radicals with an index greater than two.
- Add and subtract like radicals by first simplifying each radical.
- Multiply and divide radical expressions using the product and quotient rules for radicals.
- Rationalize the denominators of radical expressions.
- Convert between radical notation and exponential notation.
- Simplify expressions with rational exponents using the properties of exponents.
- Multiply and divide radical expressions with different indices.
- Solve equations with radicals and check for extraneous solutions.
- Add, subtract, multiply, divide, and simplify expressions using complex numbers.

Section 3.1: Square Roots

Objective: Simplify expressions with square roots.

To reverse the process of squaring a number, we find the square root of a number. In other words, a square root "un-squares" a number.

PRINCIPAL SQUARE ROOT

If a is a n*onnegative* real number, then the **principal square root** of a is the *nonnegative* number b such that $b^2 = a$.
We write $b = \sqrt{a}$.

The symbol $\sqrt{\ }$ is called the *radical sign* and the number a, under the radical sign, is called the *radicand*. An expression containing a radical sign is called a *radical expression*. Square roots are the most common type of radical expressions used.

The following example shows several square roots:

Example 1. Evaluate.

$\sqrt{1} = 1$ because $1^2 = 1$	$\sqrt{121} = 11$ because $11^2 = 121$
$\sqrt{4} = 2$ because $2^2 = 4$	$\sqrt{625} = 25$ because $25^2 = 625$
$\sqrt{9} = 3$ because $3^2 = 9$	$\sqrt{0} = 0$ because $0^2 = 0$
$\sqrt{16} = 4$ because $4^2 = 16$	$\sqrt{81} = 9$ because $9^2 = 81$
$\sqrt{25} = 5$ because $5^2 = 25$	$\sqrt{-81}$ is not a real number

Notice that $\sqrt{-81}$ is not a real number because there is no real number whose square is -81. If we square a positive number or a negative number, the result will always be positive. Thus, we can only take square roots of nonnegative numbers. In another section, we will define a method we can use to work with and evaluate square roots of negative numbers, but for now we will state they are not real numbers.

We call numbers like 1, 4, 9, 16, 25, 81, 121, and 625 *perfect squares* because they are squares of integers. Not all numbers are perfect squares. For example, 8 is not a perfect square because 8 is not the square of an integer. Using a calculator, $\sqrt{8}$ is approximately equal to 2.828427125... and that number is still a rounded approximation of the square root.

SIMPLIFYING SQUARE ROOTS

Instead of using decimal approximations, we will usually express roots in *simplest* radical form. Advantages of simplest radical form are that it is an exact answer (not an approximation) and that calculations and algebraic manipulations can be done more easily.

To express roots in simplest radical form, we will use the following property:

PRODUCT RULE OF SQUARE ROOTS

For any *nonnegative* real numbers a and b,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$$

When simplifying a square root expression, we will first find the largest perfect square factor of the radicand. Then, we will write the radicand as the product of two factors, apply the product rule, and evaluate the square root of the perfect square factor.

Example 2. Simplify.

$\sqrt{75}$	75 is divisible by the perfect square 25; Split radicand into factors
$=\sqrt{25 \cdot 3}$	Apply product rule
$=\sqrt{25} \cdot \sqrt{3}$	Take the square root of 25
$=5\sqrt{3}$	Our Answer

If there is a coefficient in front of the radical to begin with, the problem becomes a big multiplication problem.

Example 3. Simplify.

$5\sqrt{63}$	63 is divisible by the perfect square 9; Split radicand into factors
$=5\sqrt{9 \cdot 7}$	Apply product rule
$=5\sqrt{9} \cdot \sqrt{7}$	Take the square root of 9
$=5 \cdot 3\sqrt{7}$	Multiply coefficients
$=15\sqrt{7}$	Our Answer

As we simplify radicals using this method, it is important to be sure our final answer can be simplified no more.

Example 4. Simplify.

$\sqrt{72}$	72 is divisible by the perfect square 9; Split radicand into factors
$=\sqrt{9\cdot 8}$	Apply product rule
$=\sqrt{9}\cdot\sqrt{8}$	Take the square root of 9
$=3\sqrt{8}$	But 8 is also divisible by the perfect square 4; Split radicand into factors
$=3\sqrt{4\cdot 2}$	Apply product rule
$=3\sqrt{4}\cdot\sqrt{2}$	Take the square root of 4
$=3\cdot 2\sqrt{2}$	Multiply
$=6\sqrt{2}$	Our Answer

The previous example could have been done in fewer steps if we had noticed that $72=36\cdot 2$, where 36 is the largest perfect square factor of 72. Often the time it takes to discover the larger perfect square is more than it would take to simplify the radicand in several steps.

Variables are often part of the radicand as well. To simplify radical expressions involving variables, use the property below:

SIMPLIFYING $\sqrt{a^2}$

For any *nonnegative* real number a,

$$\sqrt{a^2}=a$$

Note this property only holds if *a* is *nonnegative*. For this reason, we will assume that all variables involved in a radical expression are *nonnegative*.

When simplifying with variables, variables with exponents that are divisible by 2 are perfect squares. For example, by the power of a power rule of exponents, $(x^4)^2=x^8$. So x^8 is a perfect square and $\sqrt{x^8}=\sqrt{(x^4)^2}=x^4$. A shortcut for taking the square roots of variables is to divide the exponent by 2. In our example, $\sqrt{x^8} = x^4$ because we divide the exponent 8 by 2 to get 4. When squaring, we multiply the exponent by 2, so when taking a square root, we divide the exponent by 2.

This process is shown in the following example.

Example 5. Simplify.

$-5\sqrt{18x^4y^6z^{10}}$	18 is divisible by the perfect square 9; Split radicand into factors
$=-5\sqrt{9\cdot 2x^4y^6z^{10}}$	Apply product rule
$=-5\sqrt{9}\cdot\sqrt{2}\cdot\sqrt{x^4}\cdot\sqrt{y^6}\cdot\sqrt{z^{10}}$	Simplify roots; divide exponents by 2
$=-5\cdot 3x^2y^3z^5\sqrt{2}$	Multiply coefficients
$=-15x^2y^3z^5\sqrt{2}$	Our Answer

We can't always evenly divide the exponent of a variable by 2. Sometimes we have a remainder. If there is a remainder, this means the variable with an exponent equal to the remainder will remain inside the radical sign. On the outside of the radical, the exponent of the variable will be equal to the whole number part. This process is shown in the following example.

Example 6. Simplify.

$\sqrt{20x^5y^9z^6}$	20 is divisible by the perfect square 4; Split radicand into factors
$=\sqrt{4\cdot 5x^5y^9z^6}$	Apply product rule
$=\sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^5}\cdot\sqrt{y^9}\cdot\sqrt{z^6}$	Simplify roots; divide exponents by 2, remainder is left inside
$=2x^2y^4z^3\sqrt{5xy}$	Our Answer

In the previous example, for the variable x, we divided $\frac{5}{2} = 2\,R\,1$, so x^2 came out of the radicand and $x^1 = x$ remained inside the radicand. For the variable y, we divided $\frac{9}{2} = 4\,R\,1$, so y^4 came out of the radicand and $y^1 = y$ remained inside. For the variable z, we divided $\frac{6}{2} = 3\,R\,0$, so z^3 came out of the radicand and no zs remained inside.

Practice Exercises
Section 3.1: Square Roots

Simplify. Assume that all variables represent nonnegative real numbers.

1) $\sqrt{36}$

2) $\sqrt{-100}$

3) $-\sqrt{196}$

4) $\sqrt{12}$

5) $\sqrt{125}$

6) $\sqrt{72}$

7) $\sqrt{245}$

8) $3\sqrt{24}$

9) $5\sqrt{48}$

10) $6\sqrt{128}$

11) $-8\sqrt{392}$

12) $-7\sqrt{63}$

13) $\sqrt{192n}$

14) $\sqrt{343b}$

15) $\sqrt{169v^2}$

16) $\sqrt{100n^3}$

17) $\sqrt{252x^2}$

18) $\sqrt{200a^3}$

19) $-\sqrt{100k^4}$

20) $-4\sqrt{175p^4}$

21) $-7\sqrt{64x^4}$

22) $-5\sqrt{36m}$

23) $\sqrt{45x^2y^2}$

24) $\sqrt{72a^3b^4}$

25) $\sqrt{16x^3y^3}$

26) $\sqrt{98a^4b^2}$

27) $\sqrt{320x^4y^4}$

28) $\sqrt{512m^4n^3}$

29) $6\sqrt{80xy^2}$

30) $8\sqrt{98mn}$

31) $5\sqrt{245x^2y^3}$

32) $2\sqrt{72x^2y^2}$

33) $-2\sqrt{180u^3v}$

34) $-5\sqrt{28x^3y^4}$

35) $-8\sqrt{108x^4y^2z^4}$

36) $6\sqrt{50a^4bc^2}$

37) $2\sqrt{80hj^4k}$

38) $-\sqrt{32xy^2z^3}$

39) $-4\sqrt{54mnp^2}$

40) $-8\sqrt{56m^2p^4q}$

ANSWERS to Practice Exercises
Section 3.1: Square Roots

1) 6
2) not a real number
3) -14
4) $2\sqrt{3}$
5) $5\sqrt{5}$
6) $6\sqrt{2}$
7) $7\sqrt{5}$
8) $6\sqrt{6}$
9) $20\sqrt{3}$
10) $48\sqrt{2}$
11) $-112\sqrt{2}$
12) $-21\sqrt{7}$
13) $8\sqrt{3n}$
14) $7\sqrt{7b}$
15) $13v$
16) $10n\sqrt{n}$
17) $6x\sqrt{7}$
18) $10a\sqrt{2a}$
19) $-10k^2$
20) $-20p^2\sqrt{7}$
21) $-56x^2$
22) $-30\sqrt{m}$
23) $3xy\sqrt{5}$
24) $6ab^2\sqrt{2a}$
25) $4xy\sqrt{xy}$
26) $7a^2b\sqrt{2}$
27) $8x^2y^2\sqrt{5}$
28) $16m^2n\sqrt{2n}$
29) $24y\sqrt{5x}$
30) $56\sqrt{2mn}$
31) $35xy\sqrt{5y}$
32) $12xy\sqrt{2}$
33) $-12u\sqrt{5uv}$
34) $-10xy^2\sqrt{7x}$
35) $-48x^2yz^2\sqrt{3}$
36) $30a^2c\sqrt{2b}$
37) $8j^2\sqrt{5hk}$
38) $-4yz\sqrt{2xz}$
39) $-12p\sqrt{6mn}$
40) $-16mp^2\sqrt{14q}$

Section 3.2: Higher Roots

Objective: Simplify radicals with an index greater than two.

While square roots are the most common type of radical we work with, we can take higher roots of numbers as well: cube roots, fourth roots, fifth roots, etc. Following is a definition of higher roots.

> **n^{th} ROOT**
>
> For a positive integer $n > 1$, the principal n^{th} root of a is the number b such that $b^n = a$.
> We write $b = \sqrt[n]{a}$.
>
> NOTE: If *n* is even, *a* and *b* are *nonnegative*.

We call b the n^{th} root of a. The small letter n is called the **index**. It tells us which root we are taking, or which power we are "un-doing". For square roots, the index is 2. As this is the most common root, the two is not usually written.

The following example includes several higher roots.

Example 1. Evaluate.

$\sqrt[3]{125} = 5$ because $5^3 = 125$	$\sqrt[3]{-64} = -4$ because $(-4)^3 = -64$
$\sqrt[3]{8} = 2$ because $2^3 = 8$	$\sqrt[7]{-128} = -2$ because $(-2)^7 = -128$
$\sqrt[4]{81} = 3$ because $3^4 = 81$	$\sqrt[4]{16} = 2$ because $2^4 = 16$
$\sqrt[5]{32} = 2$ because $2^5 = 32$	$\sqrt[4]{-16}$ is not a real number

We must be careful of a few things as we work with higher roots. First, it is important to check the index on the root. For example, $\sqrt{81} = 9$ because $9^2 = 81$ but $\sqrt[4]{81} = 3$ because $3^4 = 81$. Another thing to watch out for is negative numbers in the radicand. We can take an odd root of a negative number because a negative number raised to an odd power is still negative. However, the even root of a negative number is not a real number. In a later section we will discuss how to work with even roots of negative numbers, but for now we state they are not real numbers.

SIMPLIFYING HIGHER ROOTS

We can simplify higher roots in much the same way we simplified square roots, using the product rule of radicals.

PRODUCT RULE OF RADICALS

For any *nonnegative* real numbers a and b,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

Often, we are not as familiar with perfect nth powers as we are with perfect squares. It is important to remember what index we are working with as we express higher roots in simplest radical form.

Example 2. Simplify.

$\sqrt[3]{54}$	We are working with a cube root, want perfect third powers. Test 2: $2^3 = 8$; 54 is not divisible by 8. Test 3: $3^3 = 27$; 54 is divisible by 27! Split radicand into factors
$= \sqrt[3]{27 \cdot 2}$	Apply product rule
$= \sqrt[3]{27} \cdot \sqrt[3]{2}$	Take the cube root of 27
$= 3\sqrt[3]{2}$	Our Answer

Just as with square roots, if we have a coefficient, then we multiply the new coefficients together.

Example 3. Simplify.

$3\sqrt[4]{48}$	We are working with a fourth root, want perfect fourth powers. Test 2: $2^4 = 16$; 48 is divisible by 16! Split radicand into factors
$= 3\sqrt[4]{16 \cdot 3}$	Apply product rule
$= 3\sqrt[4]{16} \cdot \sqrt[4]{3}$	Take the fourth root of 16
$= 3 \cdot 2\sqrt[4]{3}$	Multiply coefficients
$= 6\sqrt[4]{3}$	Our Answer

We can also take higher roots of variables. To simplify radical expressions involving variables, use the property below:

SIMPLIFYING $\sqrt[n]{a^n}$

For any *nonnegative* real number a,

$$\sqrt[n]{a^n} = a$$

Note this property only holds if *a* is *nonnegative*. For this reason, we will assume that all variables involved in a radical expression are *nonnegative*.

As with square roots, when simplifying with variables, we will divide the variable's exponent by the index. The whole number part of the division is how many factors of that variable will come out of the nth root. Any remainder is how many factors of the variable are left behind in the radicand. This process is shown in the following examples.

Example 4. Simplify.

$\sqrt[5]{x^{25}y^{17}z^3}$	Divide each exponent by 5: whole number outside, remainder inside
$= x^5y^3\sqrt[5]{y^2z^3}$	Our Answer

In the previous example, for the variable x, we divided $\frac{25}{5} = 5R0$, so x^5 came out and no xs remained inside. For y, we divided $\frac{17}{5} = 3R2$, so y^3 came out, and y^2 remained inside. For z, when we divided $\frac{3}{5} = 0R3$, all three or z^3 remained inside.

Example 5. Simplify.

$2\sqrt[3]{40a^4b^8}$	40 is divisible by the perfect cube 8 Split radicand into factors Apply product rule
$= 2 \cdot \sqrt[3]{8} \cdot \sqrt[3]{5} \cdot \sqrt[3]{a^4} \cdot \sqrt[3]{b^8}$	Simplify roots; divide exponents by 3, remainders are left inside
$= 2 \cdot 2ab^2\sqrt[3]{5ab^2}$	Multiply coefficients
$= 4ab^2\sqrt[3]{5ab^2}$	Our Answer

Practice Exercises
Section 3.2: Higher Roots

Simplify.

1) $\sqrt[3]{64}$
2) $\sqrt[3]{-27}$
3) $\sqrt[4]{16}$
4) $\sqrt[4]{-16}$
5) $\sqrt[5]{-1}$
6) $\sqrt[8]{-1}$
7) $\sqrt[3]{625}$
8) $\sqrt[3]{375}$
9) $\sqrt[3]{750}$
10) $\sqrt[3]{250}$
11) $\sqrt[3]{24}$
12) $-4\sqrt[4]{96}$
13) $3\sqrt[4]{48}$
14) $-\sqrt[4]{112}$
15) $5\sqrt[4]{243}$
16) $\sqrt[4]{648a^2}$
17) $\sqrt[4]{64n^3}$
18) $\sqrt[5]{224n^3}$
19) $\sqrt[5]{-96x^3}$
20) $\sqrt[5]{224p^5}$
21) $\sqrt[6]{256x^6}$
22) $-8\sqrt[7]{384b^8}$
23) $-2\sqrt[3]{-48v^7}$
24) $-7\sqrt[3]{320n^6}$
25) $-\sqrt[3]{512n^6}$
26) $\sqrt[3]{-135x^5y^3}$
27) $\sqrt[3]{64u^5v^3}$
28) $\sqrt[3]{-32x^4y^4}$
29) $\sqrt[3]{1000a^4b^5}$
30) $\sqrt[3]{256x^4y^6}$
31) $7\sqrt[3]{-81x^3y^7}$
32) $-4\sqrt[3]{56x^2y^8}$
33) $8\sqrt[3]{-750xy}$
34) $-3\sqrt[3]{192ab^2}$
35) $3\sqrt[3]{135xy^3}$
36) $6\sqrt[3]{-54m^8n^3p^7}$
37) $-8\sqrt[4]{80m^4p^7q^4}$
38) $-2\sqrt[4]{405a^5b^8c}$
39) $7\sqrt[4]{128h^6j^8k^8}$
40) $5\sqrt[4]{324x^7yz^7}$

ANSWERS to Practice Exercises
Section 3.2: Higher Roots

1) 4
2) -3
3) 2
4) not a real number
5) -1
6) not a real number
7) $5\sqrt[3]{5}$
8) $5\sqrt[3]{3}$
9) $5\sqrt[3]{6}$
10) $5\sqrt[3]{2}$
11) $2\sqrt[3]{3}$
12) $-8\sqrt[4]{6}$
13) $6\sqrt[4]{3}$
14) $-2\sqrt[4]{7}$
15) $15\sqrt[4]{3}$
16) $3\sqrt[4]{8a^2}$
17) $2\sqrt[4]{4n^3}$
18) $2\sqrt[5]{7n^3}$
19) $-2\sqrt[5]{3x^3}$
20) $2p\sqrt[5]{7}$
21) $2x\sqrt[6]{4}$
22) $-16b\sqrt[7]{3b}$
23) $4v^2\sqrt[3]{6v}$
24) $-28n^2\sqrt[3]{5}$
25) $-8n^2$
26) $-3xy\sqrt[3]{5x^2}$
27) $4uv\sqrt[3]{u^2}$
28) $-2xy\sqrt[3]{4xy}$
29) $10ab\sqrt[3]{ab^2}$
30) $4xy^2\sqrt[3]{4x}$
31) $-21xy^2\sqrt[3]{3y}$
32) $-8y^2\sqrt[3]{7x^2y^2}$
33) $-40\sqrt[3]{6xy}$
34) $-12\sqrt[3]{3ab^2}$
35) $9y\sqrt[3]{5x}$
36) $-18m^2np^2\sqrt[3]{2m^2p}$
37) $-16mpq\sqrt[4]{5p^3}$
38) $-6ab^2\sqrt[4]{5ac}$
39) $14hj^2k^2\sqrt[4]{8h^2}$
40) $15xz\sqrt[4]{4x^3yz^3}$

Section 3.3: Add and Subtract Radical Expressions

Objective: Add and subtract like radicals by first simplifying each radical.

Adding and subtracting radical expressions is very similar to adding and subtracting variable expressions. If two or more radical expressions have the same indices and the same radicands, they are called *like radicals*. Consider the similarities between the following two examples.

Example 1. Perform the indicated operations.

$5x+3x-2x$	Combine like terms
$=6x$	Our Answer

Example 2. Perform the indicated operations.

$5\sqrt{11}+3\sqrt{11}-2\sqrt{11}$	Combine like radicals
$=6\sqrt{11}$	Our Answer

Notice that when we combined the radical terms with $\sqrt{11}$ it was just like combining variable terms with x. When adding and subtracting like radicals, we add and subtract the coefficients in front of the radical, and the radical stays the same.

Example 3. Perform the indicated operations.

$7\sqrt[5]{6}+4\sqrt[5]{3}-9\sqrt[5]{3}+\sqrt[5]{6}$	Combine like radicals $7\sqrt[5]{6}+\sqrt[5]{6}$ and $4\sqrt[5]{3}-9\sqrt[5]{3}$
$=8\sqrt[5]{6}-5\sqrt[5]{3}$	Our Answer

We cannot combine these radical expressions any more because the radicals are not like radical terms.

Often radical expressions do not look *like* at first. However, if we simplify the radicals, we may find we do in fact have like radicals. This process is shown in the examples on the next page.

Example 4. Perform the indicated operations.

$5\sqrt{45}+6\sqrt{18}-2\sqrt{98}+\sqrt{20}$	Simplify radicals, finding perfect square factors
$=5\sqrt{9\cdot 5}+6\sqrt{9\cdot 2}-2\sqrt{49\cdot 2}+\sqrt{4\cdot 5}$	Take square roots where possible
$=5\cdot 3\sqrt{5}+6\cdot 3\sqrt{2}-2\cdot 7\sqrt{2}+2\sqrt{5}$	Multiply coefficients
$=15\sqrt{5}+18\sqrt{2}-14\sqrt{2}+2\sqrt{5}$	Combine like radicals
$=17\sqrt{5}+4\sqrt{2}$	Our Answer

Example 5. Perform the indicated operations.

$4\sqrt[3]{54}-9\sqrt[3]{16}+5\sqrt[3]{9}$	Simplify radicals, finding perfect cube factors
$=4\sqrt[3]{27\cdot 2}-9\sqrt[3]{8\cdot 2}+5\sqrt[3]{9}$	Take cube roots where possible
$=4\cdot 3\sqrt[3]{2}-9\cdot 2\sqrt[3]{2}+5\sqrt[3]{9}$	Multiply coefficients
$=12\sqrt[3]{2}-18\sqrt[3]{2}+5\sqrt[3]{9}$	Combine like radicals $12\sqrt[3]{2}-18\sqrt[3]{2}$
$=-6\sqrt[3]{2}+5\sqrt[3]{9}$	Our Answer

Practice Exercises
Section 3.3: Add and Subtract Radical Expressions

Perform the indicated operation.

1) $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$

2) $-3\sqrt{6}-3\sqrt{3}-2\sqrt{3}$

3) $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$

4) $-2\sqrt{6}-\sqrt{3}+3\sqrt{6}$

5) $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$

6) $-3\sqrt{3}+5\sqrt{3}+2\sqrt{3}$

7) $3\sqrt{6}+3\sqrt{5}+2\sqrt{5}$

8) $-\sqrt{5}+2\sqrt{3}-2\sqrt{3}$

9) $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$

10) $-\sqrt{54}-3\sqrt{6}+3\sqrt{27}$

11) $-3\sqrt{6}-\sqrt{12}+3\sqrt{3}$

12) $4\sqrt{5}-\sqrt{5}-2\sqrt{48}$

13) $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$

14) $2\sqrt{20}+2\sqrt{20}-\sqrt{3}$

15) $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$

16) $3\sqrt{27}+2\sqrt{3}-\sqrt{12}$

17) $-2\sqrt{2}-\sqrt{2}+3\sqrt{8}+3\sqrt{6}$

18) $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$

19) $-3\sqrt{18}-\sqrt{8}+5\sqrt{8}+2\sqrt{8}$

20) $-2\sqrt{24}-2\sqrt{6}+2\sqrt{6}+2\sqrt{20}$

21) $-3\sqrt{8}-\sqrt{5}-3\sqrt{6}+2\sqrt{18}$

22) $3\sqrt{24}-3\sqrt{27}+2\sqrt{6}+2\sqrt{8}$

23) $2\sqrt{6}-\sqrt{54}-3\sqrt{27}-\sqrt{3}$

24) $-2\sqrt[3]{16}+2\sqrt[3]{16}+2\sqrt[3]{2}$

25) $3\sqrt[3]{135}-\sqrt[3]{81}-\sqrt[3]{135}$

26) $2\sqrt[4]{243}-2\sqrt[4]{243}-\sqrt[4]{3}$

27) $-3\sqrt[4]{4}+3\sqrt[4]{324}+2\sqrt[4]{64}$

28) $3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{243}$

29) $2\sqrt[4]{6}+2\sqrt[4]{4}+3\sqrt[4]{6}$

30) $-\sqrt[4]{324}+3\sqrt[4]{324}-3\sqrt[4]{4}$

31) $-2\sqrt[4]{243}-\sqrt[4]{96}+2\sqrt[4]{96}$

32) $2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{64}-\sqrt[4]{3}$

33) $2\sqrt[4]{48}-3\sqrt[4]{405}-3\sqrt[4]{48}-\sqrt[4]{162}$

34) $-3\sqrt[5]{6}-\sqrt[5]{64}+2\sqrt[5]{192}-2\sqrt[5]{64}$

ANSWERS to Practice Exercises
Section 3.3: Add and Subtract Radical Expressions

1) $6\sqrt{5}$

2) $-3\sqrt{6}-5\sqrt{3}$

3) $-3\sqrt{2}+6\sqrt{5}$

4) $\sqrt{6}-\sqrt{3}$

5) $-5\sqrt{6}$

6) $4\sqrt{3}$

7) $3\sqrt{6}+5\sqrt{5}$

8) $-\sqrt{5}$

9) $-8\sqrt{2}$

10) $-6\sqrt{6}+9\sqrt{3}$

11) $-3\sqrt{6}+\sqrt{3}$

12) $3\sqrt{5}-8\sqrt{3}$

13) $-2\sqrt{2}$

14) $8\sqrt{5}-\sqrt{3}$

15) $5\sqrt{2}$

16) $9\sqrt{3}$

17) $3\sqrt{2}+3\sqrt{6}$

18) $-12\sqrt{2}+2\sqrt{5}$

19) $3\sqrt{2}$

20) $-4\sqrt{6}+4\sqrt{5}$

21) $-\sqrt{5}-3\sqrt{6}$

22) $8\sqrt{6}-9\sqrt{3}+4\sqrt{2}$

23) $-\sqrt{6}-10\sqrt{3}$

24) $2\sqrt[3]{2}$

25) $6\sqrt[3]{5}-3\sqrt[3]{3}$

26) $-\sqrt[4]{3}$

27) $10\sqrt[4]{4}$

28) $\sqrt[4]{2}-3\sqrt[4]{3}$

29) $5\sqrt[4]{6}+2\sqrt[4]{4}$

30) $3\sqrt[4]{4}$

31) $-6\sqrt[4]{3}+2\sqrt[4]{6}$

32) $2\sqrt[4]{2}+\sqrt[4]{3}+6\sqrt[4]{4}$

33) $-2\sqrt[4]{3}-9\sqrt[4]{5}-3\sqrt[4]{2}$

34) $\sqrt[5]{6}-6\sqrt[5]{2}$

Section 3.4: Multiply and Divide Radical Expressions

Objective: Multiply and divide radical expressions using the product and quotient rules for radicals.

MULTIPLYING RADICAL EXPRESSIONS

The product rule of radicals we used previously can be generalized as follows:

PRODUCT RULE OF RADICALS

For any *nonnegative* real numbers b and d,

$$a\sqrt[n]{b}\cdot c\sqrt[n]{d}=a\cdot c\sqrt[n]{b\cdot d}$$

In words, this rule states that we are allowed to multiply the factors outside the radical and we are allowed to multiply the factors inside the radicals, as long as the indices match.

Example 1. Multiply.

$-5\sqrt{14}\cdot 4\sqrt{6}$	Multiply outside and inside the radical
$=-20\sqrt{84}$	Simplify the radical, divisible by 4
$=-20\sqrt{4\cdot 21}$	Take the square root where possible
$=-20\cdot 2\sqrt{21}$	Multiply coefficients
$=-40\sqrt{21}$	Our Answer

The same process works with higher roots.

Example 2. Multiply.

$2\sqrt[3]{18}\cdot 6\sqrt[3]{15}$	Multiply outside and inside the radical
$=12\sqrt[3]{270}$	Simplify the radical, divisible by 27
$=12\sqrt[3]{27\cdot 10}$	Take the square root where possible
$=12\cdot 3\sqrt[3]{10}$	Multiply coefficients
$=36\sqrt[3]{10}$	Our Answer

When multiplying radical expressions we can still use the distributive property or FOIL just as we could when multiplying polynomials.

Example 3. Multiply.

$7\sqrt{6}(3\sqrt{10}-5\sqrt{15})$	Distribute, following rules for multiplying radicals
$=21\sqrt{60}-35\sqrt{90}$	Simplify radicals, finding perfect square factors
$=21\sqrt{4\cdot 15}-35\sqrt{9\cdot 10}$	Take the square root where possible
$=21\cdot 2\sqrt{15}-35\cdot 3\sqrt{10}$	Multiply coefficients
$=42\sqrt{15}-105\sqrt{10}$	Our Answer

Example 4. Multiply.

$(\sqrt{5}-2\sqrt{3})(4\sqrt{10}+6\sqrt{6})$	FOIL, following rules for multiplying radicals
$=4\sqrt{50}+6\sqrt{30}-8\sqrt{30}-12\sqrt{18}$	Simplify radicals, finding perfect square factors
$=4\sqrt{25\cdot 2}+6\sqrt{30}-8\sqrt{30}-12\sqrt{9\cdot 2}$	Take the square root where possible
$=4\cdot 5\sqrt{2}+6\sqrt{30}-8\sqrt{30}-12\cdot 3\sqrt{2}$	Multiply coefficients
$=20\sqrt{2}+6\sqrt{30}-8\sqrt{30}-36\sqrt{2}$	Combine like radicals
$=-16\sqrt{2}-2\sqrt{30}$	Our Answer

Example 5. Multiply.

$(2\sqrt{5}-3\sqrt{6})(7\sqrt{2}-8\sqrt{7})$	FOIL, following rules for multiplying radicals
$=14\sqrt{10}-16\sqrt{35}-21\sqrt{12}+24\sqrt{42}$	Simplify radicals, finding perfect square factors
$=14\sqrt{10}-16\sqrt{35}-21\sqrt{4\cdot 3}+24\sqrt{42}$	Take the square root where possible
$=14\sqrt{10}-16\sqrt{35}-21\cdot 2\sqrt{3}+24\sqrt{42}$	Multiply coefficients
$=14\sqrt{10}-16\sqrt{35}-42\sqrt{3}+24\sqrt{42}$	Our Answer

The next example shows how to use FOIL to square a radical expression with two terms.

Example 6. Multiply.

$(\sqrt{2}+\sqrt{3})^2$	Write as a product
$=(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})$	FOIL, following rules for multiplying radicals
$=\sqrt{4}+\sqrt{6}+\sqrt{6}+\sqrt{9}$	Take the square root where possible
$=2+\sqrt{6}+\sqrt{6}+3$	Combine like terms
$=5+2\sqrt{6}$	Our Answer

As we are multiplying we always look at our final answer to check if all the radicals are simplified and all like radicals have been combined.

DIVIDING RADICAL EXPRESSIONS

Division with radicals is very similar to multiplication. If we think about division as reducing fractions, we can reduce the coefficients outside the radicals and reduce the values inside the radicals to get our final answer.

QUOTIENT RULE OF RADICALS

For any *nonnegative* real numbers b and d,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\frac{a}{c}\sqrt[n]{\frac{b}{d}}$$

Example 7. Divide.

$\dfrac{15\sqrt[3]{108}}{20\sqrt[3]{2}}$	Reduce $\dfrac{15}{20}$ by dividing common factor of 5; reduce $\dfrac{\sqrt[3]{108}}{\sqrt[3]{2}}$ by dividing 108 by 2
$=\dfrac{3\sqrt[3]{54}}{4}$	Simplify radical, 54 is divisible by 27
$=\dfrac{3\sqrt[3]{27\cdot 2}}{4}$	Take the cube root of 27
$=\dfrac{3\cdot 3\sqrt[3]{2}}{4}$	Multiply coefficients
$=\dfrac{9\sqrt[3]{2}}{4}$	Our Answer

Example 8. Divide.

$\dfrac{\sqrt{50x^3y^7}}{\sqrt{2xy^2}}$	Divide $\dfrac{50}{2}, \dfrac{x^3}{x}$, and $\dfrac{y^7}{y^2}$
$=\sqrt{25x^2y^5}$	Simplify radical, 25 is a perfect square, divide exponents by 2
$=5xy^2\sqrt{y}$	Our Answer

There is one catch to dividing radical expressions. It is considered bad practice to have a radical in the denominator of our final answer. We will see how to handle this situation in the next section.

Practice Exercises
Section 3.4: Multiply and Divide Radical Expressions

Perform the indicated operation.

1) $3\sqrt{5}\cdot -4\sqrt{16}$

2) $-5\sqrt{10}\cdot\sqrt{15}$

3) $\sqrt{12m}\cdot\sqrt{15m}$

4) $\sqrt{5r^3}\cdot -5\sqrt{10r^2}$

5) $\sqrt[3]{4x^3}\cdot\sqrt[3]{2x^4}$

6) $\sqrt{3}(4-\sqrt{6})$

7) $\sqrt{6}(\sqrt{2}+2)$

8) $\sqrt{10}(\sqrt{5}+\sqrt{2})$

9) $-5\sqrt{15}(3\sqrt{3}+2)$

10) $\sqrt{7}(\sqrt{3}+5\sqrt{14})$

11) $6\sqrt{10}(5n+\sqrt{2})$

12) $\sqrt{15}(\sqrt{5}-3\sqrt{3v})$

13) $(2+2\sqrt{2})(-3+\sqrt{2})$

14) $(-2+\sqrt{3})(-5+2\sqrt{3})$

15) $(\sqrt{5}-5)(2\sqrt{5}-1)$

16) $(2\sqrt{3}+\sqrt{5})(2\sqrt{3}-\sqrt{5})$

17) $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$

18) $(2\sqrt{2p}+\sqrt{q})(3\sqrt{2p}+\sqrt{q})$

19) $(-5-4\sqrt{3})(-3-4\sqrt{3})$

20) $(7-\sqrt{3})^2$

21) $\frac{\sqrt{12}}{5\sqrt{100}}$

22) $\frac{\sqrt{15}}{2\sqrt{4}}$

23) $\frac{\sqrt{5}}{4\sqrt{125}}$

24) $\frac{\sqrt{12}}{\sqrt{3}}$

ANSWERS to Practice Exercises
Section 3.4: Multiply and Divide Radical Expressions

1) $-48\sqrt{5}$

2) $-25\sqrt{6}$

3) $6m\sqrt{5}$

4) $-25r^2\sqrt{2r}$

5) $2x^2\sqrt[3]{x}$

6) $4\sqrt{3}-3\sqrt{2}$

7) $2\sqrt{3}+2\sqrt{6}$

8) $5\sqrt{2}+2\sqrt{5}$

9) $-45\sqrt{5}-10\sqrt{15}$

10) $\sqrt{21}+35\sqrt{2}$

11) $30n\sqrt{10}+12\sqrt{5}$

12) $5\sqrt{3}-9\sqrt{5v}$

13) $-2-4\sqrt{2}$

14) $16-9\sqrt{3}$

15) $15-11\sqrt{5}$

16) 7

17) $6a+a\sqrt{10}+6a\sqrt{6}+2a\sqrt{15}$

18) $12p+5\sqrt{2pq}+q$

19) $63+32\sqrt{3}$

20) $52-14\sqrt{3}$

21) $\frac{\sqrt{3}}{25}$

22) $\frac{\sqrt{15}}{4}$

23) $\frac{1}{20}$

24) 2

Section 3.5: Rationalize Denominators

Objective: Rationalize the denominators of radical expressions.

It is considered bad practice to have a radical in the denominator of a fraction in final form. If there is a radical in the denominator, we will *rationalize* it or clear out any radicals in the denominator.

RATIONALIZING DENOMINATORS CONTAINING ONE TERM

First, we will focus on rationalizing denominators with a single radical term that is a square root in the denominator. Multiply both the numerator and denominator by the same square root to produce a perfect square in the denominator. Use the property for a *nonnegative* number *a:* $(\sqrt{a})(\sqrt{a}) = (\sqrt{a^2}) = a$.

Example 1. Rationalize the denominator.

$\dfrac{\sqrt{6}}{\sqrt{5}}$	Multiply numerator and denominator by $\sqrt{5}$
$= \dfrac{\sqrt{6}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$	Multiply the numerators; multiply $\sqrt{5} \cdot \sqrt{5} = \sqrt{25} = 5$ in the denominator
$= \dfrac{\sqrt{30}}{5}$	Our Answer

Example 2. Rationalize the denominator.

$\dfrac{12}{\sqrt{3x}}$	Multiply numerator and denominator by $\sqrt{3x}$
$= \dfrac{12}{\sqrt{3x}} \cdot \dfrac{\sqrt{3x}}{\sqrt{3x}}$	Multiply the numerators; multiply $\sqrt{3x} \cdot \sqrt{3x} = \sqrt{9x^2} = 3x$ in the denominator
$= \dfrac{12\sqrt{3x}}{3x}$	Reduce the fraction
$= \dfrac{4\sqrt{3x}}{x}$	Our Answer

Example 3. Rationalize the denominator.

$\dfrac{2+\sqrt{3}}{\sqrt{7}}$ Multiply numerator and denominator by $\sqrt{7}$

$=\dfrac{(2+\sqrt{3})}{\sqrt{7}}\cdot\dfrac{\sqrt{7}}{\sqrt{7}}$ Distribute in the numerator; multiply $\sqrt{7}\cdot\sqrt{7}=\sqrt{49}=7$ in the denominator

$=\dfrac{2\sqrt{7}+\sqrt{21}}{7}$ Our Answer

Example 4. Rationalize the denominator.

$\dfrac{\sqrt{3}-9}{2\sqrt{6}}$ Multiply numerator and denominator by $\sqrt{6}$

$=\dfrac{(\sqrt{3})-9}{2\sqrt{6}}\cdot\dfrac{\sqrt{6}}{\sqrt{6}}$ Distribute in the numerator; multiply $\sqrt{6}\cdot\sqrt{6}=\sqrt{36}=6$ in the denominator

$=\dfrac{\sqrt{18}-9\sqrt{6}}{2\cdot 6}$ Simplify radicals in numerator*;* multiply denominator

$=\dfrac{\sqrt{9\cdot 2}-9\sqrt{6}}{12}$ Take square root where possible

$=\dfrac{3\sqrt{2}-9\sqrt{6}}{12}$ Factor numerator

$=\dfrac{3\left(\sqrt{2}-3\sqrt{6}\right)}{12}$ Reduce by dividing common factor of 3

$=\dfrac{\sqrt{2}-3\sqrt{6}}{4}$ Our Answer

It is important to remember that when reducing the fraction, we cannot reduce with just the 3 and 12 or just the 9 and 12. When we have addition or subtraction in the numerator or denominator, factor and then divide out any common factors.

As we rationalize denominators, it will always be important to constantly check our answer to see if it can be simplified more. We ask ourselves, can the fraction be reduced? Can the radicals be simplified? These steps may happen several times on our way to the solution.

RATIONALIZING DENOMINATORS CONTAINING TWO TERMS

Now we will focus on rationalizing denominators containing two terms with one or more square roots. We will use a different strategy to rationalize the denominator than we did when the denominator had one radical term.

Consider $\frac{2}{\sqrt{3}-5}$. If we were to multiply the denominator by $\sqrt{3}$, we would distribute and end up with $3-5\sqrt{3}$. We have not cleared the radical from the denominator so our current method will not work.

Instead, we will multiply numerator and denominator by the **conjugate** of the denominator. The conjugate has the same terms but with the opposite sign in the middle. In our example with $\sqrt{3}-5$ in the denominator, its conjugate is $\sqrt{3}+5$. When we multiply the conjugates, we get:

$$(\sqrt{3}-5)(\sqrt{3}+5)=3+5\sqrt{3}-5\sqrt{3}-25=3-25=-22$$

When multiplying conjugates, we will no longer have a radical in the denominator.

Example 5. Rationalize the denominator.

$\frac{2}{\sqrt{3}-5}$	Multiply numerator and denominator by $\sqrt{3}+5$, the conjugate of the denominator.
$=\frac{2}{\sqrt{3}-5}\left(\frac{\sqrt{3}+5}{\sqrt{3}+5}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\frac{2\sqrt{3}+10}{3+5\sqrt{3}-5\sqrt{3}-25}$	Simplify the denominator.
$=\frac{2\sqrt{3}+10}{-22}$	Factor numerator using a GCF of -2
$=\frac{-2\left(-\sqrt{3}-5\right)}{-22}$	Reduce by dividing common factor of -2
$=\frac{-\sqrt{3}-5}{11}$	Our Answer

Example 6. Rationalize the denominator.

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}$$ Multiply numerator and denominator by $\sqrt{5}-\sqrt{3}$, the conjugate of the denominator.

$$=\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)$$ Distribute in the numerator; multiply conjugates in the denominator.

$$=\frac{\sqrt{75}-\sqrt{45}}{5-\sqrt{15}+\sqrt{15}-3}$$ Simplify the radicals in the numerator; simplify the denominator.

$$=\frac{\sqrt{25}\sqrt{3}-\sqrt{9}\sqrt{5}}{2}$$ Take square roots.

$$=\frac{5\sqrt{3}-3\sqrt{5}}{2}$$ Our Answer

Example 7. Rationalize the denominator.

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}$$ Multiply numerator and denominator by $4+\sqrt{5x^3}$, the conjugate of the denominator.

$$=\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}\left(\frac{4+\sqrt{5x^3}}{4+\sqrt{5x^3}}\right)$$ Distribute in the numerator; multiply conjugates in the denominator.

$$=\frac{8\sqrt{3x}+2\sqrt{15x^4}}{16+4\sqrt{5x^3}-4\sqrt{5x^3}-5x^3}$$ Simplify the radicals in the numerator; simplify the denominator.

$$=\frac{8\sqrt{3x}+2x^2\sqrt{15}}{16-5x^3}$$ Our Answer

Example 8. Rationalize the denominator.

$\dfrac{3-\sqrt{5}}{2-\sqrt{3}}$	Multiply numerator and denominator by $2+\sqrt{3}$, the conjugate of the denominator.
$=\dfrac{3-\sqrt{5}}{2-\sqrt{3}}\left(\dfrac{2+\sqrt{3}}{2+\sqrt{3}}\right)$	Distribute in the numerator; multiply conjugates in the denominator.
$=\dfrac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4+2\sqrt{3}-2\sqrt{3}-3}$	Simplify the radicals in the numerator; simplify the denominator.
$=\dfrac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1}$	Divide each term by 1.
$=6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}$	Our Answer

Practice Exercises
Section 3.5: Rationalize Denominators

Rationalize the denominator.

1) $\frac{1}{\sqrt{2}}$

2) $\frac{5}{\sqrt{3}}$

3) $\frac{\sqrt{10}}{\sqrt{6}}$

4) $\frac{\sqrt{2}}{3\sqrt{5}}$

5) $\frac{2\sqrt{4}}{3\sqrt{3}}$

6) $\frac{4\sqrt{3}}{\sqrt{15}}$

7) $\frac{\sqrt{5}}{2-\sqrt{3}}$

8) $\frac{7}{\sqrt{3}+8}$

9) $\frac{2}{\sqrt{7}+\sqrt{2}}$

10) $\frac{2\sqrt{3}}{9+3\sqrt{11}}$

11) $\frac{1+\sqrt{3}}{5+\sqrt{2}}$

12) $\frac{6\sqrt{3}}{\sqrt{5}-3}$

13) $\frac{8}{\sqrt{13}-\sqrt{7}}$

14) $\frac{4}{2\sqrt{7}-2}$

15) $\frac{5x^2}{4\sqrt{3x^2y^3}}$

16) $\frac{4}{5\sqrt{3xy^4}}$

17) $\frac{\sqrt{2p^2}}{\sqrt{3p}}$

18) $\frac{\sqrt{8n^2}}{\sqrt{10n}}$

19) $\frac{2-5\sqrt{5}}{4\sqrt{13}}$

20) $\frac{\sqrt{5}+4}{4\sqrt{17}}$

21) $\frac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

22) $\frac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

ANSWERS to Practice Exercises
Section 3.5: Rationalize Denominators

1) $\dfrac{\sqrt{2}}{2}$

2) $\dfrac{5\sqrt{3}}{3}$

3) $\dfrac{\sqrt{15}}{3}$

4) $\dfrac{\sqrt{10}}{15}$

5) $\dfrac{4\sqrt{3}}{9}$

6) $\dfrac{4\sqrt{5}}{5}$

7) $2\sqrt{5}+\sqrt{15}$

8) $\dfrac{56-7\sqrt{3}}{61}$

9) $\dfrac{2\left(\sqrt{7}-\sqrt{2}\right)}{5}$

10) $\dfrac{\sqrt{33}-3\sqrt{3}}{3}$

11) $\dfrac{5-\sqrt{2}+5\sqrt{3}-\sqrt{6}}{23}$

12) $\dfrac{-3\left(\sqrt{15}+3\sqrt{3}\right)}{2}$

13) $\dfrac{4\left(\sqrt{13}+\sqrt{7}\right)}{3}$

14) $\dfrac{1+\sqrt{7}}{3}$

15) $\dfrac{5x\sqrt{3y}}{12y^2}$

16) $\dfrac{4\sqrt{3x}}{15xy^2}$

17) $\dfrac{\sqrt{6p}}{3}$

18) $\dfrac{2\sqrt{5n}}{5}$

19) $\dfrac{2\sqrt{13}-5\sqrt{65}}{52}$

20) $\dfrac{\sqrt{85}+4\sqrt{17}}{68}$

21) $\dfrac{\sqrt{6}-9}{3}$

22) $\dfrac{\sqrt{30}-2\sqrt{3}}{18}$

Section 3.6: Rational Exponents

Objectives: Convert between radical notation and exponential notation.
Simplify expressions with rational exponents using the properties of exponents.
Multiply and divide radical expressions with different indices.

We define rational exponents as follows:

DEFINITION OF RATIONAL EXPONENTS:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \text{ and } a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

The denominator of a rational exponent is the same as the index of our radical while the numerator serves as an exponent.

Either form of the definition can be used but we typically use the first form as it will involve smaller numbers.

Notice when the numerator of the exponent is 1, the special case of n^{th} roots follows from the definition:

$$a^{\frac{1}{n}} = (\sqrt[n]{a})^1 = \sqrt[n]{a}$$

CONVERTING BETWEEN EXPONENTIAL AND RADICAL NOTATION

We can use this definition to change any radical expression into an exponential expression.

Example 1. Rewrite with rational exponents.

$(\sqrt[5]{x})^3 = x^{\frac{3}{5}}$	$(\sqrt[6]{3x})^5 = (3x)^{\frac{5}{6}}$	Index is denominator, exponent is numerator
$\frac{1}{(\sqrt[7]{a})^3} = a^{-\frac{3}{7}}$	$\frac{1}{(\sqrt[3]{xy})^2} = (xy)^{-\frac{2}{3}}$	Negative exponents from reciprocals

We can also change any rational exponent into a radical expression by using the denominator as the index.

Example 2. Rewrite using radical notation.

$a^{\frac{5}{3}} = (\sqrt[3]{a})^5$	$(2mn)^{\frac{2}{7}} = (\sqrt[7]{2mn})^2$
$x^{-\frac{4}{5}} = \dfrac{1}{(\sqrt[5]{x})^4}$	$(xy)^{-\frac{2}{9}} = \dfrac{1}{(\sqrt[9]{xy})^2}$

Exponent is numerator; index is denominator

Negative exponent means reciprocals

The ability to change between exponential expressions and radical expressions allows us to evaluate expressions we had no means of evaluating previously.

Example 3. Use radical notation to rewrite and evaluate.

$16^{\frac{3}{2}}$ Change to radical format; numerator is exponent, denominator is index

$= (\sqrt{16})^3$ Evaluate radical

$= (4)^3$ Evaluate exponent

$= 64$ Our Answer

Example 4. Use radical notation to rewrite and evaluate.

$27^{-\frac{4}{3}}$ Negative exponent is reciprocal

$= \dfrac{1}{27^{\frac{4}{3}}}$ Change to radical format; numerator is exponent, denominator is index

$= \dfrac{1}{(\sqrt[3]{27})^4}$ Evaluate radical

$= \dfrac{1}{(3)^4}$ Evaluate exponent

$= \dfrac{1}{81}$ Our Answer

SIMPLIFY EXPRESSIONS WITH RATIONAL EXPONENTS

The largest advantage of being able to change a radical expression into an exponential expression is we are now allowed to use all our exponent properties to simplify. The following table reviews all of our exponent properties.

PROPERTIES OF EXPONENTS

$a^m a^n = a^{m+n}$	$(ab)^m = a^m b^m$	$a^{-m} = \dfrac{1}{a^m}$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\dfrac{1}{a^{-m}} = a^m$
$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\dfrac{a}{b}\right)^{-m} = \dfrac{b^m}{a^m}$

When adding and subtracting with fractions we need to have a common denominator. When multiplying we only need to multiply the numerators together and denominators together. The following examples show several different problems, using different properties to simplify the rational exponents.

Example 5. Simplify.

$a^{\frac{2}{3}}b^{\frac{1}{2}}a^{\frac{1}{6}}b^{\frac{1}{5}}$	Need common denominator for a s (6) and for b s (10)
$= a^{\frac{4}{6}}b^{\frac{5}{10}}a^{\frac{1}{6}}b^{\frac{2}{10}}$	Add exponents on a s and b s
$= a^{\frac{5}{6}}b^{\frac{7}{10}}$	Our Answer

Example 6. Simplify.

$\left(x^{\frac{1}{3}}y^{\frac{2}{5}}\right)^{\frac{3}{4}}$	Multiply each exponent by $\dfrac{3}{4}$; reduce fractions
$= x^{\frac{1}{4}}y^{\frac{3}{10}}$	Our Answer

Example 7. Simplify.

$\dfrac{x^2y^{\frac{2}{3}}}{x^{\frac{7}{2}}y^0}$	Need common denominator for xs (2) to subtract exponents
$=\dfrac{x^{\frac{4}{2}}y^{\frac{2}{3}}}{x^{\frac{7}{2}}y^0}$	Subtract exponents on x in denominator, $y^0=1$
$=x^{-\frac{3}{2}}y^{\frac{2}{3}}$	Negative exponent moves down to denominator
$=\dfrac{y^{\frac{2}{3}}}{x^{\frac{3}{2}}}$	Our Answer

MULTIPLY AND DIVIDE RADICAL EXPRESSIONS WITH DIFFERENT INDICES

We will use rational exponents to multiply or divide radical expressions having different indices. We will convert each radical expression to its equivalent exponential expression. Then, we will apply the appropriate exponent property. For our answer, we will convert the exponential expression to its equivalent radical expression. Our answer will then be written as a single radical expression.

Example 8. Multiply, writing the expression using a single radical.

$\sqrt[5]{x}\cdot\sqrt{x}$	Rewrite radical expressions using rational exponents
$=x^{\frac{1}{5}}\cdot x^{\frac{1}{2}}$	Need common denominator of 10 to add exponents
$=x^{\frac{2}{10}}\cdot x^{\frac{5}{10}}$	Add exponents
$=x^{\frac{7}{10}}$	Rewrite as a radical expression
$=\sqrt[10]{x^7}$	Our Answer

Example 9. Divide, writing the expression using a single radical.

$\dfrac{\sqrt[3]{y^2}}{\sqrt[5]{y^2}}$	Rewrite radical expressions using rational exponents
$=\dfrac{y^{\frac{2}{3}}}{y^{\frac{2}{5}}}$	Need common denominator of 15 to subtract exponents
$=\dfrac{y^{\frac{10}{15}}}{y^{\frac{6}{15}}}$	Subtract exponents
$=y^{\frac{4}{15}}$	Rewrite as a radical expression
$=\sqrt[15]{y^4}$	Our Answer

It is important to remember that as we simplify with rational exponents, we are using the exact same properties we used when simplifying integer exponents. The only difference is we need to follow our rules for fractions as well. It may be worth reviewing your notes on exponent properties to be sure you are comfortable with using the properties.

Practice Exercises
Section 3.6: Rational Exponents

Write each expression in radical form.

1) $m^{\frac{3}{5}}$

2) $(10r)^{-\frac{3}{4}}$

3) $(7x)^{\frac{3}{2}}$

4) $(6b)^{-\frac{3}{4}}$

Write each expression in exponential form.

5) $\dfrac{1}{\left(\sqrt{6x}\right)^3}$

6) $\sqrt{v}$

7) $\dfrac{1}{(\sqrt[4]{n})^7}$

8) $\sqrt{5a}$

Evaluate.

9) $8^{\frac{2}{3}}$

10) $16^{\frac{1}{4}}$

11) $4^{\frac{3}{2}}$

12) $100^{-\frac{3}{2}}$

13) $27^{-\frac{1}{3}}$

14) $32^{\frac{3}{5}}$

15) $81^{-\frac{3}{4}}$

16) $25^{\frac{3}{2}}$

Simplify. Your answer should contain only positive exponents.

17) $x^{\frac{1}{3}}y \cdot xy^{\frac{3}{2}}$

18) $4v^{\frac{2}{3}} \cdot v^{-1}$

19) $\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^{-1}$

20) $\left(x^{\frac{5}{3}}y^{-2}\right)^0$

21) $\left(x^0 y^{\frac{1}{3}}\right)^{\frac{3}{2}} x^0$

22) $u^{-\frac{5}{4}}v^2 \cdot \left(u^{\frac{3}{2}}\right)^{-\frac{3}{2}}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 3.6 (continued)

Simplify. Your answer should contain only positive exponents.

23) $\dfrac{a^{\frac{3}{4}}b^{-1}\cdot b^{\frac{7}{4}}}{3b^{-1}}$

24) $\dfrac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}}\cdot xy^{\frac{1}{2}}}$

25) $\dfrac{3y^{-\frac{5}{4}}}{y^{-1}\cdot 2y^{-\frac{1}{3}}}$

26) $\dfrac{ab^{\frac{1}{3}}\cdot 2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{\frac{2}{3}}}$

27) $\left(\dfrac{m^{\frac{3}{2}}n^{-2}}{\left(mn^{\frac{4}{3}}\right)^{-1}}\right)^{\frac{7}{4}}$

28) $\dfrac{\left(y^{-\frac{1}{2}}\right)^{\frac{3}{2}}}{x^{\frac{3}{2}}y^{\frac{1}{2}}}$

29) $\dfrac{\left(m^2n^{\frac{1}{2}}\right)^0}{n^{\frac{3}{4}}}$

30) $\dfrac{y^0}{\left(x^{\frac{3}{4}}y^{-1}\right)^{\frac{1}{3}}}$

31) $\dfrac{\left(x^{-\frac{4}{3}}y^{-\frac{1}{3}}\cdot y\right)^{-1}}{x^{\frac{1}{3}}y^{-2}}$

32) $\dfrac{\left(x^{\frac{1}{2}}y^0\right)^{-\frac{4}{3}}}{y^4\cdot x^{-2}y^{-\frac{2}{3}}}$

Perform the indicated operation, writing the expression using a single radical.

33) $\sqrt{x}\cdot\sqrt[4]{x}$

34) $\dfrac{\sqrt[5]{x^2}}{\sqrt[6]{x}}$

ANSWERS to Practice Exercises
Section 3.6: Rational Exponents

1) $\left(\sqrt[5]{m}\right)^3$

2) $\dfrac{1}{\left(\sqrt[4]{10r}\right)^3}$

3) $\left(\sqrt{7x}\right)^3$

4) $\dfrac{1}{\left(\sqrt[4]{6b}\right)^3}$

---

5) $(6x)^{-\frac{3}{2}}$

6) $v^{\frac{1}{2}}$

7) $n^{-\frac{7}{4}}$

8) $(5a)^{\frac{1}{2}}$

---

9) 4

10) 2

11) 8

12) $\dfrac{1}{1000}$

13) $\dfrac{1}{3}$

14) 8

15) $\dfrac{1}{27}$

16) 125

---

17) $x^{\frac{4}{3}}y^{\frac{5}{2}}$

18) $\dfrac{4}{v^{\frac{1}{3}}}$

19) $\dfrac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

20) 1

21) $y^{\frac{1}{2}}$

22) $\dfrac{v^2}{u^{\frac{7}{2}}}$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 3.6 (continued)

23) $\dfrac{a^{\frac{3}{4}}b^{\frac{7}{4}}}{3}$

24) $\dfrac{2y^{\frac{17}{6}}}{x^{\frac{7}{4}}}$

25) $\dfrac{3y^{\frac{1}{12}}}{2}$

26) $\dfrac{a^{\frac{3}{2}}}{2b^{\frac{19}{12}}}$

27) $\dfrac{m^{\frac{35}{8}}}{n^{\frac{7}{6}}}$

28) $\dfrac{1}{x^{\frac{3}{2}}y^{\frac{5}{4}}}$

29) $\dfrac{1}{n^{\frac{3}{4}}}$

30) $\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{4}}}$

31) $xy^{\frac{4}{3}}$

32) $\dfrac{x^{\frac{4}{3}}}{y^{\frac{10}{3}}}$

---

33) $\sqrt[4]{x^3}$

34) $\sqrt[30]{x^7}$

Section 3.7: Solving Radical Equations

Objective: Solve equations with radicals and check for extraneous solutions.

In this section, we solve equations that have roots in the problem. As you might expect, to clear a root we can raise both sides to an exponent. Thus, to clear a square root, we can raise both sides to the second power. To clear a cube root, we can raise both sides to the third power.

There is one catch to solving radical equations. Sometimes we end up with proposed solutions that do not actually work in the original equation. This will only happen if the index on the root is even, and it will not happen all the time for those roots. So, for radical equations solved by raising both sides to an *even* power, we must check our answers by substituting each result into the original equation. If a proposed solution does not work, it is called an *extraneous* solution, and is not included in the final solution.

NOTE: When solving a radical equation with an *even* index, always check your answers!

Example 1. Solve the equation.

$\sqrt{7x+2}=4$	Even index; we will have to check all results
$\left(\sqrt{7x+2}\right)^2=4^2$	Square both sides, simplify exponents
$7x+2=16$	Solve
$\underline{-2\quad -2}$	Subtract 2 from both sides
$\dfrac{7x}{7}=\dfrac{14}{7}$	Divide both sides by 7.
$x=2$	Need to check this result in the original equation
$\sqrt{7(2)+2}=4$	Multiply
$\sqrt{14+2}=4$	Add
$\sqrt{16}=4$	Square root
$4=4$	True, it works
$x=2$	Our Solution

Example 2. Solve the equation.

$\sqrt[3]{x-1}=-4$	Odd index; we don't need to check our results
$\left(\sqrt[3]{(x-1)}\right)^3=(-4)^3$	Cube both sides, simplify exponents
$x-1=-64$	Solve
$\underline{\quad+1\quad\ +1}$	Add 1 to both sides
$x=-63$	Our Solution

Example 3. Solve the equation.

$\sqrt[4]{3x+6}=-3$	Even index; we will have to check all results
$\left(\sqrt[4]{3x+6}\right)^4=(-3)^4$	Raise both sides to the fourth power
$3x+6=81$	Solve
$\underline{\quad-6\quad-6}$	Subtract 6 from both sides
$\dfrac{3x}{3}=\dfrac{75}{3}$	Divide both sides by 3
$x=25$	Need to check this result in the original equation
$\sqrt[4]{3(25)+6}=-3$	Multiply
$\sqrt[4]{75+6}=-3$	Add
$\sqrt[4]{81}=-3$	Simplify the radical
$3=-3$	False, extraneous solution; thus, $x=25$ is not a solution
No Solution	Our Solution

If the radical is not alone on one side of the equation, we will have to isolate the radical before we raise it to an exponent.

Example 4. Solve the equation.

$x+\sqrt{4x+1}=5$	Even index, we will have to check all results
$\underline{-x\qquad\qquad\quad -x}$	Isolate radical by subtracting x from both sides
$\sqrt{4x+1}=5-x$	Square both sides
$\left(\sqrt{4x+1}\right)^2=(5-x)^2$	Evaluate exponents, recall $(a-b)^2=a^2-2ab+b^2$

$4x+1=25-10x+x^2$	Rewrite equation equal to zero
$\underline{-4x-1 \quad\quad -4x-1}$	Subtract $4x$ and 1 from both sides; reorder terms
$0=x^2-14x+24$	Factor
$0=(x-12)(x-2)$	Set each factor equal to zero
$x-12=0$ or $x-2=0$	Solve each equation
$\underline{+12 \quad +12}$ $\quad$ $\underline{+2 \quad +2}$	
$x=12$ or $x=2$	Need to check both results by substituting each into the original equation
$(12)+\sqrt{4(12)+1}=5$	Check $x=12$ first; multiply inside the radical
$12+\sqrt{48+1}=5$	Add inside the root sign
$12+\sqrt{49}=5$	Take the square root
$12+7=5$	Add
$19=5$	False, extraneous solution; thus $x=12$ is not a solution
$(2)+\sqrt{4(2)+1}=5$	Check $x=2$ second; multiply inside the radical
$2+\sqrt{8+1}=5$	Add inside the root sign
$2+\sqrt{9}=5$	Take the square root
$2+3=5$	Add
$5=5$	True, it works
$x=2$	Our Solution

The above example illustrates that as we square both sides of the equation we could end up with a quadratic equation. In this case, we must set the equation to zero and solve by factoring. We will have to check both solutions if the root in the problem was even (for example, a square root or a fourth root). Sometimes both values work, sometimes only one value works, and sometimes neither value works.

If there is more than one square root in a problem we will clear all the roots at the same time. This means we must first make sure that one root is isolated on one side of the equal sign before squaring both sides.

Example 5. Solve the equation.

$\sqrt{3x-8}-\sqrt{x}=0$	Even index, we will have to check all results
$\underline{+\sqrt{x}\quad +\sqrt{x}}$	Isolate first root by adding $\sqrt{x}$ to both sides
$\sqrt{3x-8}=\sqrt{x}$	Square both sides
$\left(\sqrt{3x-8}\right)^2=\left(\sqrt{x}\right)^2$	Evaluate exponents
$3x-8=x$	Solve the equation
$\underline{-3x\quad -3x}$	Subtract $3x$ from both sides
$\dfrac{-8}{-2}=\dfrac{-2x}{-2}$	Divide both sides by -2
$4=x$	Need to check result in original equation
$\sqrt{3(4)-8}-\sqrt{4}=0$	Multiply inside the root sign
$\sqrt{12-8}-\sqrt{4}=0$	Subtract inside the root sign
$\sqrt{4}-\sqrt{4}=0$	Take roots
$2-2=0$	Subtract
$0=0$	True, it works
$x=4$	Our Solution

When the index of the roots is not 2, we need to raise both sides of the equation to the power that corresponds to that index.

Example 6. Solve the equation.

$\sqrt[4]{x-1}=\sqrt[4]{8}$	Even index, we will have to check all results
$\left(\sqrt[4]{x-1}\right)^4=\left(\sqrt[4]{8}\right)^4$	Raise both sides to the fourth power
$x-1=8$	Evaluate exponents
$\underline{+1\quad +1}$	Add 1 to both sides of the equation
$x=9$	Need to check result in original equation
$\sqrt[4]{(9)-1}=\sqrt[4]{8}$	Subtract
$\sqrt[4]{8}=\sqrt[4]{8}$	True, it works
$x=9$	Our Solution

Example 7. Solve the equation.

$\sqrt[3]{x^2+5}=\sqrt[3]{x^2-4x+1}$	Odd index, we don't need to check our results
$\left(\sqrt[3]{x^2+5}\right)=\left(\sqrt[3]{x^2-4x+1}\right)^3$	Raise both sides to the third power
$x^2+5=x^2-4x+1$	Subtract x^2 on both sides of the equation
$\underline{-x^2 \qquad -x^2}$	
$5=-4x+1$	Subtract 1 from both sides of the equation
$\underline{-1 \qquad -1}$	
$\dfrac{4}{-4}=\dfrac{-4x}{-4}$	Divide both sides by -4
$-1=x$	
$x=-1$	Our Solution

Practice Exercises
Section 3.7: Solving Radical Equations

Solve.

1) $\sqrt{2x+3}-3=0$

2) $\sqrt{5x+1}-4=0$

3) $\sqrt{6x-5}-x=0$

4) $\sqrt{x+2}-\sqrt{3x}=0$

5) $3+x=\sqrt{6x+13}$

6) $x-1=\sqrt{7-x}$

7) $\sqrt{3-3x}-1=2x$

8) $\sqrt{2x+2}=\sqrt{5x-1}$

9) $\sqrt{4x+5}-\sqrt{x+4}=0$

10) $\sqrt{3x+4}-\sqrt{x+2}=0$

11) $\sqrt{2x-4}-\sqrt{x+3}=0$

12) $\sqrt[3]{3x+1}=-2$

13) $\sqrt[4]{x-3}=2$

14) $\sqrt[4]{7x-5}=-2$

15) $\sqrt[5]{6x-2}=-2$

16) $\sqrt[3]{2x-1}=\sqrt[3]{7x+9}$

ANSWERS to Practice Exercises
Section 3.7: Solving Radical Equations

1) 3

2) 3

3) $1, 5$

4) 1

5) ± 2

6) 3

7) $\frac{1}{4}$

8) 1

9) $-\frac{1}{3}$

10) -1

11) 7

12) -3

13) 19

14) no solution

15) -5

16) -2

Section 3.8: Complex Numbers

Objective: Add, subtract, multiply, divide, and simplify expressions using complex numbers.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, mathematicians create new ways for dealing with the problem. Throughout history this has been the case with the need for a number that is nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for the square roots of negative numbers. To work with the square root of negative numbers and to solve equations such as $x^2 = -1$; mathematicians have created a new number system called the complex numbers. First, we define the imaginary unit:

DEFINITION OF THE IMAGINARY UNIT *i*

$$i = \sqrt{-1} \quad \text{where } i^2 = -1$$

With this definition, the square root of a negative number can now be expressed as a multiple of i. We will use the product rule of radicals and simplify the negative as a factor of negative one. This process is shown in the following examples.

Example 1. Write in terms of i.

$\sqrt{-16}$	Consider the negative as a factor of -1
$=\sqrt{-1 \cdot 16}$	Take each root, square root of -1 is i
$=4i$	Our Answer

Example 2. Write in terms of i.

$\sqrt{-24}$	Find perfect square factors, including -1
$=\sqrt{-1 \cdot 4 \cdot 6}$	Square root of -1 is i, square root of 4 is 2
$=2i\sqrt{6}$	Our Answer

Then, mathematicians created a new number system called the set of complex numbers. A **complex number** is one that contains both a real and imaginary part.

DEFINITION OF COMPLEX NUMBERS

$$a+bi$$

where a and b are real numbers

We call a the real part and b the imaginary part. Examples of complex numbers include $2+5i, -3+i\sqrt{5}, -6i$ because $-6i = 0-6i$ *;* and 3 because $3 = 3+0i$.

ADDING AND SUBTRACTING COMPLEX NUMBERS

The operations of addition, subtraction, and multiplication of complex numbers are performed very similarly to how they are done with polynomials. A new technique will be needed for division though.

When adding and subtracting complex numbers, we combine like terms by adding or subtracting the real parts, adding or subtracting the imaginary parts, and expressing the answer in the form of a complex number $a+bi$.

Example 3. Add.

$(2+5i)+(4-7i)$	Combine real parts, $2+4$ imaginary parts $5i-7i$
$=6-2i$	Our Answer

It is important to notice what operation we are doing. Students often see the parentheses and think that means FOIL. We only use FOIL to multiply. This problem is an addition problem, so we simply add the real and imaginary parts.

For subtraction of complex numbers, the idea is the same, but we need to remember to first distribute the negative onto all the terms in the parentheses.

Example 4. Subtract.

$(4-8i)-(3-5i)$	Distribute the negative
$=4-8i-3+5i$	Combine real parts, $4-3$ imaginary parts $-8i+5i$
$=1-3i$	Our Answer

Addition and subtraction can appear together in one problem.

Example 5. Perform the indicated operations.

$(5i)-(3+8i)+(-4+7i)$	Distribute the negative
$=5i-3-8i-4+7i$	Combine real parts, $-3-4$ imaginary parts $5i-8i+7i$
$=-7+4i$	Our Answer

MULTIPLYING COMPLEX NUMBERS

Multiplying complex numbers is the same as multiplying polynomials, but we replace i^2 with -1.

Example 6. Multiply.

$(3i)(7i)$	Multiply
$=21i^2$	Replace i^2 with -1
$=21(-1)$	Multiply
$=-21$	Our Answer

When multiplying complex radicals, it is important that we first rewrite as multiples of i.

Example 7. Multiply.

$\sqrt{-6}\sqrt{-3}$	Simplify each root using $i=\sqrt{-1}$
$=(i\sqrt{6})(i\sqrt{3})$	Multiply
$=i^2\sqrt{18}$	Replace i^2 with -1
$=-\sqrt{18}$	Simplify the radical
$=-\sqrt{9\cdot 2}$	Take square root of 9
$-3\sqrt{2}$	Our Answer

Example 8. Multiply.

$5i(3i-7)$	Distribute
$=15i^2-35i$	Replace i^2 with -1
$=15(-1)-35i$	Multiply
$=-15-35i$	Our Answer

Example 9. Multiply.

$(2-4i)(3+5i)$	FOIL
$=6+10i-12i-20i^2$	Replace i^2 with -1
$=6+10i-12i-20(-1)$	Multiply
$=6+10i-12i+20$	Combine real parts, $6+20$ imaginary parts $10i-12i$
$=26-2i$	Our Answer

Remember when squaring a binomial, we write as a product of two same binomials and then FOIL.

Example 10. Multiply.

$(4-5i)^2$	Write as a product of two same complex numbers
$=(4-5i)(4-5i)$	FOIL
$=16-20i-20i+25i^2$	Replace i^2 with -1
$=16-20i-20i+25(-1)$	Multiply
$=16-20i-20i-25$	Combine real parts, $16-25$ imaginary parts $-20i-20i$
$=-9-40i$	Our Answer

Example 11. Multiply.

$(2+3i)(2-3i)$	FOIL
$=4-6i+6i-9i^2$	Replace i^2 with -1
$=4-6i+6i-9(-1)$	Multiply
$=4-6i+6i+9$	Combine real parts, $4+9$ imaginary parts $-6i+6i$
$=13$	Our Answer

Notice how the product of the two complex numbers above resulted in a real number.

The complex numbers $a+bi$ and $a-bi$ are called **complex conjugates** of each other. Notice that $(a+bi)\,(a-bi)=a^2+b^2$. When we multiply complex conjugates, the result is always a real number.

DIVIDING COMPLEX NUMBERS

Dividing complex numbers also has one thing we need to be careful of. If i is $\sqrt{-1}$, and it is in the denominator of a fraction, then we have a radical in the denominator! This means we will want to rationalize our denominator so there are no i s. This is done by multiplying numerator and denominator by the conjugate of the denominator.

Example 12. Divide.

$\dfrac{2-6i}{4+8i}$	Multiply by conjugate of denominator, $4-8i$
$=\dfrac{(2-6i)}{(4+8i)}\cdot\dfrac{(4-8i)}{(4-8i)}$	FOIL in numerator, denominator is difference of squares
$=\dfrac{8-16i-24i+48i^2}{16-64(-1)}$	Replace i^2 with -1
$=\dfrac{8-16i-24i+48(-1)}{16-64(-1)}$	Multiply
$=\dfrac{8-16i-24i-48}{16+64}$	Combine real and imaginary parts
$=\dfrac{-40-40i}{80}$	Reduce, dividing each term by 80
$=-\dfrac{40}{80}-\dfrac{40i}{80}$	Reduce and write in the form of a complex number
$=-\dfrac{1}{2}-\dfrac{1}{2}i$	Our Answer

Example 13. Divide.

$=\dfrac{7+3i}{-5i}$	Multiply by conjugate of denominator, $5i$
$=\dfrac{(7+3i)}{-5i}\cdot\dfrac{5i}{5i}$	Distribute $5i$ in numerator
$=\dfrac{35i+15i^2}{-25i^2}$	Replace i^2 with -1

$= \dfrac{35i + 15(-1)}{-25(-1)}$ Multiply

$= \dfrac{35i - 15}{25}$ Reduce, dividing each term by 25

$= -\dfrac{15}{25} + \dfrac{35}{25}i$ Reduce and write in the form of a complex number

$= -\dfrac{3}{5} + \dfrac{7}{5}i$ Our Answer

Practice Exercises
Section 3.8: Complex Numbers

Write in terms of i.

1) $\sqrt{-81}$

2) $\sqrt{-45}$

Multiply.

3) $\sqrt{-4}\cdot\sqrt{-9}$

4) $\sqrt{-10}\cdot\sqrt{-2}$

5) $\sqrt{-12}\cdot\sqrt{-2}$

6) $\sqrt{-3}\cdot\sqrt{27}$

Perform the indicated operation, writing the answer in the form of a complex number $a+bi$.

7) $3-(-8+4i)$

8) $(-8i)-(7i)-(5-3i)$

9) $(7i)-(3-2i)$

10) $(-4-i)+(1-5i)$

11) $(-6i)-(3+7i)$

12) $(5-4i)+(8-4i)$

13) $(3-3i)+(-7-8i)$

14) $(i)-(2+3i)-6$

15) $(3i)(-8i)$

16) $(16i)(-2i)$

17) $(6i)(-9i)$

18) $(-7i)^2$

19) $(-5i)(-10i)$

20) $(-7-4i)(-8+6i)$

21) $(6+5i)^2$

22) $(8-6i)(-4+2i)$

23) $(-4+5i)(2-7i)$

24) $(-2+i)(3-5i)$

25) $(1+5i)(2+i)$

26) $\dfrac{9}{i}$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 3.8 (continued)

Perform the indicated operation, writing the answer in the form of a complex number $a+bi$**.**

27) $\frac{5}{6i}$

28) $\frac{-3+2i}{-3i}$

29) $\frac{-3-6i}{4i}$

30) $\frac{-4+2i}{3i}$

31) $\frac{10-i}{-i}$

32) $\frac{4i}{-10+i}$

33) $\frac{8}{7-6i}$

34) $\frac{9i}{1-5i}$

35) $\frac{7}{10-7i}$

36) $\frac{4}{4+6i}$

37) $\frac{5-3i}{3+2i}$

38) $\frac{1+7i}{1+i}$

39) $\frac{6-i}{4-3i}$

40) $\frac{3+8i}{2-5i}$

ANSWERS to Practice Exercises
Section 3.8: Complex Numbers

1) $9i$

2) $3i\sqrt{5}$

---

3) -6

4) $-2\sqrt{5}$

5) $-2\sqrt{6}$

6) $9i$

---

7) $11-4i$

8) $-5-12i$

9) $-3+9i$

10) $-3-6i$

11) $-3-13i$

12) $13-8i$

13) $-4-11i$

14) $-8-2i$

15) 24

16) 32

17) 54

18) -49

19) -50

20) $80-10i$

21) $11+60i$

22) $-20+40i$

23) $27+38i$

24) $-1+13i$

25) $-3+11i$

26) $-9i$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 3.8 (continued)

27) $-\frac{5}{6}i$

28) $-\frac{2}{3}-i$

29) $-\frac{3}{2}+\frac{3}{4}i$

30) $\frac{2}{3}+\frac{4}{3}i$

31) $1+10i$

32) $\frac{4}{101}-\frac{40}{101}i$

33) $\frac{56}{85}+\frac{48}{85}i$

34) $-\frac{45}{26}+\frac{9}{26}i$

35) $\frac{70}{149}+\frac{49}{149}i$

36) $\frac{4}{13}-\frac{6}{13}i$

37) $\frac{9}{13}-\frac{19}{13}i$

38) $4+3i$

39) $\frac{27}{25}+\frac{14}{25}i$

40) $-\frac{34}{29}+\frac{31}{29}i$

Review: Chapter 3

Simplify. Assume that all variables represent positive real numbers.

1) $\sqrt{49}$

2) $\sqrt{48}$

3) $3\sqrt{45}$

4) $\sqrt{80p}$

5) $\sqrt{108x^2}$

6) $-2\sqrt{36y^4}$

7) $\sqrt{32m^2n^2}$

8) $\sqrt{147x^4y^4}$

9) $8\sqrt{243a^2b^3}$

10) $-3\sqrt{125q^4r^2s^4}$

11) $-7\sqrt{28xyz^2}$

Simplify.

12) $\sqrt[4]{-81}$

13) $\sqrt[3]{128}$

14) $-3\sqrt[4]{32}$

15) $\sqrt[6]{192x^6}$

16) $\sqrt[3]{128x^3y^6}$

17) $-4\sqrt[3]{135st^3}$

Perform the indicated operation.

18) $3\sqrt{2}+2\sqrt{2}+7\sqrt{2}$

19) $4\sqrt{7}-3\sqrt{3}+2\sqrt{3}$

20) $5\sqrt{2}+\sqrt{18}-2\sqrt{32}$

21) $-2\sqrt{32}-3\sqrt{8}-3\sqrt{12}+6\sqrt{12}$

22) $-6\sqrt[3]{54}+3\sqrt[3]{54}+11\sqrt[3]{2}$

23) $-\sqrt[4]{64}+3\sqrt[4]{4}-5\sqrt[4]{64}$

Perform the indicated operation.

24) $7\sqrt{8}\cdot\sqrt{12}$

25) $\sqrt{14}(\sqrt{7}+\sqrt{2})$

26) $(-4+\sqrt{5})(-3+7\sqrt{5})$

27) $(4+\sqrt{2})^2$

28) $\frac{\sqrt{63}}{\sqrt{7}}$

Rationalize the denominator.

29) $\dfrac{\sqrt{14}}{\sqrt{6}}$

30) $\dfrac{4\sqrt{5}}{\sqrt{15}}$

31) $\dfrac{-8}{\sqrt{11}+\sqrt{3}}$

32) $\dfrac{2\sqrt{6}}{\sqrt{2}-7}$

33) $\dfrac{6x^3}{7\sqrt{5x^3y^2}}$

34) $\dfrac{\sqrt{18n^2}}{\sqrt{12n}}$

35) $\dfrac{-\sqrt{5}-5\sqrt{2}}{\sqrt{2}}$

Write the expression in radical form.

36) $(13r)^{\frac{3}{2}}$

Write the expression in exponential form.

37) $\sqrt{ps}$

Evaluate.

38) $27^{-\frac{2}{3}}$

39) $121^{\frac{3}{2}}$

40) $16^{\frac{3}{4}}$

Simplify. Your answer should contain only positive exponents.

41) $x^{\frac{1}{2}}y^2 \cdot xy^{\frac{3}{2}}$

42) $\left(x^4y^{-3}\right)^0$

43) $\dfrac{a^{\frac{2}{5}}b^{-2} \cdot b^{\frac{5}{2}}}{5b^{-3}}$

44) $\dfrac{3b^{-\frac{3}{4}}(ab^{\frac{2}{3}})}{6a^{-2}b^{\frac{5}{3}}}$

45) $\dfrac{\left(m^2n^{\frac{1}{3}}p\right)^0}{n^{\frac{1}{3}}}$

46) $\dfrac{\left(x^{\frac{1}{2}}y^0\right)^{-4}}{y^3 \cdot x^{-2}}$

Write the expression using a single radical.

47) $\dfrac{\sqrt[3]{x^2}}{\sqrt[4]{x}}$

Solve.

48) $\sqrt{2x-3}-5=0$

49) $\sqrt{6x-8}-x=0$

50) $4+x=\sqrt{8x+25}$

51) $\sqrt{4+7x}+1=2x$

52) $\sqrt{3x-2}-\sqrt{x+5}=0$

53) $\sqrt[4]{x+2}=1$

54) $\sqrt[3]{5x-4}=-3$

Write in terms of i.

55) $\sqrt{-48}$

Multiply.

56) $\sqrt{-9}\cdot\sqrt{-25}$

57) $\sqrt{-18}\cdot\sqrt{-7}$

Perform the indicated operation, writing the answer in the form $a+bi$.

58) $(-5i)+(3i)+(2-9i)$
59) $-4i-(5-6i)$
60) $-(1+2i)-7+i$
61) $(4i)(-7i)$
62) $(-9+i)(-6-2i)$
63) $(2+3i)(4-5i)$
64) $\dfrac{11}{i}$
65) $\dfrac{3+6i}{2i}$
66) $\dfrac{3i}{-8+2i}$
67) $\dfrac{10}{7-10i}$
68) $\dfrac{2-9i}{3+i}$

ANSWERS to Review: Chapter 3

1) 7
2) $4\sqrt{3}$
3) $9\sqrt{5}$
4) $4\sqrt{5p}$
5) $6x\sqrt{3}$
6) $-12y^2$
7) $4mn\sqrt{2}$
8) $7x^2y^2\sqrt{3}$
9) $72ab\sqrt{3b}$
10) $-15q^2rs^2\sqrt{5}$
11) $-14z\sqrt{7xy}$

---

12) Not a real number
13) $4\sqrt[3]{2}$
14) $-6\sqrt[4]{2}$
15) $2x\sqrt[6]{3}$
16) $4xy^2\sqrt[3]{2}$
17) $-12t\sqrt[3]{5s}$

---

18) $12\sqrt{2}$
19) $4\sqrt{7}-\sqrt{3}$
20) 0
21) $-14\sqrt{2}+6\sqrt{3}$
22) $2\sqrt[3]{2}$
23) $-9\sqrt[4]{4}$

---

24) $28\sqrt{6}$
25) $7\sqrt{2}+2\sqrt{7}$
26) $47-31\sqrt{5}$
27) $18+8\sqrt{2}$
28) 3

29) $\dfrac{\sqrt{21}}{3}$

30) $\dfrac{4\sqrt{3}}{3}$

31) $\sqrt{3}-\sqrt{11}$

32) $\dfrac{-4\sqrt{3}-14\sqrt{6}}{47}$

33) $\dfrac{6x\sqrt{5x}}{35y}$

34) $\dfrac{\sqrt{6n}}{2}$

35) $\dfrac{-10-\sqrt{10}}{2}$

---

36) $\left(\sqrt{13r}\right)^3$

---

37) $(ps)^{\frac{1}{2}}$

---

38) $\dfrac{1}{9}$

39) 1331

40) 8

---

41) $x^{\frac{3}{2}}y^{\frac{7}{2}}$

42) 1

43) $\dfrac{a^{\frac{2}{5}}b^{\frac{7}{2}}}{5}$

44) $\dfrac{a^3}{2b^{\frac{7}{4}}}$

45) $\dfrac{1}{n^{\frac{1}{3}}}$

46) $\dfrac{1}{y^3}$

---

47) $\sqrt[12]{x^5}$

48) 14

49) $2, 4$

50) $-3, 3$

51) 3

52) $\frac{7}{2}$

53) -1

54) $-\frac{23}{5}$

---

55) $4i\sqrt{3}$

---

56) -15

57) $-3\sqrt{14}$

---

58) $2-11i$

59) $-5+2i$

60) $-8-i$

61) 28

62) $56+12i$

63) $23+2i$

64) $-11i$

65) $3-\frac{3}{2}i$

66) $\frac{3}{34}-\frac{6}{17}i$

67) $\frac{70}{149}+\frac{100}{149}i$

68) $-\frac{3}{10}-\frac{29}{10}i$

CHAPTER 4

Quadratic Equations and Graphs

Objectives
Chapter 4

- Solve quadratic equations by applying the square root property.
- Solve quadratic equations by completing the square**.**
- Solve quadratic equations by using the quadratic formula.
- Graph parabolas using the vertex, x-intercepts, and y-intercept.
- Solve quadratic application problems.

Section 4.1: Square Root Property

Objective: Solve quadratic equations by applying the square root property.

In an earlier chapter, we learned how to solve equations by factoring. The next example reviews how we solved a quadratic equation $ax^2 + bx + c = 0$ by factoring.

Example 1. Solve the equation.

$$x^2 + 5x + 6 = 0 \quad \text{Factor using } \textit{ac method}$$
$$(x+3)(x+2) = 0 \quad \text{Set each factor equal to zero}$$
$$x+3=0 \quad \text{or} \quad x+2=0$$
$$\underline{-3=-3} \qquad \underline{-2=-2}$$
$$x=-3 \quad \text{or} \quad x=-2 \quad \text{Our Solutions}$$

However, not every quadratic equation can be solved by factoring. For example, consider the equation $x^2 - 2x - 7 = 0$. The trinomial on the left side, $x^2 - 2x - 7$, cannot be factored; however, we will see in a later section that the equation $x^2 - 2x - 7 = 0$ has two solutions: $1 + 2\sqrt{2}$ and $1 - 2\sqrt{2}$.

SQUARE ROOT PROPERTY

In this chapter, we will learn additional methods besides factoring for solving quadratic equations. We will start with a method that makes use of the following property:

> **SQUARE ROOT PROPERTY:**
>
> If k is a real number and $x^2 = k$,
> then $x = \sqrt{k}$ or $x = -\sqrt{k}$
>
> Often this property is written using shorthand notation:
>
> If $x^2 = k$, then $x = \pm\sqrt{k}$.

To solve a quadratic equation by applying the square root property, we will first need to isolate the squared expression on one side of the equation and the constant term on the other side.

SOLVING QUADRATIC EQUATIONS BY APPLYING THE SQUARE ROOT PROPERTY

Example 2. Solve the equation.

$x^2 = 16$	The squared term is already isolated; Apply the square root property $(\pm)$
$\sqrt{x^2} = \pm\sqrt{16}$	Simplify radicals
$x = \pm 4$	Our Solutions

Example 3. Solve the equation.

$x^2 - 7 = 0$	Isolate the squared term
$x^2 = 7$	Apply the square root property $(\pm)$
$\sqrt{x^2} = \pm\sqrt{7}$	Simplify radicals
$x = \pm\sqrt{7}$	Our Solutions

Example 4. Solve the equation.

$2x^2 + 36 = 0$	Isolate the squared term:
$\dfrac{2x^2}{2} = \dfrac{-36}{2}$	Subtract 36 from both sides Divide by 2
$x^2 = -18$	Apply the square root property $(\pm)$
$\sqrt{x^2} = \pm\sqrt{-18}$	Simplify radicals: $\pm\sqrt{-18} = \pm\sqrt{9}\cdot\sqrt{-1}\cdot\sqrt{2} = \pm 3i\sqrt{2}$
$x = \pm 3i\sqrt{2}$	Our Solutions

Example 5. Solve the equation.

		$(2x+4)^2 = 36$	A squared expression is already isolated on the left side; Apply the square root property $(\pm)$
		$\sqrt{(2x+4)^2} = \pm\sqrt{36}$	Simplify radicals
		$2x + 4 = \pm 6$	To avoid sign errors, separate into two equations
$2x + 4 = 6$	or	$2x + 4 = -6$	with one equation for $+$, one equation for $-$
$-4 = -4$		$-4 = -4$	Subtract 4 from both sides
$\dfrac{2x}{2} = \dfrac{2}{2}$	or	$\dfrac{2x}{2} = \dfrac{-10}{2}$	Divide both sides by 2
$x = 1$	or	$x = -5$	Our Solutions

In the previous example we used two separate equations to simplify, because when we took the root, our solutions were two rational numbers, 6 and -6. If the roots do not simplify to rational numbers, we may keep the $\pm$ in the equation.

Example 6. Solve the equation.

$(6x-9)^2 = 45$ A squared expression is already isolated on the left side; Apply the square root property $(\pm)$

$\sqrt{(6x-9)^2} = \pm\sqrt{45}$ Simplify radicals: $\pm\sqrt{45} = \pm\sqrt{9}\cdot\sqrt{5} = \pm\, 3\sqrt{5}$

$6x-9 = \pm 3\sqrt{5}$ Use one equation because radical did not simplify to a rational number

$\underline{\;+9 = +9\;}$

$\dfrac{6x}{6} = \dfrac{9\pm 3\sqrt{5}}{6}$ Divide both sides by 6

$x = \dfrac{9\pm 3\sqrt{5}}{6}$ Factor numerator and denominator

$x = \dfrac{3\left(3\pm\sqrt{5}\right)}{3\cdot 2}$ Divide out common factor of 3

$x = \dfrac{3\pm\sqrt{5}}{2}$ Our Solutions

Example 7. Solve the equation.

$2(3x-1)^2 + 7 = 23$ Isolate the squared expression on the left side; Subtract 7 from both sides

$\underline{\;-7 \quad = -7\;}$

$2\,(3x-1)^2 = 16$ Divide both sides by 2

$\dfrac{2\,(3x-1)^2}{2} = \dfrac{16}{2}$

$(3x-1)^2 = 8$ Apply the square root property $(\pm)$

$\sqrt{(3x-1)^2} = \pm\sqrt{8}$ Simplify radicals: $\pm\sqrt{8} = \pm\sqrt{4}\cdot\sqrt{2} = \pm\, 2\sqrt{2}$

$3x-1 = \pm 2\sqrt{2}$ Use one equation because radical did not simplify to a rational

$3x-1 = \pm 2\sqrt{2}$ Add 1 to both sides

$\underline{\;+1 = +1\;}$ Divide both sides by 3

$\dfrac{3x}{3} = \dfrac{1\pm 2\sqrt{2}}{3}$

$x = \dfrac{1\pm 2\sqrt{2}}{3}$ Our Solutions

Example 8. Solve the equation.

$(x+3)^2+9=7$ $-9 \quad =-9$	Isolate the squared expression on the left side; Subtract 9 from both sides
$(x+3)^2=-2$	Apply the square root property ($\pm$)
$\sqrt{(x+3)^2}=\pm\sqrt{-2}$	Simplify radicals: $\pm\sqrt{-2}=\pm\sqrt{-1}\cdot\sqrt{2}=\pm\, i\sqrt{2}$
$x+3=\pm i\sqrt{2}$	Use one equation because radical did not simplify to a rational
$x+3=\pm i\sqrt{2}$ $-3=-3$	Subtract 3 from both sides
$x=-3\pm i\sqrt{2}$	Our Solutions

Example 9. Solve the equation.

$\left(x+\frac{1}{3}\right)^2=\frac{2}{9}$	Apply the square root property ($\pm$)
$\sqrt{\left(x+\frac{1}{3}\right)^2}=\pm\sqrt{\frac{2}{9}}$	Simplify radicals: $\pm\sqrt{\frac{2}{9}}=\pm\frac{\sqrt{2}}{\sqrt{9}}=\pm\frac{\sqrt{2}}{3}$
$x+\frac{1}{3}=\pm\frac{\sqrt{2}}{3}$	Use one equation because radical did not simplify to a rational
$x+\frac{1}{3}=\pm\frac{\sqrt{2}}{3}$	Subtract $\frac{1}{3}$ from both sides
$x=-\frac{1}{3}\pm\frac{\sqrt{2}}{3}$	Add fractions
$x=\frac{-1\pm\sqrt{2}}{3}$	Our Solutions

Practice Exercises
Section 4.1: Square Root Property

Solve each equation using the square root property.

1) $x^2 = 64$

2) $x^2 = 75$

3) $x^2 + 5 = 13$

4) $x^2 - 7 = 20$

5) $x^2 + 50 = 0$

6) $5x^2 - 7 = 18$

7) $(x-4)^2 = 9$

8) $(2x+1)^2 = 25$

9) $(x+1)^2 = 3$

10) $(x-3)^2 = 12$

11) $(x+2)^2 = -9$

12) $(2x+1)^2 + 3 = 21$

13) $(9x-3)^2 = 72$

14) $(2x-8)^2 - 5 = 15$

15) $-2(x-6)^2 - 13 = 7$

16) $-3(4x-5)^2 + 8 = -19$

17) $\left(x - \frac{5}{2}\right)^2 = \frac{81}{4}$

18) $\left(x + \frac{3}{4}\right)^2 = \frac{10}{16}$

ANSWERS to Practice Exercises
Section 4.1: Square Root Property

1) ± 8
2) $\pm 5\sqrt{3}$
3) $\pm 2\sqrt{2}$
4) $\pm 3\sqrt{3}$
5) $\pm 5i\sqrt{2}$
6) $\pm\sqrt{5}$
7) $1, 7$
8) $-3, 2$
9) $-1 \pm \sqrt{3}$
10) $3 \pm 2\sqrt{3}$
11) $-2 \pm 3i$
12) $\dfrac{-1 \pm 3\sqrt{2}}{2}$
13) $\dfrac{1 \pm 2\sqrt{2}}{3}$
14) $4 \pm \sqrt{5}$
15) $6 \pm i\sqrt{10}$
16) $\dfrac{1}{2}, 2$
17) $7, -2$
18) $\dfrac{-3 \pm \sqrt{10}}{4}$

Section 4.2: Completing the Square

Objective: Solve quadratic equations by completing the square.

In this section, we continue to address the question of how to solve any quadratic equation $ax^2 + bx + c = 0$. Now, we will learn a method known as **completing the square**. When completing the square, we will change the quadratic into a perfect square that can then be solved by applying the square root property. The next example reviews the square root property.

Example 1. Solve the equation.

$(x+5)^2 = 18$	Use square root property
$\sqrt{(x+5)^2} = \pm\sqrt{18}$	Simplify radicals
$x+5 = \pm 3\sqrt{2}$	Subtract 5 from both sides
$-5 = -5$	
$x = -5 \pm 3\sqrt{2}$	Our Solutions

COMPLETING THE SQUARE

To complete the square, or make our problem into the form of the previous example, we will be searching for the third term of a trinomial. If a quadratic is of the form $x^2 + bx + c$, and a perfect square, the third term, c, can be easily found by the formula $(\frac{1}{2} \cdot b)^2$. This is shown in the following examples, where we find the number that completes the square, and then factor that perfect square trinomial.

Example 2. Find the value of c that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 + 8x + c$	$c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = 8$
	The third term that completes the square is 16: $\left(\frac{1}{2} \cdot 8\right)^2 = (4)^2 = 16$
$x^2 + 8x + 16$	Our expression is a perfect square; factor
$= (x+4)(x+4)$	
$= (x+4)^2$	Our Answer

Example 3. Find the value of c that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 - 7x + c$ $\qquad$ $c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = -7$

The third term that completes the square is $\frac{49}{4}$:

$$\left(\frac{1}{2} \cdot -7\right)^2 = \left(-\frac{7}{2}\right)^2 = \frac{49}{4}$$

$x^2 - 7x + \frac{49}{4}$ $\qquad$ Our expression is a perfect square; factor

$= \left(x - \frac{7}{2}\right)\left(x - \frac{7}{2}\right)$

$= \left(x - \frac{7}{2}\right)^2$ $\qquad$ Our Answer

Example 4. Find the value of c that makes this expression a perfect square trinomial. Then, factor that perfect square trinomial.

$x^2 + \frac{5}{3}x + c$ $\qquad$ $c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = \frac{5}{3}$

The third term that completes the square is $\frac{25}{36}$:

$$\left(\frac{1}{2} \cdot \frac{5}{3}\right)^2 = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$x^2 + \frac{5}{3}x + \frac{25}{36}$ $\qquad$ Our expression is a perfect square; factor

$= \left(x + \frac{5}{6}\right)^2$ $\qquad$ Our Answer

SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

The process in the previous examples, combined with the square root property, is used to solve quadratic equations by completing the square. The following six steps describe the process used to solve a quadratic equation by completing the square, along with a practice example to demonstrate each step.

Steps for Solving by Completing the Square	**Example**
	$3x^2+18x-6=0$
1. Separate the constant term from the variable terms.	$+6=+6$ $3x^2+18x \quad =6$
2. If $a\neq 1$, then divide each term by a.	$\frac{3}{3}x^2+\frac{18}{3}x \quad =\frac{6}{3}$ $x^2+6x \quad =2$
3. Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$.	$\left(\frac{1}{2}\cdot 6\right)^2=3^2=9$
4. Add the resulting value to both sides of the equation.	$x^2+6x \quad =2$ $+9=+9$ $x^2+6x+9=11$
5. Factor the perfect square trinomial.	$(x+3)^2=11$
6. Solve by applying the square root property.	$\sqrt{(x+3)^2}=\pm\sqrt{11}$ $x+3=\pm\sqrt{11}$ $-3 \;=-3$ $x=-3\pm\sqrt{11}$

The advantage of this method is that it can be used to solve **any** quadratic equation. The following examples show how completing the square can give us rational solutions, irrational solutions, and even complex solutions.

Example 5. Solve the equation by completing the square.

$2x^2+20x+48=0$ Separate the constant term from variable terms

$\underline{-48=-48}$ Subtract 48 from both sides of the equation

$\frac{2x^2}{2}+\frac{20x}{2} \quad =\frac{-48}{2}$ Divide each term by 2

$x^2+10x \quad =-24$ Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$

Our $b=10$; $\left(\frac{1}{2}\cdot 10\right)^2=(5)^2=25$

$x^2+10x \quad =-24$ Add 25 to both sides of the equation

$\underline{+25=+25}$

$x^2+10x+25=1$ Factor the perfect square trinomial

$(x+5)^2=1$ Solve by applying the square root property

$\sqrt{(x+5)^2} = \pm\sqrt{1}$	Simplify radicals
$x+5=\pm 1$	One equation for $+$, one equation for $-$
$x+5=1$ or $x+5=-1$	Subtract 5 from both sides
$\underline{\quad -5=-5} \qquad \underline{\quad -5=-5}$	
$x=-4$ or $x=-6$	Our Solutions

Example 6. Solve the equation by completing the square.

$x^2-3x-2=0$	Separate the constant term from variable terms
$\underline{\quad +2 = +2}$	Add 2 to both sides
$x^2-3x \quad = 2$	No need to divide since $a=1$
	Find the value that completes the square: $\left(\frac{1}{2}\cdot b\right)^2$
	Our $b=-3$; $\left(\frac{1}{2}\cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$
	Add $\frac{9}{4}$ to both sides of the equation
$x^2-3x+\frac{9}{4}=2+\frac{9}{4}$	Need common denominator (4) on right side of equation
	$\frac{2}{1}\left(\frac{4}{4}\right)+\frac{9}{4}=\frac{8}{4}+\frac{9}{4}=\frac{17}{4}$
$x^2-3x+\frac{9}{4}=\frac{17}{4}$	Factor the perfect square trinomial
$\left(x-\frac{3}{2}\right)^2=\frac{17}{4}$	Solve by applying the square root property
$\sqrt{\left(x-\frac{3}{2}\right)^2}=\pm\sqrt{\frac{17}{4}}$	Simplify radicals
$x-\frac{3}{2}=\pm\frac{\sqrt{17}}{2}$	Add $\frac{3}{2}$ to both sides of the equation
$\underline{\quad +\frac{3}{2} \quad = +\frac{3}{2}}$	Notice that we already have a common denominator
$x=\frac{3\pm\sqrt{17}}{2}$	Our Solutions

As Example 6 has shown, when solving by completing the square, we will often need to use fractions and be comfortable finding common denominators and adding fractions together.

Sometimes when solving quadratic equations, the solutions are complex numbers, as is the case in Example 7.

Example 7. Solve the equation by completing the square.

$x^2 - 6x + 30 = 0$	Separate the constant term from variable terms
$\underline{\quad -30 = -30}$	Subtract 30 from both sides of the equation
$x^2 - 6x \quad = -30$	No need to divide since $a = 1$
	Find the value that completes the square: $\left(\frac{1}{2} \cdot b\right)^2$
	Our $b = -6$; $\left(\frac{1}{2} \cdot -6\right)^2 = (-3)^2 = 9$
$x^2 - 6x \quad = -30$ $\underline{\quad +9 = +9}$	Add 9 to both sides of the equation
$x^2 - 6x + 9 = -21$	Factor the perfect square trinomial
$(x-3)^2 = -21$	Solve by applying the square root property
$\sqrt{(x-3)^2} = \pm\sqrt{-21}$	Simplify radicals
$x - 3 = \pm i\sqrt{21}$ $\underline{\quad +3 = +3}$	Add 3 to both sides
$x = 3 \pm i\sqrt{21}$	Our Solutions

We can use completing the square to solve any quadratic equation so we want to get comfortable using the six steps of this method.

Practice Exercises
Section 4.2: Completing the Square

Find the value of c that makes each expression a perfect square trinomial; then, factor that perfect square trinomial.

1) $x^2 - 30x + c$
2) $a^2 + 24a + c$
3) $m^2 - 36m + c$
4) $x^2 + 34x + c$
5) $x^2 - 15x + c$
6) $r^2 + \frac{1}{9}r + c$
7) $y^2 + y + c$
8) $p^2 - 17p + c$

Solve each equation by completing the square.

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 9 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $x^2 + 5x = 7$
14) $3k^2 + 2k - 4 = 0$
15) $-4z^2 + z + 1 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 12 = 0$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$
27) $3k^2 + 9 = 6k$
28) $5n^2 = -10n + 15$
29) $p^2 - 8p = -55$
30) $x^2 + 8x + 15 = 8$
31) $7n^2 - n + 7 = 7n + 6n^2$
32) $n^2 + 4n = 12$
33) $8n^2 + 16n = 64$
34) $b^2 + 7b - 33 = 0$
35) $-5x^2 - 8x + 40 = -8$
36) $m^2 = -15 + 9m$
37) $4b^2 - 15b + 56 = 3b^2$
38) $10v^2 - 15v = 27 + 4v^2 - 6v$
39) $n^2 = -21 + 10n$
40) $a^2 - 56 = -10a$

ANSWERS to Practice Exercises
Section 4.2: Completing the Square

1) $225; (x-15)^2$
2) $144; (a+12)^2$
3) $324; (m-18)^2$
4) $289; (x+17)^2$
5) $\frac{225}{4}; \left(x-\frac{15}{2}\right)^2$
6) $\frac{1}{324}; \left(r+\frac{1}{18}\right)^2$
7) $\frac{1}{4}; \left(y+\frac{1}{2}\right)^2$
8) $\frac{289}{4}; \left(p-\frac{17}{2}\right)^2$

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9) $11, 5$
10) $9, -1$
11) $4+i\sqrt{29}, 4-i\sqrt{29}$
12) $-1+i\sqrt{42}, -1-i\sqrt{42}$
13) $\frac{-5+\sqrt{53}}{2}, \frac{-5-\sqrt{53}}{2}$
14) $\frac{-1+\sqrt{13}}{3}, \frac{-1-\sqrt{13}}{3}$
15) $\frac{1+\sqrt{17}}{8}, \frac{1-\sqrt{17}}{8}$
16) $\frac{-4+3\sqrt{2}}{4}, \frac{-4-3\sqrt{2}}{4}$
17) $-5+\sqrt{86}, -5-\sqrt{86}$
18) $8+2\sqrt{29}, 8-2\sqrt{29}$
19) $9, 7$
20) $4+2\sqrt{7}, 4-2\sqrt{7}$
21) $-1+i\sqrt{21}, -1-i\sqrt{21}$
22) $1, -3$
23) $\frac{3}{2}, -\frac{7}{2}$
24) $3, -1$
25) $-5+2i, -5-2i$
26) $7+\sqrt{85}, 7-\sqrt{85}$
27) $1+i\sqrt{2}, 1-i\sqrt{2}$
28) $1, -3$
29) $4+i\sqrt{39}, 4-i\sqrt{39}$
30) $-1, -7$
31) $7, 1$
32) $2, -6$
33) $2, -4$
34) $\frac{-7+\sqrt{181}}{2}, \frac{-7-\sqrt{181}}{2}$
35) $\frac{12}{5}, -4$
36) $\frac{9+\sqrt{21}}{2}, \frac{9-\sqrt{21}}{2}$
37) $8, 7$
38) $3, -\frac{3}{2}$
39) $7, 3$
40) $4, -14$

Section 4.3: Quadratic Formula

Objective: Solve quadratic equations using the quadratic formula.

In this section, we will develop a formula to solve any quadratic equation $ax^2+bx+c=0$, where a, b, and c are real numbers and $a \neq 0$. Solve for this general equation for x by completing the square:

$$ax^2+bx+c=0$$ Separate the constant term from variable terms

$$-c=-c$$ Subtract c from both sides

$$\frac{ax^2}{a}+\frac{bx}{a} = \frac{-c}{a}$$ Divide each term by a

$$x^2+\frac{b}{a}x = \frac{-c}{a}$$ Find the value that completes the square:

$$\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2=\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$$

Add that value to both sides of the equation

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$ Subtract fractions on the right side of the equation using a common denominator of $4a^2$:

$$\frac{b^2}{4a^2}-\frac{c}{a}\left(\frac{4a}{4a}\right)=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$ Factor the perfect square trinomial

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$ Solve by applying the square root property

$$\sqrt{\left(x+\frac{b}{2}\right)^2}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$ Simplify radicals

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$ Subtract $\frac{b}{2a}$ from both sides

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ Our Solutions

QUADRATIC FORMULA

This result is a very important one to us because we can use this formula to solve any quadratic equation. Once we identify the values of *a; b;* and *c* in the quadratic equation, we can substitute those values into $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and we get our solutions. This formula is known as the **quadratic formula**.

QUADRATIC FORMULA:

The solutions to the quadratic equation $ax^2 + bx + c = 0$ for $a \neq 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

SOLVING QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

We can use the quadratic formula to solve any quadratic equation. This method is demonstrated in the following examples.

Example 1. Solve the equation using the quadratic formula.

$x^2 + 3x + 2 = 0$ — $a = 1$, $b = 3$, $c = 2$; use quadratic formula

$x = \dfrac{-(3) \pm \sqrt{(3)^2 - 4(1)(2)}}{2(1)}$ — Evaluate the exponent and multiply

$x = \dfrac{-3 \pm \sqrt{9 - 8}}{2}$ — Evaluate the subtraction under radical sign

$x = \dfrac{-3 \pm \sqrt{1}}{2}$ — Evaluate the root

$x = \dfrac{-3 \pm 1}{2}$ — Evaluate $+$ and $-$ to get the two answers

$x = \dfrac{-3+1}{2}$ or $x = \dfrac{-3-1}{2}$

$x = \dfrac{-2}{2}$ or $x = \dfrac{-4}{2}$ — Simplify the fractions, if possible

$x = -1$ or -2 — Our Solutions

As we are solving a quadratic equation using the quadratic formula, it is important to remember that the equation must first be set equal to zero.

Example 2. Solve the equation using the quadratic formula.

$$\begin{array}{r} 25x^2 = 30x + 11 \\ -30x - 11 = -30x - 11 \\ \hline 25x^2 - 30x - 11 = 0 \end{array}$$

First set the equation equal to zero

Subtract $30x$ and 11 from both sides of the equation

$a = 25$, $b = -30$, and $c = -11$; use quadratic formula

$$x = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)}$$ Evaluate the exponent and multiply

$$x = \frac{30 \pm \sqrt{900 + 1100}}{50}$$ Evaluate the addition under radical sign

$$x = \frac{30 \pm \sqrt{2000}}{50}$$ Simplify the root

$$x = \frac{30 \pm 20\sqrt{5}}{50}$$ Factor numerator and denominator

$$x = \frac{10\left(3 \pm 2\sqrt{5}\right)}{10 \cdot 5}$$ Divide out common factor of 10

$$x = \frac{3 \pm 2\sqrt{5}}{5}$$ Our Solutions

Example 3. Solve the equation using the quadratic formula.

$$\begin{array}{r} 3x^2 + 4x + 8 = 2x^2 + 6x - 5 \\ -2x^2 - 6x + 5 = -2x^2 - 6x + 5 \\ \hline x^2 - 2x + 13 = 0 \end{array}$$

First set the equation equal to zero

Subtract $2x^2$ and $6x$, and add 5

$a = 1$, $b = -2$, and $c = 13$; use quadratic formula

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)}$$ Evaluate the exponent and multiply

$$x = \frac{2 \pm \sqrt{4 - 52}}{2}$$ Evaluate the subtraction under radical sign

$x = \dfrac{2 \pm \sqrt{-48}}{2}$ Simplify the root

$x = \dfrac{2 \pm 4i\sqrt{3}}{2}$ Factor numerator

$x = \dfrac{2\left(1 \pm 2i\sqrt{3}\right)}{2 \cdot 1}$ Divide out common factor of 2

$x = 1 \pm 2i\sqrt{3}$ Our Solutions

Notice this equation has two imaginary solutions and they are complex conjugates.

When we solve quadratic equations, we don't necessarily get two unique solutions. We can end up with only one real number solution if the square root simplifies to zero.

Example 4. Solve the equation using the quadratic formula.

$4x^2 - 12x + 9 = 0$ $a = 4$, $b = -12$, $c = 9$; use quadratic formula

$x = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$ Evaluate the exponent and multiply

$x = \dfrac{12 \pm \sqrt{144 - 144}}{8}$ Evaluate the subtraction under radical sign

$x = \dfrac{12 \pm \sqrt{0}}{8}$ Simplify the root

$x = \dfrac{12 \pm 0}{8}$ Evaluate $+$ and $-$ to get the two answers;

They are identical values; so, only one instance needs to be considered

$x = \dfrac{12}{8}$ Reduce fraction

$x = \dfrac{3}{2}$ Our Solution

When solving a quadratic equation, if the term with x or the constant term is missing, we can still solve the equation using the quadratic formula. We simply use zero for the coefficient of the missing term. If the term with x is missing, we have $b = 0$ and if the constant term is missing, we have $c = 0$. Note if $a = 0$, the term with x^2 is missing, meaning the equation is a linear equation, not a quadratic equation.

Example 5. Solve the equation using the quadratic formula.

$3x^2 + 7 = 0$	$a = 3$, $b = 0$ (missing term), $c = 7$; use quadratic formula
$x = \dfrac{-(0) \pm \sqrt{(0)^2 - 4(3)(7)}}{2(3)}$	Evaluate the exponent and multiply
$x = \pm\dfrac{\sqrt{-84}}{6}$	Simplify the root
$x = \pm\dfrac{2i\sqrt{21}}{6}$	Reduce the fraction; divide $2i$ and 6 by 2
$x = \pm\dfrac{i\sqrt{21}}{3}$	Our Solutions

SELECTING A METHOD FOR SOLVING A QUADRATIC EQUATION

We have covered four different methods that can be used to solve a quadratic equation: factoring, applying the square root property, completing the square, and using the quadratic formula. It is important to be familiar with all four methods as each has its own advantages when solving quadratic equations.

Some of the examples in this section could have been solved using a method other than the quadratic formula. In Example 1, we used the quadratic formula to solve the equation $x^2 + 3x + 2 = 0$. We could have chosen to solve this equation factoring instead:

$$x^2 + 3x + 2 = 0$$
$$(x+1)(x+2) = 0$$
$$x + 1 = 0 \qquad x + 2 = 0$$
$$x = -1 \qquad x = -2$$

In Example 5, we could have chosen to solve $3x^2 + 7 = 0$ by applying the square root property since there is no x term and we can isolate the squared term.

The following table walks you through a suggested process and an example of each method to decide which would be best to use when solving a quadratic equation.

1. If ax^2+bx+c can be factored easily, solve by **factoring**:	$x^2-5x+6=0$ $(x-2)(x-3)=0$ $x=2$ or $x=3$
2. If the equation can be written with a squared term or expression on one side and a constant term on the other, solve by applying the **square root property**:	$x^2-7=0$ $x^2=7$ $x=\pm\sqrt{7}$
3. If $a=1$ and b is even, solve by **completing the square**:	$x^2+2x=4$ $\left(\frac{1}{2}\cdot 2\right)^2=1^2=1$ $x^2+2x+1=4+1$ $(x+1)^2=5$ $x+1=\pm\sqrt{5}$ $x=-1\pm\sqrt{5}$
4. Otherwise, solve by the **quadratic formula**:	$x^2-3x+4=0$ $x=\dfrac{3\pm\sqrt{(-3)^2-4(1)(4)}}{2(1)}$ $x=\dfrac{3\pm i\sqrt{7}}{2}$

The above table offers a suggestion for deciding how to solve a quadratic equation. Remember that the methods of completing the square and the quadratic formula will always work to solve any quadratic equation. Solving a quadratic equation by factoring only works if the expression can be factored.

Practice Exercises
Section 4.3: Quadratic Formula

Solve each equation using the quadratic formula.

1) $x^2 - 4x + 3 = 0$
2) $m^2 + 4m - 48 = -3$
3) $4x^2 - 7 = -5x$
4) $3k^2 = -3k + 11$
5) $2x^2 + 4x = -4$
6) $5p^2 + 2p + 6 = 0$
7) $3r^2 - 2r - 1 = 0$
8) $2x^2 - 2x - 15 = 0$
9) $4n^2 - 36 = 0$
10) $3b^2 + 6 = 0$
11) $2x^2 + 3x + 8 = 0$
12) $3x = 6x^2 + 7$
13) $2x^2 = 8x + 2$
14) $-16t^2 + 32t + 48 = 0$
15) $7x^2 + 3x = 14$
16) $6n^2 - 1 = 0$
17) $2p^2 + 6p - 16 = 4$
18) $9m^2 - 16 = 0$
19) $3n^2 + 3n = -3$
20) $3t^2 - 3 = 8t$
21) $2x^2 = -7x + 49$
22) $-3r^2 + 4 = -6r$
23) $5x^2 = 7x + 7$
24) $6a^2 = -5a + 13$
25) $8n^2 = -3n - 8$
26) $6v^2 = 4 + 6v$
27) $2x^2 + 5x = -3$
28) $x^2 = 8$
29) $4a^2 - 64 = 0$
30) $2k^2 + 6k - 16 = 2k$
31) $4p^2 + 5p - 36 = 3p^2$
32) $12x^2 + x + 7 = 5x^2 + 5x$
33) $-5n^2 - 3n - 52 = 2 - 7n^2$
34) $7m^2 - 6m + 6 = -m$
35) $7r^2 - 12 = -3r$
36) $3x^2 - 3 = x^2$
37) $2n^2 - 9 = 4$
38) $6t^2 = t^2 + 7 - t$

ANSWERS to Practice Exercises
Section 4.3: Quadratic Formula

1) $1, 3$
2) $5, -9$
3) $\frac{-5+\sqrt{137}}{8}, \frac{-5-\sqrt{137}}{8}$
4) $\frac{-3+\sqrt{141}}{6}, \frac{-3-\sqrt{141}}{6}$
5) $-1+i, -1-i$
6) $\frac{-1+i\sqrt{29}}{5}, \frac{-1-i\sqrt{29}}{5}$
7) $1, -\frac{1}{3}$
8) $\frac{1+\sqrt{31}}{2}, \frac{1-\sqrt{31}}{2}$
9) $3, -3$
10) $i\sqrt{2}, -i\sqrt{2}$
11) $\frac{-3+i\sqrt{55}}{4}, \frac{-3-i\sqrt{55}}{4}$
12) $\frac{3+i\sqrt{159}}{12}, \frac{3-i\sqrt{159}}{12}$
13) $2+\sqrt{5}, 2-\sqrt{5}$
14) $3, -1$
15) $\frac{-3+\sqrt{401}}{14}, \frac{-3-\sqrt{401}}{14}$
16) $\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}$
17) $2, -5$
18) $\frac{4}{3}, -\frac{4}{3}$
19) $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$
20) $3, -\frac{1}{3}$
21) $\frac{7}{2}, -7$
22) $\frac{3+\sqrt{21}}{3}, \frac{3-\sqrt{21}}{3}$
23) $\frac{7+3\sqrt{21}}{10}, \frac{7-3\sqrt{21}}{10}$
24) $\frac{-5+\sqrt{337}}{12}, \frac{-5-\sqrt{337}}{12}$
25) $\frac{-3+i\sqrt{247}}{16}, \frac{-3-i\sqrt{247}}{16}$
26) $\frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}$
27) $-1, -\frac{3}{2}$
28) $2\sqrt{2}, -2\sqrt{2}$
29) $4, -4$
30) $2, -4$
31) $4, -9$
32) $\frac{2+3i\sqrt{5}}{7}, \frac{2-3i\sqrt{5}}{7}$
33) $6, -\frac{9}{2}$
34) $\frac{5+i\sqrt{143}}{14}, \frac{5-i\sqrt{143}}{14}$
35) $\frac{-3+\sqrt{345}}{14}, \frac{-3-\sqrt{345}}{14}$
36) $\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}$
37) $\frac{\sqrt{26}}{2}, -\frac{\sqrt{26}}{2}$
38) $\frac{-1+\sqrt{141}}{10}, \frac{-1-\sqrt{141}}{10}$

Section 4.4: Parabolas

Objective: Graph parabolas using the vertex, *x*-intercepts, and *y*-intercept.

Just as the graph of a linear equation $y = mx + b$ can be drawn, the graph of a quadratic equation $y = ax^2 + bx + c$ can be drawn. The graph is simply a picture showing what pairs of values x and y can be used to make the equation true. For a linear equation, the graph is a line but for a quadratic equation, the graph is a U shaped curve called a **parabola**.

GRAPHING A PARABOLA BY CREATING A TABLE OF VALUES

One way to draw the graph of a quadratic equation is to make a table of values and evaluate the equation for each *x*-value we choose. The completed table gives us a set of points to graph. Remember that points are ordered pairs in the form of (x, y); so, each *x*-value and its corresponding *y*-value are a point to be graphed.

Example 1. Graph the parabola $y = x^2 - 4x + 3$.

Make a table of values. We will test five *x*-values to get an idea of the shape of the graph:

$$y = x^2 - 4x + 3$$

x	y
0	
1	
2	
3	
4	

Plug 0 in for x and evaluate: $y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 = 3$

Plug 1 in for x and evaluate: $y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 = 0$

Plug 2 in for x and evaluate: $y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$

Plug 3 in for x and evaluate: $y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 = 0$

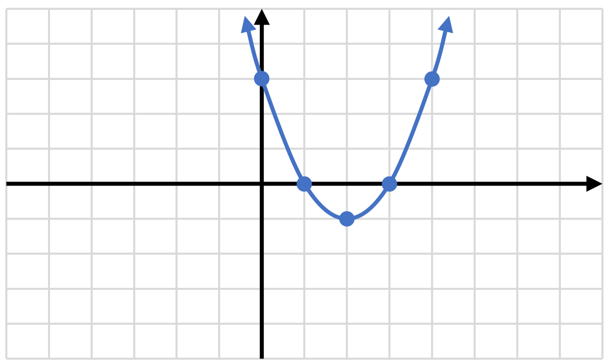
Plug 4 in for x and evaluate: $y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 = 3$

The completed table is shown below:

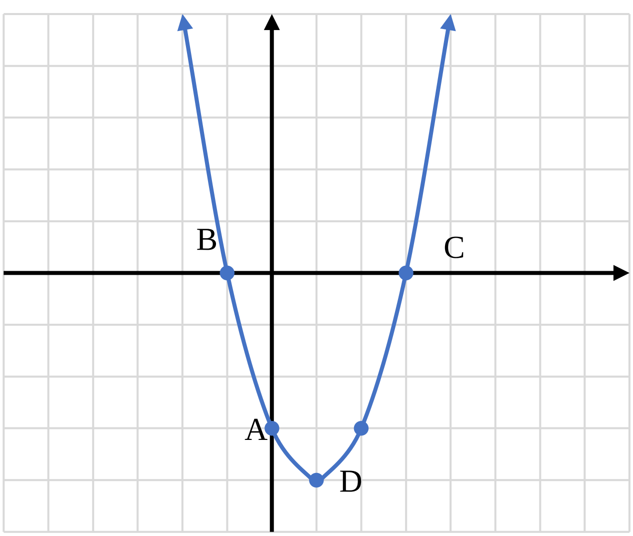
$$y = x^2 - 4x + 3$$

x	y
0	3
1	0
2	-1
3	0
4	3

Graph by plotting the points $(0,3)$, $(1,0)$, $(2,-1)$, $(3,0)$ and $(4,3)$. Connect the points with a smooth, U shaped curve.



The above method to graph a parabola works for any quadratic equation; however, it can be very tedious to find all the points that would be necessary to get the correct bend and shape. For this reason, we identify several key points on a graph to help us graph parabolas more efficiently. These key points are described below.



y **-intercept** (Point A): where the graph crosses the vertical y-axis.

x **-intercepts** (Points B and C): where the graph crosses the horizontal x-axis.

Vertex (Point D): the turning point where the graph changes directions.

GRAPHING A PARABOLA USING THE VERTEX, *X*-INTERCEPTS, AND *Y*-INTERCEPT

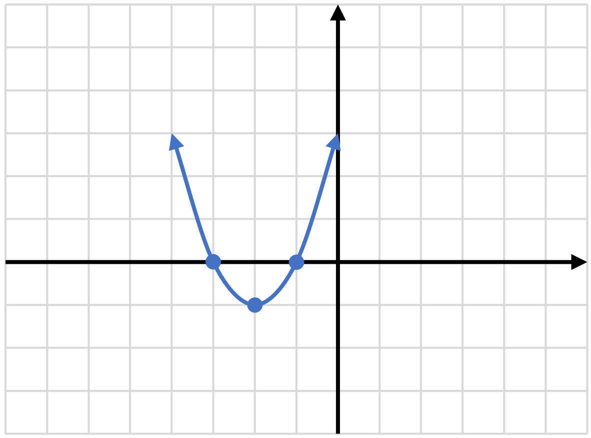
We will use the following method to find each of the points on our parabola.

To graph the parabola $y = ax^2 + bx + c$:

1. **Find the y-intercept:** Find the y-intercept by evaluating when $x = 0$; this always simplifies to $y = c$.
2. **Find the x-intercepts:** Find any x-intercepts by setting $y = 0$ and solving the equation $0 = ax^2 + bx + c$.
   If the solutions are real numbers, there are two x-intercepts. It is also possible to have only one x-intercept or no x-intercepts (if the solutions are complex numbers).
3. **Find the vertex:** Let $x = \frac{-b}{2a}$ to find the x-coordinate of the vertex. Then plug this x-value into the equation to find its corresponding *y*-value, which is the *y*-coordinate of the vertex.
4. **Determine whether the parabola opens upward or downward:**
   If a is a *positive* number, then the vertex will be the *minimum* point of the parabola and the graph will open *upward* (U-shaped).
   If a is a *negative* number, then the vertex will be the *maximum* point of the parabola and the graph will open *downward* (upside down U-shaped).
5. **Plot the points and connect with a smooth U-shaped curve.**

Example 2. Graph the parabola

$y = x^2 + 4x + 3$	Find the key points
$y = 3$	*y*-intercept is $y = c$, point $(0,3)$
$0 = x^2 + 4x + 3$	To find the x-intercepts, we solve the equation
$0 = (x+3)(x+1)$	Factor completely
$x + 3 = 0$ or $x + 1 = 0$	Set each factor equal to zero
$-3 = -3$ $-1 = -1$	Solve each equation
$x = -3$ or $x = -1$	Our x-intercepts, points $(-3,0)$ and $(-1,0)$
$x = \frac{-4}{2(1)} = \frac{-4}{2} = -2$	To find the vertex, first use $x = \frac{-b}{2a}$
$y = (-2)^2 + 4(-2) + 3$	Plug this value into the equation to find the y-coordinate
$y = 4 - 8 + 3$	Evaluate
$y = -1$	y-value of vertex
$(-2,-1)$	Vertex as a point



Graph points $(0,3)$, $(-3,0)$, and $(-1,0)$, as well as the vertex at $(-2,-1)$.

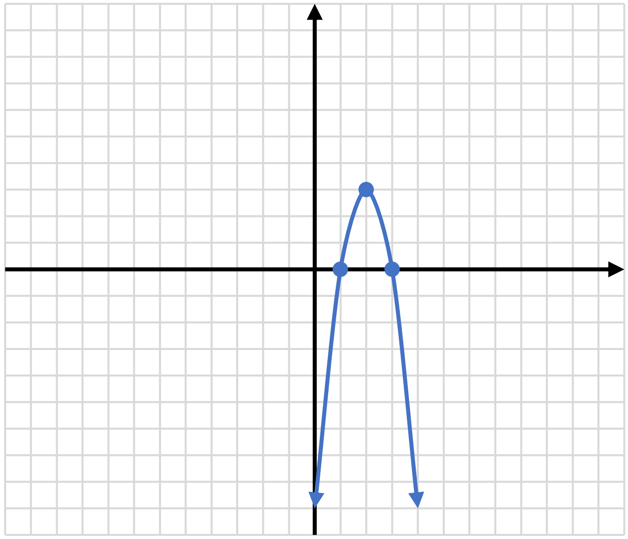
Connect the dots with a smooth curve in a U shape to get our parabola.

Our Graph

If the leading coefficient a in $y = ax^2 + bx + c$ is *negative*, the parabola will end up having an upside-down U shape. The process to graph it is identical, we just need to be very careful of how our signs operate. Remember, if a is negative, then ax^2 will also be negative because we only square the x, not the a.

Example 3. Graph the parabola

$y = -3x^2 + 12x - 9$	Find the key points
$y = -9$	y-intercept is $y = c$, point $(0,-9)$
$0 = -3x^2 + 12x - 9$	To find the x-intercepts, this equation
$0 = -3(x^2 - 4x + 3)$	Factor out GCF first, then factor rest
$0 = -3(x-3)(x-1)$	Set each factor with a variable equal to zero
$x - 3 = 0$ or $x - 1 = 0$	Solve each equation
$+3 = +3$ $\quad$ $+1 = +1$	
$x = 3$ or $x = 1$	Our x-intercepts, points $(3,0)$ and $(1,0)$
$x = \dfrac{-12}{2(-3)} = \dfrac{-12}{-6} = 2$	To find the vertex, first use $x = \dfrac{-b}{2a}$
$y = -3(2)^2 + 12(2) - 9$	Plug this value into the equation to find the y-coordinate
$y = -3(4) + 24 - 9$	Evaluate
$y = -12 + 24 - 9$	
$y = 3$	y-value of vertex
$(2,3)$	Vertex as a point



Graph the points $(0,-9)$, $(3,0)$, and $(1,0)$, as well as the vertex at $(2,3)$.

Connect the dots with a smooth curve in an upside-down U shape to get our parabola.

Our Graph

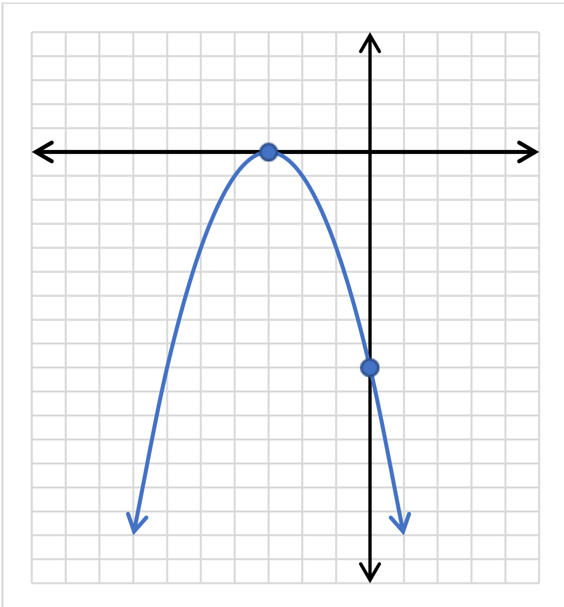
It is important to remember the graph of all quadratics is a parabola with the same U shape (either opening up or opening down). If you plot your points and they cannot be connected in the correct U shape, then at least one of your points must be wrong. Go back and check your work!

Whenever you have a perfect square trinomial quadratic equation, you will have only one unique *x*-intercept, and that *x*-intercept will also be the vertex of the parabola.

Example 4. Graph the parabola

$y=-x^2-6x-9$	Find the key points
$y=-9$	*y*-intercept is $y=c$, point $(0,-9)$
$0=-x^2-6x-9$	To find the *x*-intercept in this equation,
$0=-1(x^2+6x+9)$	Factor out GCF first, then factor the trinomial
$0=-1(x+3)(x+3)$	Set each factor with a variable equal to zero
$x+3=0$ or $x+3=0$	Solve each equation
$\underline{-3=-3}$ $\underline{-3=-3}$	
$x=-3$ or $x=-3$	Since they are the same value, the *x*-intercept is $(-3,0)$
$x=-\frac{-6}{2(-1)}=-\frac{-6}{-2}=-3$	To find the vertex, first use $x=\frac{-b}{2a}$
$y=-(-3)^2-6(-3)-9$	Plug this value into the equation to find the *y*-coordinate
$y=-(9)+18-9$	Evaluate
$y=-9+18-9$	
$y=0$	*y*-value of vertex
$(-3,0)$	Vertex as a point

Notice that the *x*-intercept and the vertex are the same point $(-3,0)$. This occurs whenever you have a perfect square trinomial as your quadratic equation. This is because whenever you factor a perfect square trinomial, both factors are identical. By setting each factor equal to zero there is only one unique solution.



Graph the *y*-intercept $(0,-9)$ and the vertex $(-3,0)$.

Connect the dots with a smooth curve in an upside-down U shape to get our parabola.

Our Graph

It is important to remember the graphs of all quadratics are parabolas with the same basic U shape. The differences come from the vertex being shifted to a different location, the curve opening up or down, and how quickly the curve opens.

Practice Exercises
Section 4.4: Parabolas

Find the vertex and intercepts. Use this information to graph each parabola.

1) $y = x^2 - 2x - 8$

2) $y = x^2 - 2x - 3$

3) $y = 2x^2 - 12x + 10$

4) $y = 2x^2 - 12x + 16$

5) $y = -2x^2 + 12x - 18$

6) $y = -2x^2 + 12x - 10$

7) $y = -3x^2 + 24x - 45$

8) $y = -3x^2 + 12x - 9$

9) $y = -x^2 + 4x + 5$

10) $y = -x^2 + 4x - 3$

11) $y = -x^2 + 6x - 5$

12) $y = -2x^2 + 16x - 30$

13) $y = -2x^2 + 16x - 24$

14) $y = 2x^2 + 4x - 6$

15) $y = 3x^2 + 12x + 9$

16) $y = 5x^2 + 30x + 45$

17) $y = 5x^2 - 40x + 75$

18) $y = 5x^2 + 20x + 15$

19) $y = -5x^2 - 60x - 175$

20) $y = -5x^2 + 20x - 15$

21) $y = 3x^2 - 6x + 1$

22) $y = 9x^2 - 18x + 4$

23) $y = -6x^2 - 18x - 11$

24) $y = x^2 - 4x + 5$

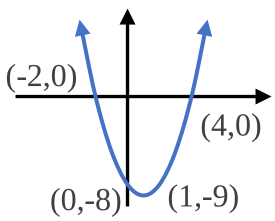
25) $y = -3x^2 + 6x - 5$

26) $y = x^2 + 6x + 10$

27) $y = x^2 + 8x + 16$

28) $y = -x^2 + 10x - 25$

ANSWERS to Practice Exercises
Section 4.4: Parabolas

1)


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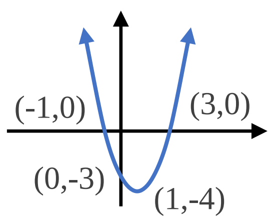

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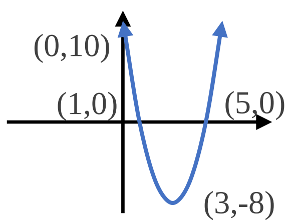

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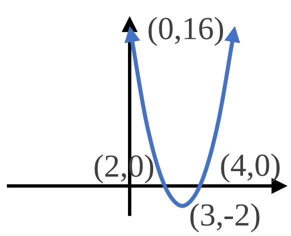

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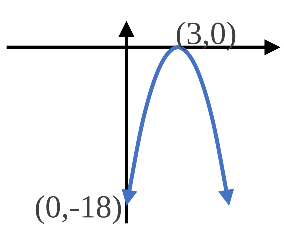

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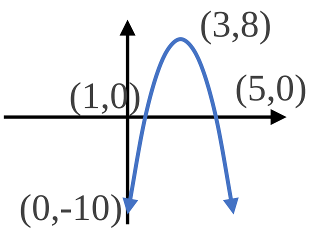

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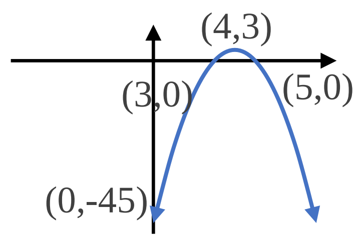

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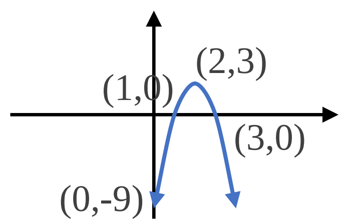

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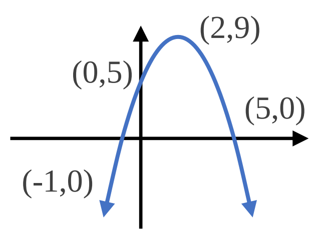

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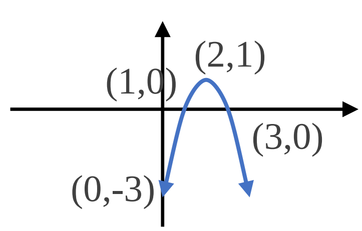

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 4.4 (continued)

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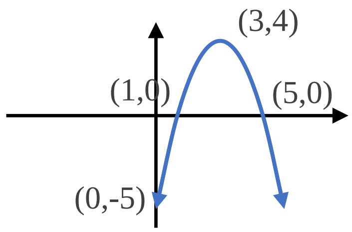

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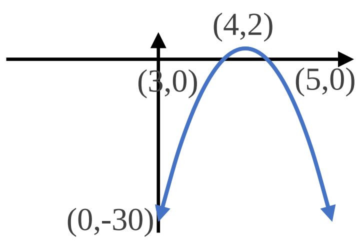

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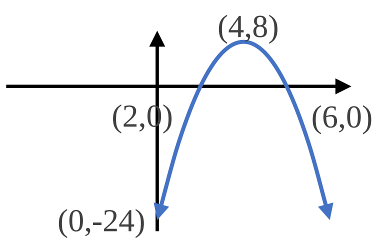

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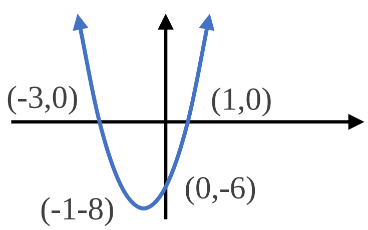

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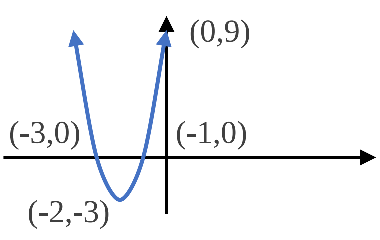

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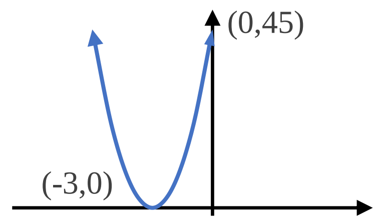

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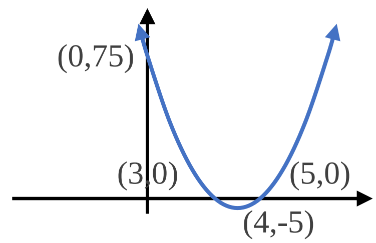

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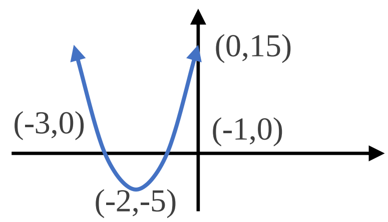

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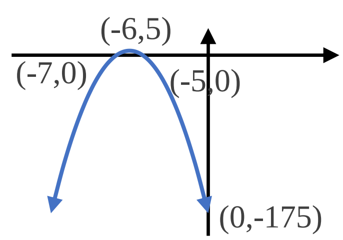

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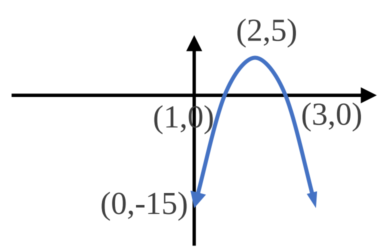

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 4.4 (continued)

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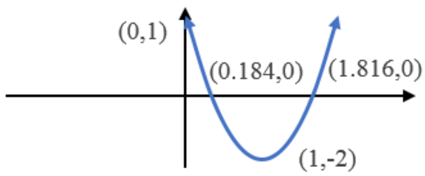

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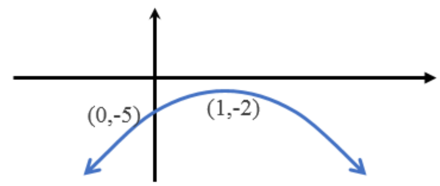

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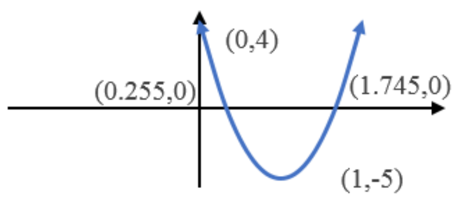

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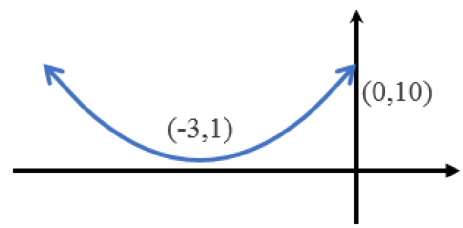

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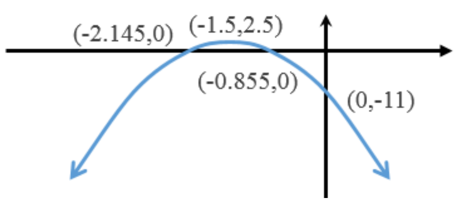

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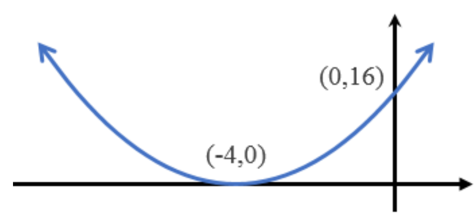

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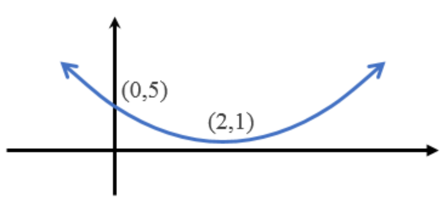

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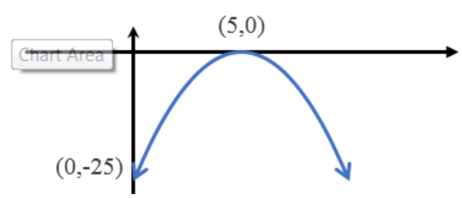


Section 4.5: Quadratic Applications

Objective: Solve quadratic application problems.

The vertex of the parabola formed by the graph of a quadratic equation is either a maximum point or a minimum point, depending on the sign of *a*. If *a* is a *positive* number, then the vertex is the *minimum*; if *a* is a *negative* number, then the vertex is a *maximum*.

An example of a maximum would be the highest height of a ball that has been thrown into the air. An example of a minimum would be the minimum average cost to a company for a product that has been produced.

Example 1. Answer each of the following questions.

Terry is on the balcony of her apartment, which is 150 feet above the ground. She tosses a ball vertically upward. The ball's height above the ground as it travels is modeled by the quadratic equation $h=-16t^2+64t+150$, where t is the amount of time (in seconds) the ball has been in flight and h is the height of the ball (in feet) at any particular time.

a. How many seconds will it take for the ball to reach its maximum height above the ground?

$h=-16t^2+64t+150$ The time t is unknown; $a=-16$, $b=64$, $c=150$

$t=\dfrac{-(64)}{2(-16)}=\dfrac{-64}{-32}=2$ Use $\dfrac{-b}{2a}$ to find the amount of time, t, that has passed when the ball reaches its maximum height

2 seconds The ball reaches its maximum height after 2 seconds

b. What is the ball's maximum height above the ground?

$h=-16t^2+64t+150$ The time that has passed, t, is 2 seconds

$h=-16(2)^2+64(2)+150$ Substitute the value 2 in for t everywhere in the equation

$h=-16(4)+64(2)+150$ Simplify

$h=-64+128+150=214$

214 feet So, the maximum height of the ball is 214 feet

c. How long does it take for the ball to hit the ground? Round to the nearest tenth of a second, if necessary.

$h=-16t^2+64t+150$	The time that has passed, t, is unknown; When the ball hits the ground, its height *h* is zero
$0=-16t^2+64t+150$	Set the quadratic equation equal to zero and solve the equation; $a=-16$, $b=64$, $c=150$
$t=\dfrac{-(64)\pm\sqrt{(64)^2-4(-16)(150)}}{2(-16)}$	Use the quadratic formula to determine the time $t=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
$t=\dfrac{-64\pm\sqrt{4096+9600}}{-32}$	Simplify
$t=\dfrac{-64\pm\sqrt{13696}}{-32}$	Use a calculator to evaluate each of solutions
$t\approx-1.7$ or $t\approx5.7$	Time cannot be negative; so $t\approx-1.7$ is extraneous
5.7 seconds	So, the time lapsed when the ball hits the ground is approximately 5.7 seconds

Example 2. Solve.

The price to charge for a product is a key business decision. If the price is low, then the business may sell many items but will not make much profit per sale. If the price is high, the business will make a large profit per sale but they will have fewer sales. Some value in the middle is "just right" and will maximize profit.

A company has determined that if they charge a price x, in dollars, then their profit P, in thousands of dollars, is given by the equation $P=-x^2+120x-2000$. To maximize profit, what price should the company choose? What is the maximum profit?

$P=-x^2+120x-2000$	The price, x, is unknown; $a=-1$, $b=120$, $c=-2000$
$x=\dfrac{-b}{2a}=\dfrac{-120}{2(-1)}=\dfrac{-120}{-2}=60$	Use $\frac{-b}{2a}$ to find the price x that will maximize the profit.
$x=60$ dollars	The company should choose \$60 as the price to maximize profit.

$P=-(60)^2+120(60)-2000$	Substitute the value 60 for x in the equation.
$P=-3600+7200-2000$	Simplify using the Order of Operations.
$P=1600$	The maximum profit is 1600 thousand dollars or \$1,600,000.

Example 3. Solve.

Arthur sells used cell phones. He has determined that his average cost to package and ship cell phones to customers is given by the equation $C=2x^2-60x+1700$, where x is the number of cell phones packaged and shipped every two weeks, and C is the average cost. How many cell phones must Arthur package and ship during the two-week period in order to minimize the average cost? What is the minimum average cost?

$C=2x^2-60x+1700$	The number of cell phones, x, is unknown; $a=2$, $b=-60$, $c=1700$
$x=-\frac{b}{2a}=-\frac{(-60)}{2(2)}=\frac{60}{4}=15$	Use $\frac{-b}{2a}$ to find the number of cell phones that will minimize the cost.
$x=15$	15 cell phones must be shipped to minimize the average cost.
$C=2(15)^2-60(15)+1700$	Substitute the value 15 for x in the equation.
$C=2(15)^2-60(15)+1700$ $C=2(225)-60(15)+1700$ $C=450-900+1700$	Simplify using the Order of Operations.
$C=1250$	The minimum average cost is 1250 dollars.

Practice Exercises
Section 4.5: Quadratic Applications

Use the following information to answer questions 1 and 2.

George is standing on the top of a 275 foot building. He throws a ball straight up into the air. The ball's initial velocity is given as 48 ft/sec. The height h, in feet, of the ball after t seconds is given by the equation $h=-16t^2+48t+275$.

1) How long will it take for the ball to reach its maximum height above the ground?
2) What is the maximum height that the ball reaches?

Use the following information to answer questions 3-5.

Shelly is standing on a platform 100 feet above the ground. She tosses a baseball straight up into the air. The equation $h=-16t^2+64t+100$ models the ball's height h, in feet, above the ground t seconds after it was thrown.

3) How long will it take for the ball to reach its maximum height above the ground?
4) What is the maximum height that the ball reaches?
5) How many seconds does it take for the ball to finally hit the ground (rounded to the nearest tenth of a second)?

Use the following information to answer questions 6-8.

Les is standing on the ground. He launches a model rocket straight up into the air. The equation $h=-16t^2+64t$ models the rocket's height *h*, in feet, above the ground t seconds after it was launched.

6) How long will it take for the rocket to reach its maximum height above the ground?
7) What is the maximum height that the rocket reaches?
8) How many seconds does it take for the ball to finally hit the ground (rounded to the nearest tenth of a second)?

Use the following information to answer questions 9-10.

The equation $P=-0.001x^2+2.45x-525$ models the profit P, in dollars, for x lasagne meals sold each week at Mama Anna's Restaurant

9) How many lasagne meals should the restaurant sell each week in order to maximize its profit?
10) What would be the maximum weekly profit if they sell the necessary number of meals (rounded to the nearest cent)?

Use the following information to answer questions 11-12.

The equation $P=0.001t^2-0.24t+59.90$ closely models common stock XYZ's closing price, P, in dollars, after t days of trading on the market for the calendar year of 2015.

11) After how many days was XYZ stock at its lowest value?
12) What was the stock's lowest price for 2015?

ANSWERS to Practice Exercises
Section 4.5: Quadratic Applications

1) 1.5 seconds
2) 311 feet
3) 2 seconds
4) 164 feet
5) 5.2 seconds
6) 2 seconds
7) 64 feet
8) 4 seconds
9) 1225 meals
10) $975.63
11) 120 days
12) $45.50

Review: Chapter 4

Solve the equation using the square root property.

1) $(8s+5)^2=49$

2) $7(5x-6)^2-3=172$

Solve the equation by completing the square.

3) $5x^2+10x-40=0$

4) $-3z^2-18z+5=-1$

5) $4p^2-3p-9=0$

6) $m^2+4m-39=-19$

Solve the equation by using the quadratic formula.

7) $4m^2-3m-5=0$

8) $-9x^2-3x+2=0$

9) $7x^2-4x=-7$

10) $5x^2=-22x-8$

11) $3w^2-w+14=8$

12) $3x^2-15=0$

Determine the *x*-intercept(s), *y*-intercept, and vertex of the graph of each quadratic equation.

13) $y=-2x^2+24x-40$

14) $y=2x^2-20x+32$

15) $y=x^2-12x+36$

Determine the vertex and intercepts, then use this information to sketch the graph.

16) $y=x^2+6x+8$.

17) $y=2x^2-16x+24$.

Solve.

18) NASA launches a rocket at $t=0$ seconds. Its height, in meters, above sea-level in terms of time is given by the equation $h=-4.9t^2+58t+241$.

a) How high is the rocket after 8 seconds?

b) How high was the rocket when it was initially launched?

Solve.

19) A large explosion causes wood and metal debris to rise vertically into the air with an initial velocity of 128 feet per second. The equation $h = 128t - 16t^2$ gives the height of the falling debris above the ground, in feet, t seconds after the explosion.
    a) Use the given equation to find the height of the debris one second after the explosion.
    b) How many seconds after the explosion will the debris hit the ground?

20) If the equation $P = 4 + 5x - 2x^2$ represents the profit, in thousands of dollars, in selling x thousand Bassblast speakers, how many speakers should be sold to maximize profit? What is the maximum profit?

21) We are standing on the top of a 512 feet tall building and launch a small object upward. The object's height, measured in feet, after t seconds is given by the equation $h = -16t^2 + 224t + 512$.
    a) After how many seconds does the object hit the ground?
    b) What is the highest point that the object reaches?

22) We are standing on the top of a 1024 feet tall building and launch a small object upward. The object's vertical position, measured in feet, after t seconds is given by the equation $h = -16t^2 + 192t + 1024$. How long does it take for the object to reach the highest point? What is the highest point that the object reaches?

ANSWERS to Review: Chapter 4

1) $-\frac{3}{2}, \frac{1}{4}$

2) $\frac{11}{5}, \frac{1}{5}$

---

3) $2, -4$

4) $-3+\sqrt{11}, -3-\sqrt{11}$

5) $\frac{3+\sqrt{153}}{8}, \frac{3-\sqrt{153}}{8}$

6) $-2+2\sqrt{6}, -2-2\sqrt{6}$

---

7) $\frac{3+\sqrt{89}}{8}, \frac{3-\sqrt{89}}{8}$

8) $\frac{1}{3}, -\frac{2}{3}$

9) $\frac{2+3i\sqrt{5}}{7}, \frac{2-3i\sqrt{5}}{7}$

10) $-\frac{2}{5}, -4$

11) $\frac{1+i\sqrt{71}}{6}, \frac{1-i\sqrt{71}}{6}$

12) $-\sqrt{5}, \sqrt{5}$

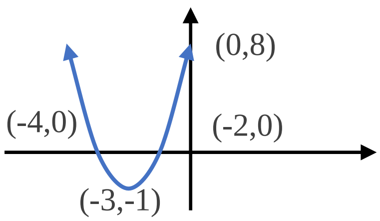
---

13) y-intercept: $(0,-40)$; x-intercepts: $(2,0)$, $(10,0)$; vertex: $(6,32)$

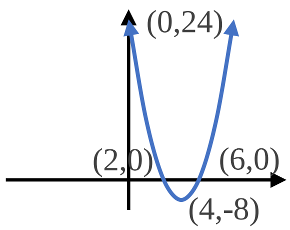
14) y-intercept: $(0,32)$; x-intercepts: $(2,0)$, $(8,0)$; vertex: $(5,-18)$

15) y-intercept: $(0,36)$; x-intercept: $(6,0)$; vertex: $(6,0)$

16)



17)



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18) 391.4 meters; 241 meters

19) 112 feet; 8 seconds

20) 1250 speakers; $7,125

21) 16 seconds; 1296 feet

22) 6 seconds; 1600 feet

CHAPTER 5

Functions and

Exponential and Logarithmic Equations

Objectives
Chapter 5

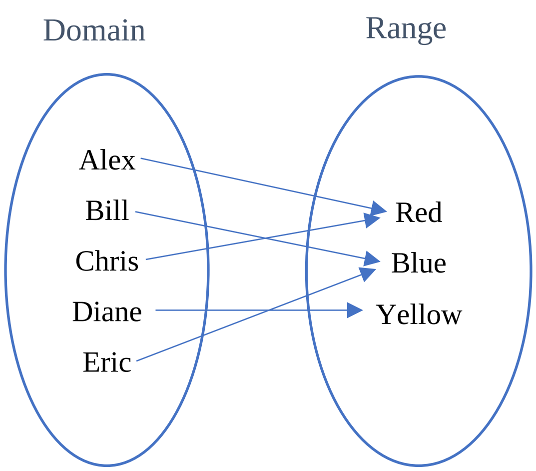
- Identify functions and use correct notation to evaluate functions at numerical and variable values.
- Combine functions using sum, difference, product, and quotient of functions.
- Graph exponential functions.
- Solve exponential equations by finding a common base.
- Convert between logarithms and exponents and use that relationship to solve basic logarithmic equations.
- Graph logarithmic functions.
- Calculate final account balances using the formulas for compound and continuous interest.

Section 5.1: Functions

Objective: Identify functions and use correct notation to evaluate functions at numerical and variable values.

A relationship is a matching of elements between two sets with the first set called the **domain** and the second set called the **range**. The following examples show several ways that relationships can be given.

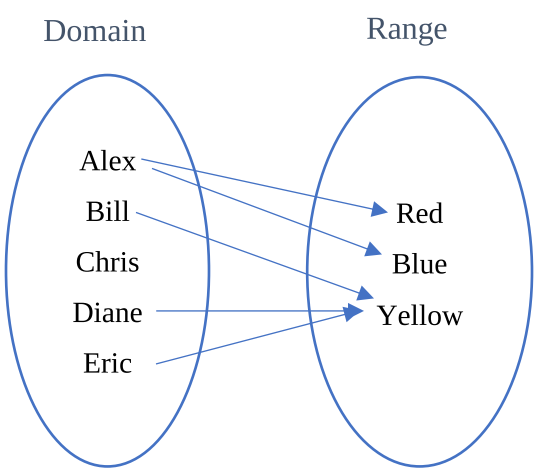
Example 1. Suppose that the domain is five people in a classroom, {Alex, Bill, Chris, Diane, Eric} and the range is the colors, {red, blue, yellow}. Suppose the relationship shows each person's favorite color from those three choices. This can be shown with a **mapping**.



Example 2. Another way to express a relationship is to give a set of ordered pairs. The set {(Alex, Red), (Bill, Blue), (Chris, Red), (Diane, Yellow), (Eric, Blue)} gives the exact same information as the mapping above. In a relationship, some values might be matched more than once. Note that in the above example that each person is matched with just one color but some colors are used more than once.

Example 3. Suppose the following mapping uses the same domain and range as the first two examples but now is showing the information about which colors each person is wearing.

a) Who is wearing Red? (Answer: Alex)
b) Which of those colors is Bill wearing? (Answer: Yellow)
c) Who is wearing Yellow? (Answer: Bill and Diane)



Notice that Alex is wearing both red and blue clothing. Chris is not wearing any of the colors listed in the range.

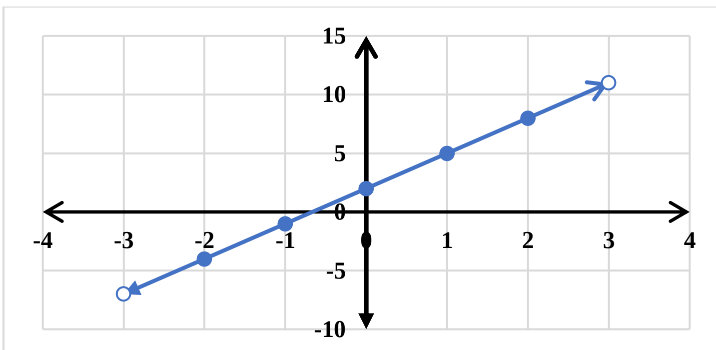
Another way to express a relationship is with an **equation**. Normally *y* denotes the range and *x* the domain.

Example 4. If we have the equation $y = 3x + 2$ then we have a relationship where the domain and range are both the set of all real numbers. Here we can find the matching *y* for any value of *x*. For example, if $x = 0$ then $y = 3(0) + 2 = 2$. This can be expressed as the ordered pair $(0, 2)$. Other ordered pairs that satisfy the equation include $(1, 5)$ and $(7, 23)$.

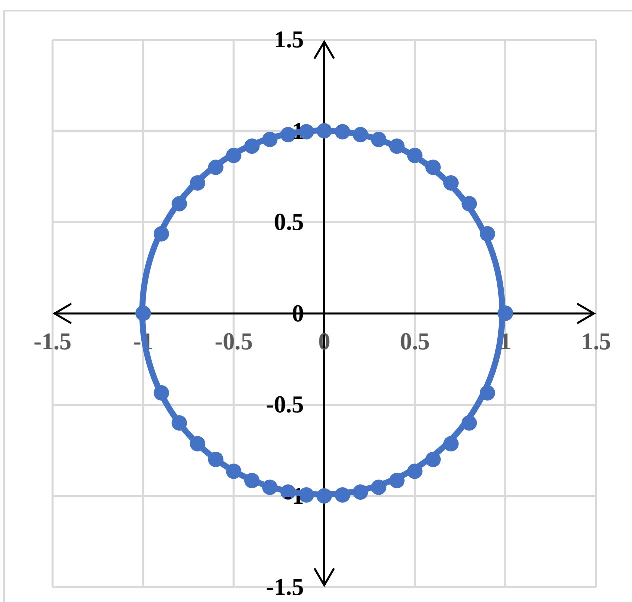
Example 5. If we have the equation $y^2 + x^2 = 1$ then again we have a relationship. Here note that for a given value of *x* that there might be more than one matching value of *y*. For example, if $x = 0$, then $y = -1$ or $y = 1$. Both make the equation true.

Another way to express a relationship is with a **graph**.

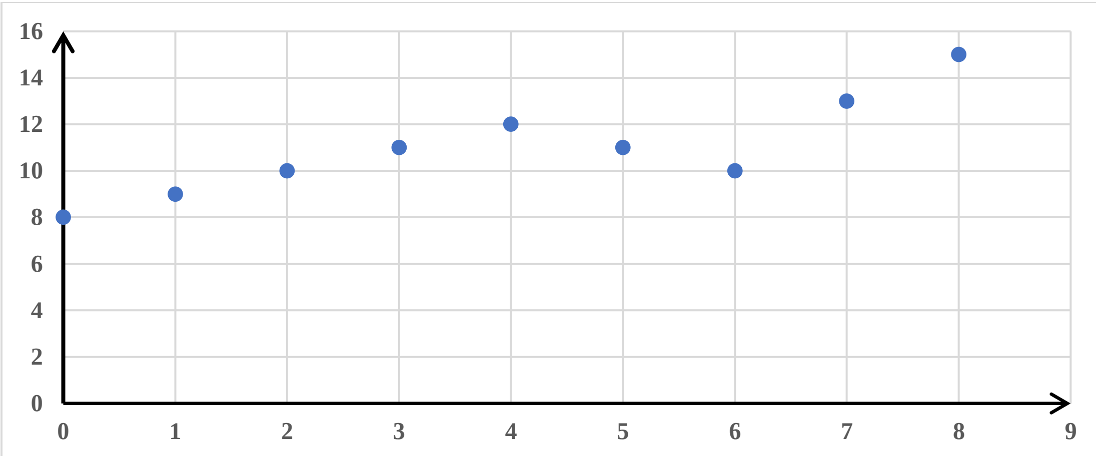
Example 6. This is the graph of the equation $y = 3x + 2$ from Example 4. The graph shows the combination of *x* and *y* values which make the equation true.



Example 7. This is the graph of the equation $y^2 + x^2 = 1$ from Example 5. The graph shows the combination of *x* and *y* values which make the equation true.



Example 8. A graph does not need to come from a "nice" equation. In the following example suppose that the vertical value gives the temperature in degrees Celsius and the horizontal value gives the time in hours after midnight on a certain day. It visually shows how the values are related. For example we can see that at 8 a.m. the temperature was 15 degrees Celsius. The temperature was 10 degrees Celsius at two different times: once at 2 a.m. and again at 6 a.m.



DEFINITION OF A FUNCTION

There is a special classification of relationships known as functions. **Functions** are relationships in which each value of the domain corresponds with exactly one value of the range.

Generally x is the variable that we put into an equation to evaluate and find y. For this reason x is considered an input variable and y is considered an output variable. This means the definition of a function, in terms of equations in x and y could be stated as: there is exactly one y value corresponding with any x value.

The box below summarizes this definition and vocabulary.

DEFINITION OF A FUNCTION

A relationship represents a **function** if each input value x is matched with exactly one output value y.

- The **domain** is the set of all input values.
- The **range** is the set of all output values.

Let's revisit some of the examples above to determine whether each relationship represents a function.

- Example 1: Each name is matched with exactly one favorite color so the relationship **is** a function.
- Example 2: This example shows the same relationship as Example 1 but as a set of ordered pairs rather than as a diagram. Notice that no two ordered pairs have the same first component but different second component. So as we already know, the relationship in Example 2 **is** a function.
- Example 3: Alex is wearing two different colors so one input (Alex) is matched with more than one output (red and blue). The relationship **is not** a function.
- Example 5: We saw if $x = 0$, then $y = -1$ or $y = 1$. Since we know of at least one value of *x* having more than one matching value *y*, the relationship **is not** a function.

Now we will consider some new examples and determine if the relationship represents a function.

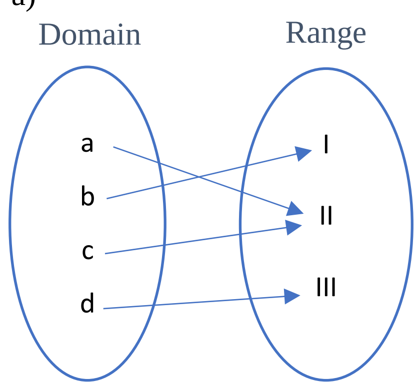
Example 9. Which of the following sets of ordered pairs represents a function?

a) $\{(1,a),(2,b),(3,c),(4,a)\}$ is a function, since there are no two pairs with the same first component. The domain is the set $\{1, 2, 3, 4\}$ and the range is $\{a, b, c\}$.

b) $\{(1,a),(2,b),(1,c),(4,a)\}$ is not a function, since there are two pairs with the same first component but different second components: $(1,a)$ and $(1,c)$.

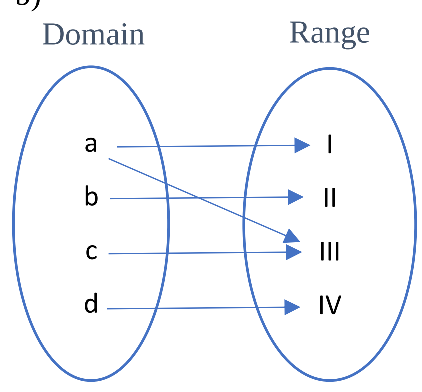
Example 10. Which of the following mapping diagrams represents a function?

a)



This diagram represents a function since each element in the domain corresponds to exactly one element in the range.

b)

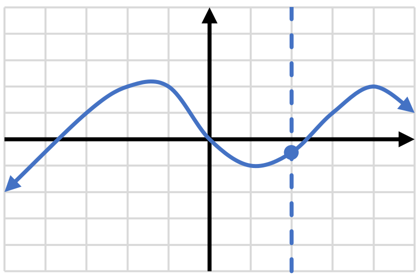


This diagram does not represent a function since one element of the domain corresponds with more than one element of the range; (a, I), (a, III).

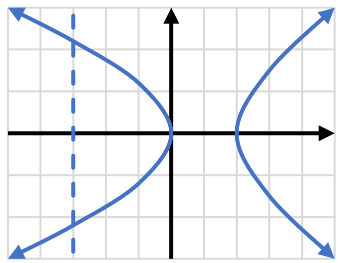
VERTICAL LINE TEST

There is a test called the **Vertical Line Test** to determine if a relationship given graphically is a function. If there is even one vertical line that can be drawn which intersects the graph more than once, then there is an x-value which is matched with more than one y-value, meaning the graph does **not** represent a function.

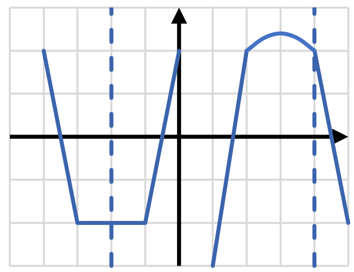
Example 11. Which of the following graphs are graphs of functions?



Drawing a vertical line through this graph will only cross the graph once. It *is* a function.



Drawing a vertical line through this graph will cross the graph twice. This *is not* a function.



Drawing a vertical line through this graph will cross the graph at most once. It *is* a function.

FUNCTION NOTATION

Once we know a relationship represents a function, we often change the notation used to emphasize the fact that it is a function. In accordance with the idea of corresponding input and output values, these equations are often represented with the notation $f(x)$ for the output value. This notation is read as "f of x" and signifies the y value that corresponds to a given x value. Writing $y = f(x)$ represents only a change in how the output is referenced. In Example 4 above, instead of writing the function as $y = 3x + 2$, we could have written $f(x) = 3x + 2$.

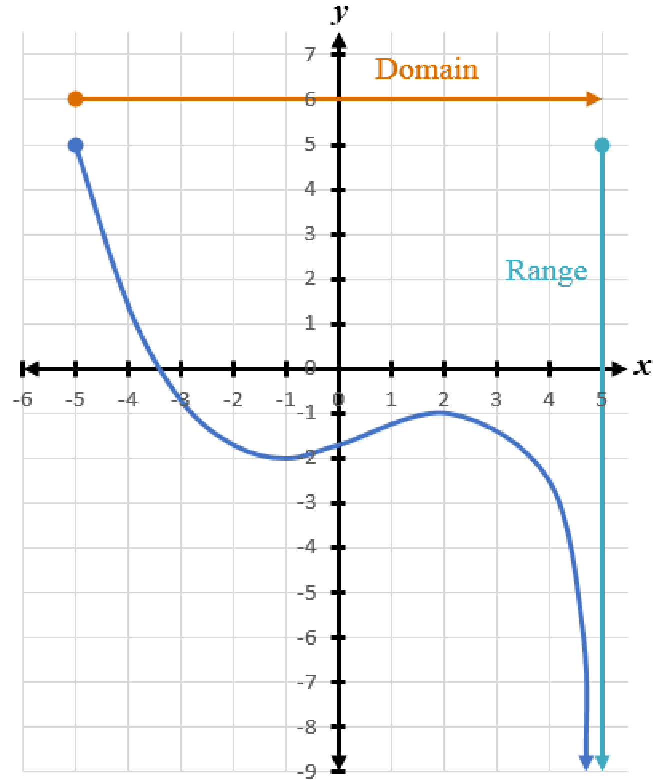
It is important to point out that $f(x)$ does not mean f times x". It is a notation that names the function with the first letter (function f) and then in parentheses, we are given information about what variable is used as the input in the function (variable x). The first letter naming the function can be anything we want it to be. For example, you will often see $g(x)$, read g of x.

FINDING DOMAIN AND RANGE

We will now identify the domain and range of functions that are represented by graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis.Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values.

Note that the domain and range are always written from smaller to larger values: from left to right for domain, and from the bottom of the graph to the top of the graph for range.

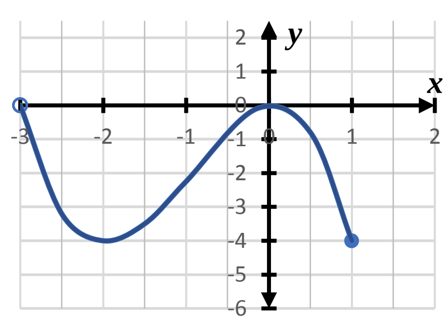
Consider the graph below.



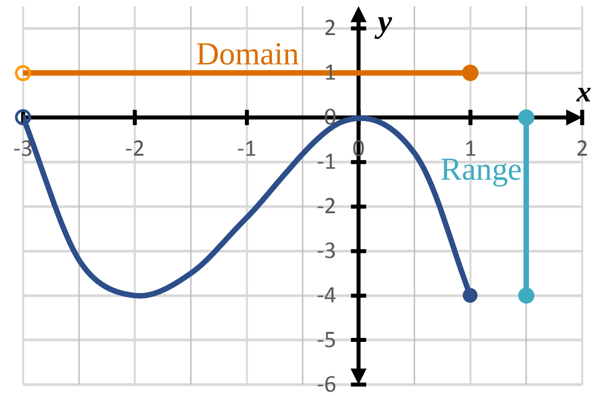
We observe that the graph extends horizontally from -5 to the right without bound, so the domain is "all real numbers greater than or equal to -5". We write that domain in interval notation as $[-5,\infty)$ or in set builder notation as $\{x \mid x \geq -5\}$.

The vertical extent of the graph is all range values 5 and below or "all real numbers less than or equal to 5". So the range is written in interval notation as $(-\infty,5]$ or in set builder notation as $\{y \mid y \leq 5\}$.

Example 12. Find the domain and range given the following graph.



Answer:



Domain: $(-3,1]$ or $-3 < x \leq 1$

Range: $[-4,0]$ or $-4 \leq y \leq 0$

Notice that the open circle at $(-3,0)$ indicates that the point $(-3,0)$ does not belong to the graph. The closed circle at $(1,-4)$ indicates that the point $(1,-4)$ does belong to the graph.

Example 13. Find the domain of the function.

$f(x) = \dfrac{3x-1}{x^2+x-6}$	Find values of x for which the denominator is equal to 0.
$x^2+x-6=0$	Solve by factoring
$(x+3)(x-2)=0$	Set each factor equal to zero
$x+3=0 \quad \text{or} \quad x-2=0$	Solve each equation
$\underline{-3 \quad -3} \qquad \underline{+2 \quad +2}$	
$x=-3 \quad \text{or} \quad x=2$	Exclude these values from the domain of f
All real numbers except $x=-3$ and 2	Our Answer

The notation in the previous example tells us that x can be any value except for -3 and -2. If x were one of those two values, the function would be undefined.

Example 14. Find the domain of the function.

$f(x) = 3x^2 - x$	With this function, there are no excluded values
All real numbers or $\mathbb{R}$ or $(-\infty,\infty)$	Our Solution

In the above example there are no real numbers that make the function undefined. This means any number can be used for x.

Example 15. Find the domain of the function.

$f(x) = \sqrt{2x-3}$	Square roots of negative numbers are not real numbers, so we restrict the domain to all real numbers for which the radicand is nonnegative

$$
\begin{array}{rl}
2x-3\geq 0 & \text{Set up an inequality} \\
\underline{+3\quad +3} & \text{Solve} \\
\dfrac{2x}{2}\geq\dfrac{3}{2} & \\
x\geq\dfrac{3}{2} & \\
x\geq\dfrac{3}{2} & \text{Our Answer}
\end{array}
$$

The notation in the above example states that our variable can be any number greater than or equal to $\frac{3}{2}$. Any number smaller than $\frac{3}{2}$ would make the function undefined because the radicand would have a value less than zero.

EVALUATING A FUNCTION USING FUNCTION NOTATION

Function notation can be used when we want to evaluate a function. A numerical value or an expression is substituted in the place of the input variable in the equation and the output value or expression is determined. This process is shown in the following examples.

Example 16. Evaluate the function.

$$
\begin{array}{rl}
\text{Let } f(x)=3x^2-4x\text{; find } f(-2). & \text{Substitute } -2 \text{ for } x \text{ in the function} \\
f(-2)=3(-2)^2-4(-2) & \text{Evaluate, using order of operations} \\
=3(4)-4(-2) & \text{Multiply} \\
=12+8 & \text{Add} \\
f(-2)=20 & \text{Our Answer}
\end{array}
$$

One advantage of using function notation is that instead of asking "what is the value of y if $x=-2$?", we can now just write "Find $f(-2)$". The answer statement $f(-2)=20$ above tells us the value of the function is 20 when $x=-2$.

Example 17. Evaluate the function.

$$
\begin{array}{rl}
\text{Let } h(x)=3^{2x-6}\text{; find } h(4). & \text{Substitute } 4 \text{ for } x \text{ in the function} \\
h(4)=3^{2(4)-6} & \text{Simplify exponent, multiplying first} \\
=3^{8-6} & \text{Subtract in exponent} \\
=3^2 & \text{Evaluate exponent} \\
h(4)=9 & \text{Our Answer}
\end{array}
$$

Example 18. Evaluate the function.

Let $k(a)=2\|a+4\|$; find $k(-7)$.	Substitute -7 for a in the function
$k(-7)=2\|-7+4\|$	Add inside absolute value
$=2\|-3\|$	Evaluate absolute value
$=2(3)$	Multiply
$k(-7)=6$	Our Answer

As the above examples show, the function can take many different forms, but the way to evaluate the function is always the same: replace the variable with what is in parentheses and simplify.

We can also substitute expressions into functions using the same process. Often the expressions use the same variable, so it is important to remember each occurrence of the variable is replaced by whatever is in parentheses.

Example 19. Evaluate the function.

Let $g(x)=x^4+1$; find $g(3x)$.	Replace x in the function with $(3x)$
$g(3x)=(3x)^4+1$	Simplify exponent
$g(3x)=81x^4+1$	Our Answer

Example 20. Evaluate the function.

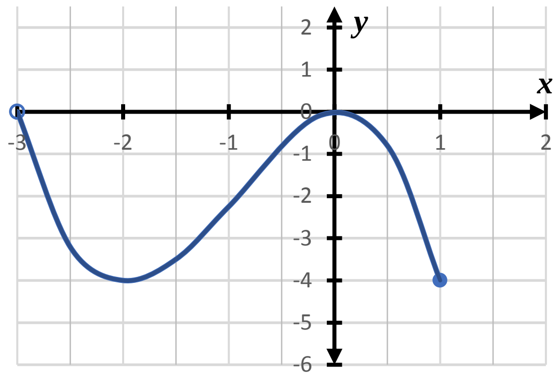
Let $p(t)=t^2-t$; find $p(t+1)$.	Replace each t in the function with $(t+1)$
$p(t+1)=(t+1)^2-(t+1)$	Square binomial
$=t^2+2t+1-(t+1)$	Distribute negative sign
$=t^2+2t+1-t-1$	Combine like terms
$p(t+1)=t^2+t$	Our Answer

It is important to become comfortable with function notation and learn how to use it as we transition into more advanced algebra topics.

EVALUATING A FUNCTION FROM ITS GRAPH

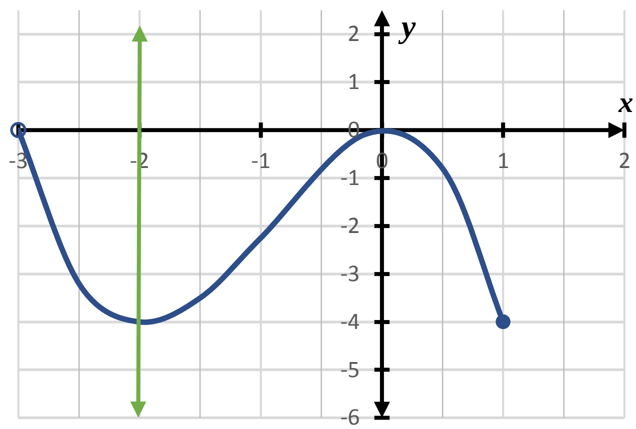
Finally, in addition to being evaluated algebraically, functions can be evaluated by identifying input and output values on its graph. To do this, we only need to find the desired value for the input variable on the *x*-axis, and then moving along a vertical path through that value, note where this path intersects the graph of the function. Upon finding this intersection we now should move along a horizontal path toward the y-axis. The value at which this horizontal path intersects the *y*-axis represents the output value for the desired input value.

Example 21. Using the graph of the function *f* below, find $f(-2)$.



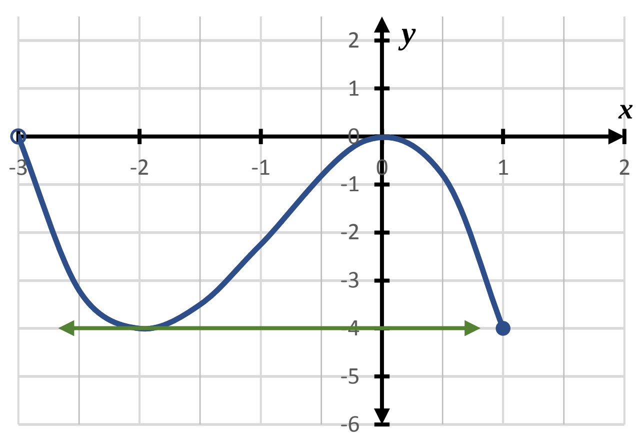
Locate -2 on the *x*-axis and visualize a vertical path passing through that value.

Remember that this is where the input value is equal to -2.

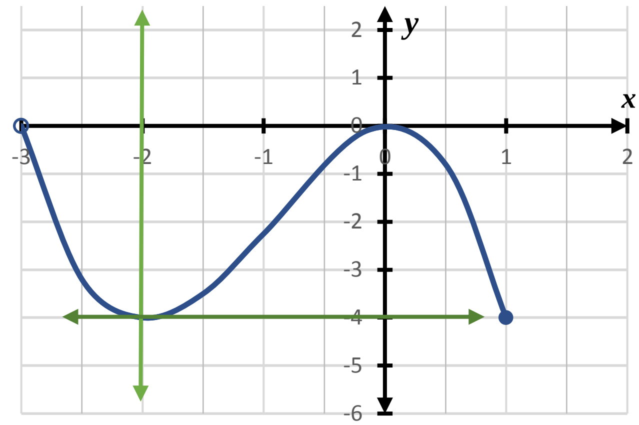


Now, notice where this line intersects the graph of the given function.

At that point of intersection, visualize a horizontal line passing through and in the direction of the y-axis.



At this point it should be noted that the horizontal line intersects the y-axis at -4. This is the value of the output variable. Therefore, visualizing both the vertical and horizontal lines at the same time identifies the values of both the input and output variables, and in effect finds what we are looking for, $f(-2)$. This is expressly seen in the graph on the next page.

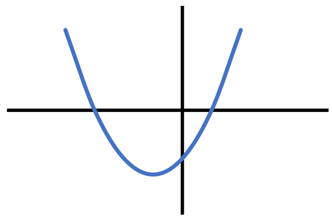


Therefore, according to the procedure followed, and the vertical and horizontal lines, we see that $f(-2) = -4$. In other words, the function evaluated at -2 is equal to -4.

Practice Exercises
Section 5.1: Functions

Determine which of the following represent functions.

1)



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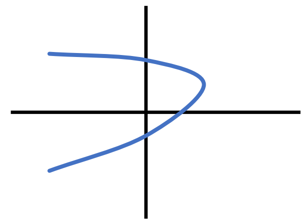

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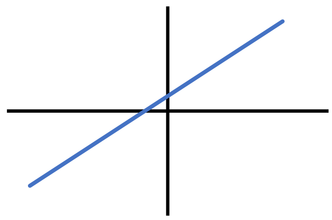
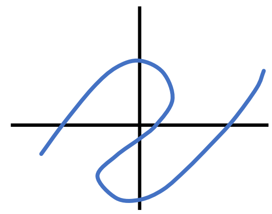


4)



Specify the domain of each of the following functions.

5) $f(x) = -5x + 1$

6) $f(x) = \sqrt{8 - 4x}$

7) $s(t) = \dfrac{1}{t^2}$

8) $s(t) = \dfrac{1}{t^2 + 1}$

9) $f(x) = x^2 - 3x - 4$

10) $f(x) = \sqrt{x - 16}$

11) $f(x) = \dfrac{-2}{x^2 - 3x - 4}$

12) $y(x) = \dfrac{x}{x^2 - 25}$

Evaluate each function.

13) $g(x) = 3x^2 + 4x - 4$; Find $g(0)$

14) $g(x) = 5x - 3$; Find $g(2)$

15) $f(x) = 3x + 1$; Find $f(0)$

16) $f(x) = 2x^2 + 9x + 4$; Find $f(-9)$

17) $f(n) = n^2 - 9n - 3$; Find $f(10)$

18) $f(t) = 3^t - 2$; Find $f(4)$

19) $w(x) = x^2 + 4x$; Find $w(-5)$

20) $w(n) = 4n + 3$; Find $w(2)$

21) $p(n) = -3|n|$; Find $p(-7)$

22) $h(n) = 4n + 2$; Find $h(n + 3)$

23) $g(x) = x + 1$; Find $g(3x)$

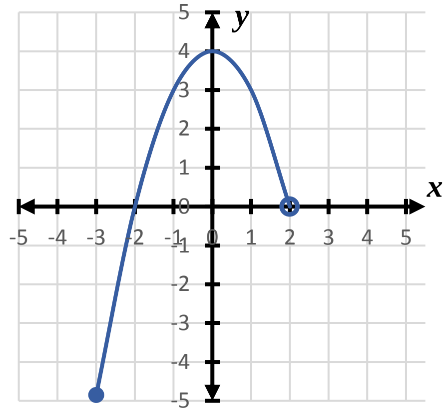
24) $h(t) = t^2 + t$; Find $h(t + 1)$

Practice Exercises: Section 5.1 (continued)

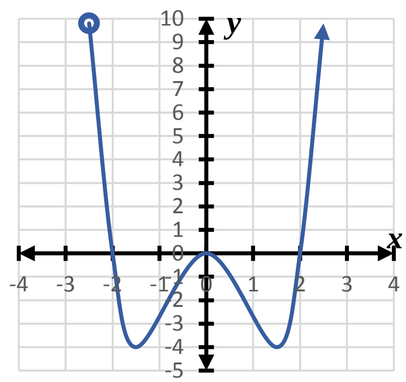
For each of the functions in problems 25 - 27, find the following:

a) Domain
b) Range
c) $f(-1)$
d) $f(0)$
e) the x value(s) such that $f(x)=0$

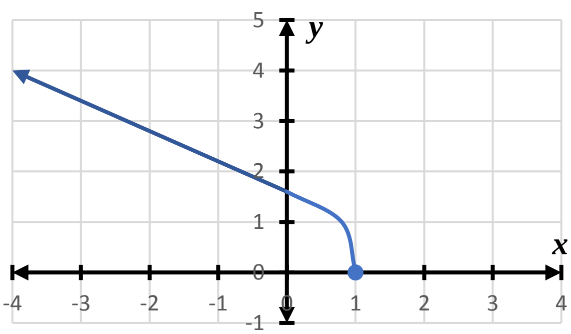
25)



26)



27)



Determine whether each relation is a function. Identify the domain and range for each relation.

28) $\{(-2,4),(3,6),(4,6),(9,0)\}$
29) $\{(0,-5),(2,-3),(4,-1),(6,1),(8,3)\}$
30) $\{(5,6),(7,0),(9,-1),(7,1)\}$

ANSWERS to Practice Exercises
Section 5.1: Functions

1) Yes
2) Yes
3) No
4) No

---

5) $(-\infty, \infty)$
6) $(-\infty, 2]$
7) all real numbers except $t = 0$
8) $(-\infty, \infty)$
9) $(-\infty, \infty)$
10) $[16, \infty)$
11) all real numbers except $x = -1$ and $x = 4$
12) all real numbers except $x = -5$ and $x = 5$

---

13) -4
14) 7
15) 1
16) 85
17) 7
18) 79
19) 5
20) 11
21) -21
22) $4n + 14$
23) $3x + 1$
24) $t^2 + 3t + 1$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 5.1 (continued)

25)

a) $-3 \le x < 2$
b) $-5 \le y \le 4$
c) $f(-1) = 3$
d) $f(0) = 4$
e) $x = -2$

26)

a) $x > -2.5$
b) $y \ge -4$
c) $f(-1) = -3$
d) $f(0) = 0$
e) $x = -2, 0, 2$

27)

a) $x > -2.5$
b) $y \ge -4$
c) $f(-1) = -3$
d) $f(0) = 1.5$
e) $x = 1$

28)

Function;
Domain $\{-2, 3, 4, 9\}$
Range $\{0, 4, 6\}$

29)

Function;
Domain $\{0, 2, 4, 6, 8\}$
Range $\{0, 4, 6\}$

30)

Not a Function;
Domain $\{5, 7, 9\}$
Range $\{-1, 0, 1, 6\}$

Section 5.2: Operations on Functions

Objective: Combine functions using sum, difference, product, and quotient of functions.

We can combine functions using four common operations. The four basic operations on functions are addition, subtraction, multiplication, and division. The notation for these functions is as follows.

Addition	$(f+g)(x) = f(x) + g(x)$
Subtraction	$(f-g)(x) = f(x) - g(x)$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$
Division	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ for $g(x) \neq 0$

EVALUATING FUNCTIONS

When we evaluate the sum/difference/product/quotient of two functions, we evaluate each function independently and then perform the given operation with both results.

Example 1. Perform the indicated operation.

Let $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

Find $(f+g)(-3)$ — Evaluate $(f+g)$ at $x = -3$

$= f(-3) + g(-3)$

Evaluate f at $x = -3$;

$f(-3) = (-3)^2 - (-3) - 2$

$f(-3) = 9 + 3 - 2$

$f(-3) = 10$

Evaluate g at $x = -3$:

$g(-3) = (-3) + 1$

$g(-3) = -2$

$= 10 + (-2)$ — Add the two results together

$= 8$ — Our Answer

The process is the same regardless of the operation being performed.

Example 2. Perform the indicated operation.

Let $h(x) = 2x - 4$ and $k(x) = -3x + 1$.

Find $(h \cdot k)(5)$	Evaluate $(h \cdot k)$ at $x = 5$
$= h(5) \cdot k(5)$	Evaluate h at $x = 5$; $h(5) = 2(5) - 4$ $h(5) = 10 - 4$ $h(5) = 6$ Evaluate k at $x = 5$: $k(5) = -3(5) + 1$ $k(5) = -15 + 1$ $k(5) = -14$
$= (6)(-14)$	Multiply the two results together
$= -84$	Our Answer

COMBINING FUNCTIONS

In the examples above, we evaluated the sum and quotient of two functions at a given value of the variable. Often, instead of evaluating, we are asked to create a new function by performing an operation on the two given functions. We combine the two functions using the indicated operation, writing each in parentheses, and then simplifying the expression.

Example 3. Perform the indicated operation.

Let $f(x) = 2x - 4$ and $g(x) = x^2 - x + 5$.

Find $(f - g)(x)$	Find the difference of two functions
$= f(x) - g(x)$	Replace $f(x)$ with $(2x - 4)$ and $g(x)$ with $(x^2 - x + 5)$
$= (2x - 4) - (x^2 - x + 5)$	Subtract by distributing the negative
$= 2x - 4 - x^2 + x - 5$	Combine like terms
$= -x^2 + 3x - 9$	Our Answer

The parentheses are very important when we are combining $f(x)$ and $g(x)$ using a given operation. In the previous example, we needed the parentheses to know to distribute the negative.

Example 4. Perform the indicated operation.

Let $f(x) = x^2 - 4x - 5$ and $g(x) = x - 5$.

Find $\left(\frac{f}{g}\right)(x)$.	Find the quotient of two functions
$= \dfrac{f(x)}{g(x)}$	Replace $f(x)$ with $(x^2 - 4x - 5)$ and $g(x)$ with $(x-5)$
$= \dfrac{(x^2 - 4x - 5)}{(x-5)}$	Simplify the fraction; we must first factor
$= \dfrac{(x-5)(x+1)}{(x-5)}$	Divide out common factor of $x-5$
$= x+1$	Our Answer

In the examples below, we will combine and evaluate functions. Notice the input value in these examples is a variable expression.

Example 5. Perform the indicated operation.

Let $f(x) = 2x - 1$ and $g(x) = x + 4$.

Find $(f+g)(x^2)$.	Find the sum of two functions
$= f(x^2) + g(x^2)$	Evaluate $f(x)$ at x^2 and evaluate $g(x)$ at x^2
$= [2(x^2) - 1] + [(x^2) + 4]$	Distributing the $+$ does not change the problem
$= 2x^2 - 1 + x^2 + 4$	Combine like terms
$= 3x^2 + 3$	Our Answer

Example 6. Perform the indicated operation.

Let $f(x) = 2x - 1$ and $g(x) = x + 4$.

Find $(f \cdot g)(3x)$.	Find the product of two functions
$= f(3x) \cdot g(3x)$	Evaluate $f(x)$ at $3x$ and evaluate $g(x)$ at $3x$
$= [2(3x) - 1] \cdot [(3x) + 4]$	Multiply $2(3x)$
$= (6x - 1)(3x + 4)$	FOIL
$= 18x^2 + 24x - 3x - 4$	Combine like terms
$= 18x^2 + 21x - 4$	Our Answer

APPLICATIONS OF OPERATIONS ON FUNCTIONS

Example 7. A college has two campuses that opened at the same time. The function $A(x)=200x+500$ gives the enrollment at campus A x years after opening. The function $B(x)=100x+1000$ gives the enrollment at campus B x years after opening.

Find $(A+B)(x)$.	Find the sum of two functions
$=A(x)+B(x)$	Replace $A(x)$ with $(200x+500)$ and $B(x)$ with $(100x+1000)$
$=(200x+500)+(100x+1000)$	Combine like terms
$=300x+1500$	Our Answer

This combined function $(A+B)(x)=300x+1500$ gives the total enrollment at both campuses of the college x years after opening.

Example 8. Use the functions from Example 7 to answer the question.

Find $(A+B)(10)$.	Since we already combined $A(x)+B(x)$ to get $(A+B)(x)=300x+1500$ in Example 7, evaluate $(A+B)(x)$ when x = 10.
$=300(10)+1500$	Multiply
$=3000+1500$	Add
$=4500$	Our Answer

This answer tells us that the total enrollment at both campuses 10 years after opening is 4500 students.

Practice Exercises
Section 5.2: Operations on Functions

Perform the indicated operations.

1) Let $f(x) = -4x + 1$ and $g(x) = -2x - 1$. Find $(f + g)(5)$.

2) Let $g(x) = 3x + 3$ and $f(x) = 2x - 2$. Find $(g + f)(9)$.

3) Let $f(x) = x^3 + 5x^2$ and $g(x) = 2x + 4$. Find $(f + g)(3)$.

4) Let $g(x) = 3x + 1$ and $f(x) = x^3 + 3x^2$. Find $(g \cdot f)(2)$.

5) Let $f(x) = -3x^2 + 3x$ and $g(x) = 2x + 5$. Find $\left(\dfrac{f}{g}\right)(-4)$.

6) Let $g(x) = 4x + 3$ and $h(x) = x^3 - 2x^2$. Find $(g - h)(-1)$.

7) Let $g(x) = x + 3$ and $f(x) = -x + 4$. Find $(g - f)(3)$.

8) Let $g(x) = x^2 + 2$ and $f(x) = 2x + 6$. Find $(g - f)(0)$.

9) Let $g(t) = t - 3$ and $h(t) = -3t^3 + 6t$. Find $(g + h)(1)$.

10) Let $f(n) = n - 5$ and $g(n) = 4n + 2$. Find $(f + g)(-8)$.

11) Let $h(t) = t + 5$ and $g(t) = 3t - 5$. Find $(h \cdot g)(5)$.

12) Let $g(a) = 3a - 2$ and $h(a) = 4a - 2$. Find $(g + h)(-10)$.

13) Let $h(n) = 2n - 1$ and $g(n) = 3n - 5$. Find $\left(\dfrac{h}{g}\right)(0)$.

14) Let $g(x) = x^2 - 2$ and $h(x) = 2x + 5$. Find $(g + h)(-6)$.

15) Let $f(a) = -2a - 4$ and $g(a) = a^2 + 3$. Find $\left(\dfrac{f}{g}\right)(7)$.

The Practice Exercises are continued on the next page.

Practice Exercises: Section 5.2 (continued)

Perform the indicated operations.

16) Let $f(x) = x^2 - 5x$ and $g(x) = x + 5$. Find $(f + g)(x)$.

17) Let $f(x) = 4x - 4$ and $g(x) = 3x^2 - 5$. Find $(f + g)(x)$.

18) Let $f(x) = -3x + 2$ and $g(x) = x^2 + 5x$. Find $(f - g)(x)$.

19) Let $g(n) = n^2 - 3$ and $h(n) = 2n - 3$. Find $(g - h)(n)$.

20) Let $g(x) = 2x - 3$ and $h(x) = x^3 - 2x^2 + 2x$. Find $(g - h)(x)$.

21) Let $g(t) = t - 4$ and $h(t) = 2t$. Find $(g \cdot h)(t)$.

22) Let $g(x) = 4x + 5$ and $h(x) = x^2 + 5x$. Find $(g \cdot h)(x)$.

23) Let $g(n) = n^2 + 5$ and $f(n) = 3n + 5$. Find $\left(\frac{g}{f}\right)(n)$.

24) Let $f(x) = 2x + 4$ and $g(x) = 4x - 5$. Find $(f - g)(x)$.

25) Let $g(a) = -2a + 5$ and $f(a) = 3a + 5$. Find $\left(\frac{g}{f}\right)(a)$.

26) Let $g(t) = t^3 + 3t^2$ and $h(t) = 3t - 5$. Find $(g - h)(t)$.

27) Let $h(n) = n^3 + 4n$ and $g(n) = 4n + 5$. Find $(h + g)(n)$.

28) Let $f(x) = 4x + 2$ and $g(x) = x^2 + 2x$. Find $\left(\frac{f}{g}\right)(x)$.

ANSWERS to Practice Exercises
Section 5.2: Operations on Functions

1) -30

2) 46

3) 82

4) 140

5) 20

6) 2

7) 5

8) -4

9) 1

10) -43

11) 100

12) -74

13) $\frac{1}{5}$

14) 27

15) $-\frac{9}{26}$

The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 5.2 (continued)

16) $x^2 - 4x + 5$

17) $3x^2 + 4x - 9$

18) $-x^2 - 8x + 2$

19) $n^2 - 2n$

20) $-x^3 + 2x^2 - 3$

21) $2t^2 - 8t$

22) $4x^3 + 25x^2 + 25x$

23) $\dfrac{n^2 + 5}{3n + 5}$

24) $-2x + 9$

25) $\dfrac{-2a + 5}{3a + 5}$

26) $t^3 + 3t^2 - 3t + 5$

27) $n^3 + 8n + 5$

28) $\dfrac{4x + 2}{x^2 + 2x}$

Section 5.3: Exponential Functions and Equations

Objectives: Graph exponential functions.
Solve exponential equations by finding a common base.

As our study of algebra gets more advanced, we begin to study more involved functions. One pair of inverse functions we will look at are exponential functions and logarithmic functions. Here we will look at exponential functions and then we will consider logarithmic functions in another section.

GRAPHING EXPONENTIAL FUNCTIONS

Exponential functions have the form $f(x)=b^x$ where $b>0$ and $b\neq 1$. Notice that exponential functions have the variable in the exponent. It is important not to confuse exponential functions with polynomial functions where the variable is in the base such as $f(x)=x^2$.

Example 1.

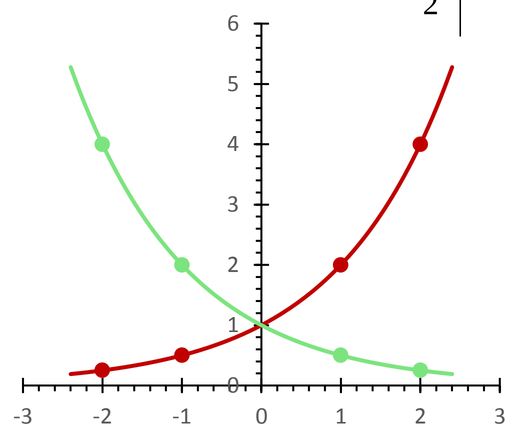
Evaluate $f(x)=2^x$ and $g(x)=\left(\frac{1}{2}\right)^x$ at $x=-2,-1,0,1,2$ and graph.

$f(x)=2^x$

x	$y=f(x)$
-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$
-1	$2^{-1}=\frac{1}{2^1}=\frac{1}{2}$
0	$2^0=1$
1	$2^1=2$
2	$2^2=4$

$g(x)=\left(\frac{1}{2}\right)^x$

x	$y=g(x)$
-2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$
-1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$
0	$\left(\frac{1}{2}\right)^0=1$
1	$\left(\frac{1}{2}\right)^1=\frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$



The domain of $f(x) = b^x$ consists of all real numbers, written in interval notation as $(-\infty, \infty)$. The range of $f(x) = b^x$ consists of all positive real numbers, written in interval notation as $(0, \infty)$.

If $b > 1$, then the graph goes up to the right and is an increasing function, called an exponential **growth** function. If $0 < b < 1$, then the graph goes down to the right and is a decreasing function, called an exponential **decay** function.

The graphs of all exponential functions of the form $f(x) = b^x$ pass through the point $(0,1)$.

The graph of an exponential function $f(x) = b^x$ approaches, but does not touch, the x-axis. We say the x-axis, or the line $y = 0$, is a *horizontal asymptote* of the graph of the function.

SOLVING EXPONENTIAL EQUATIONS

Solving exponential equations cannot be done using the skill set we have seen in the past. For example, if $3^x = 9$, we cannot take the x^{th}-root of 9 because we do not know what the index is and this doesn't get us any closer to finding x. However, we may notice that 9 is 3^2. We can then conclude that if $3^x = 3^2$ then $x = 2$.

We will use this process to solve exponential equations. If we can rewrite an equation so that the bases match, then the exponents must also match.

Example 2. Solve the equation.

$4^{2x} = 4^{x+3}$	Same bases, set exponents equal
$2x = x + 3$	Solve
$2x = x + 3$	Subtract x from both sides
$\underline{-x \quad -x}$	
$x = 3$	Our Solution

Example 3. Solve the equation.

$5^{2x+1} = 125$	Rewrite 125 as 5^3
$5^{2x+1} = 5^3$	Same bases, set exponents equal
$2x + 1 = 3$	Solve
$\underline{-1 \quad -1}$	Subtract 1 from both sides
$\dfrac{2x}{2} = \dfrac{2}{2}$	Divide both sides by 2
$x = 1$	Our Solution

Sometimes we may have to do work on both sides of the equation to get a common base. As we do so, we will use various exponent properties to help. First we will use the exponent property that states $(a^x)^y = a^{xy}$.

Example 4. Solve the equation.

$$
\begin{array}{rl}
8^{3x} = 32 & \text{Rewrite } 8 \text{ as } 2^3 \text{ and } 32 \text{ as } 2^5 \\
(2^3)^{3x} = 2^5 & \text{Multiply exponents } 3 \text{ and } 3x \\
2^{9x} = 2^5 & \text{Same bases, set exponents equal} \\
\dfrac{9x}{9} = \dfrac{5}{9} & \begin{array}{l}\text{Solve} \\ \text{Divide both sides by } 9\end{array} \\
x = \dfrac{5}{9} & \text{Our Solution}
\end{array}
$$

As we multiply exponents, we may need to distribute if there are several terms involved.

Example 5. Solve the equation.

$$
\begin{array}{rl}
3^{3x+5} = 81^{4x+1} & \text{Rewrite } 81 \text{ as } 3^4 \\
3^{3x+5} = (3^4)^{4x+1} & \text{Multiply exponents } 4(4x+1) \text{ by distributing} \\
3^{3x+5} = 3^{16x+4} & \text{Same bases, set exponents equal} \\
3x+5 = 16x+4 & \text{Solve} \\
\underline{-16x \qquad -16x} & \text{Subtract } 16x \text{ from both sides} \\
-13x+5 = 4 & \\
\underline{-5 \quad -5} & \text{Subtract } 5 \text{ from both sides} \\
\dfrac{-13x}{-13} = \dfrac{-1}{-13} & \text{Divide both sides by } -13 \\
x = \dfrac{1}{13} & \text{Our Solution}
\end{array}
$$

Another useful property of exponents is that negative exponents will give us a reciprocal: $a^{-n} = \dfrac{1}{a^n}$.

Example 6. Solve the equation.

$$
\begin{array}{rl}
\left(\dfrac{1}{9}\right)^{2x} = 3^{7x-1} & \text{Rewrite } \dfrac{1}{9} \text{ as } 3^{-2} \text{ (Use the negative exponent property)} \\
(3^{-2})^{2x} = 3^{7x-1} & \text{Multiply exponents } -2 \text{ and } 2x \\
3^{-4x} = 3^{7x-1} & \text{Same bases, set exponents equal}
\end{array}
$$

$-4x = 7x - 1$	Subtract $7x$ from both sides
$\underline{-7x \quad -7x}$	
$\dfrac{-11x}{-11} = \dfrac{-1}{-11}$	Divide by both sides by -11
$x = \dfrac{1}{11}$	Our Solution

If we have several factors with the same base on one side of the equation, we can add the exponents using the property that states $a^x a^y = a^{x+y}$.

Example 7. Solve the equation.

$5^{4x} \cdot 5^{2x-1} = 5^{3x+11}$	Add exponents on left, combing like terms
$5^{6x-1} = 5^{3x+11}$	Same bases, set exponents equal
$6x - 1 = 3x + 11$	Solve
$\underline{-3x \qquad -3x}$	Subtract $3x$ from both sides
$3x - 1 = 11$	Add 1 to both sides
$\underline{+1 \quad +1}$	
$\dfrac{3x}{3} = \dfrac{12}{3}$	Divide both sides by 3
$x = 4$	Our Solution

It may take a bit of practice to get used to knowing which base to use. We will use our properties of exponents to help us simplify. Again, below are the properties we used:

$$(a^x)^y = a^{xy} \quad \text{and} \quad \frac{1}{a^n} = a^{-n} \quad \text{and} \quad a^x a^y = a^{x+y}$$

In all of the equation we solved here, we were able to find a common base. However, this is not always possible. For example, $2 = 10^x$ cannot be written using a common base. To solve equations like this, we will need to use the inverse of an exponential function. The inverse is called a logarithmic function, which we will discuss in another section.

Practice Exercises
Section 5.3: Exponential Functions and Equations

Graph the following exponential functions and write the domain and range in interval notation.

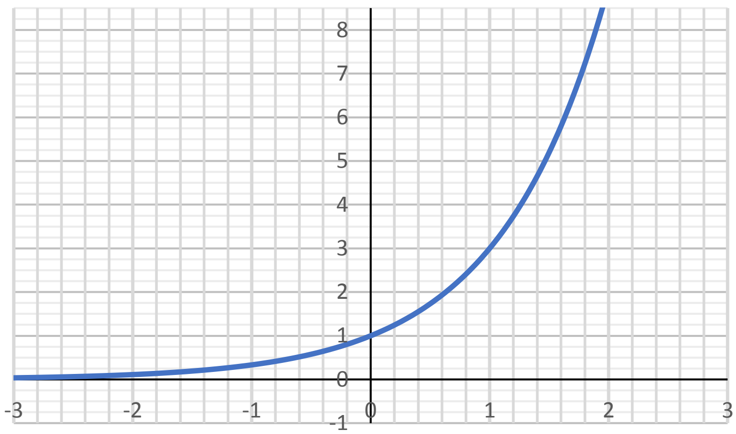
1. $f(x) = 3^x$
2. $f(x) = \left(\frac{1}{3}\right)^x$
3. $f(x) = 2^x + 3$

Solve.

4. $3^{1-2n} = 3^{1-3n}$
5. $4^{2x} = \frac{1}{16}$
6. $4^{2a} = 1$
7. $16^{-3p} = 64^{-3p}$
8. $\left(\frac{1}{25}\right)^{-k} = 125^{-2k-2}$
9. $625^{-n-2} = \frac{1}{125}$
10. $6^{2m+1} = \frac{1}{36}$
11. $6^{2r-3} = 6^{r-3}$
12. $6^{-3x} = 36$
13. $5^{2n} = 5^{-n}$
14. $64^b = 2^5$
15. $216^{-3v} = 36^{3v}$
16. $\left(\frac{1}{4}\right)^x = 16$
17. $27^{-2n-1} = 9$
18. $4^{3a} = 4^3$
19. $4^{-3v} = 64$
20. $64^{x+2} = 16$
21. $9^{2n+3} = 243$
22. $16^{2k} = \frac{1}{64}$
23. $243^p = 27^{-3p}$
24. $3^{-2x} = 3^3$
25. $4^{2n} = 4^{2-3n}$
26. $5^{m+2} = 5^{-m}$
27. $625^{2x} = 25$
28. $\left(\frac{1}{36}\right)^{b-1} = 216$
29. $216^{2n} = 36$
30. $6^2 \cdot 6^{2x} = 6^2$
31. $3^{2m} \cdot 3^4 = 1$
32. $2^{3x} \cdot 2^{5x} = 2^{4x+12}$

ANSWERS to Practice Exercises
Section 5.3: Exponential Functions and Equations

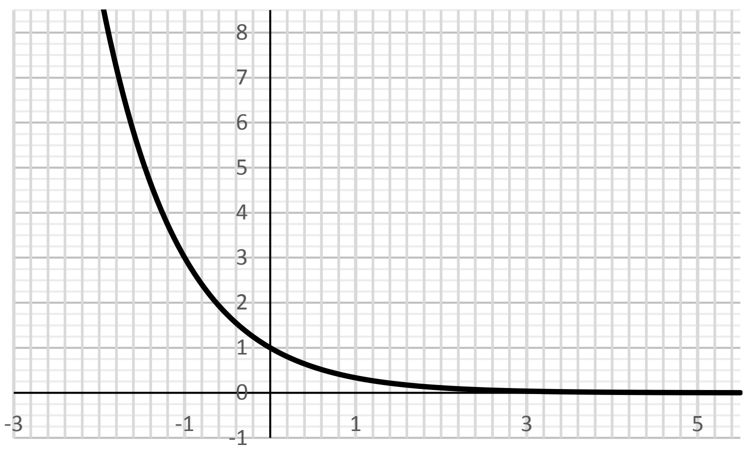
1)



Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

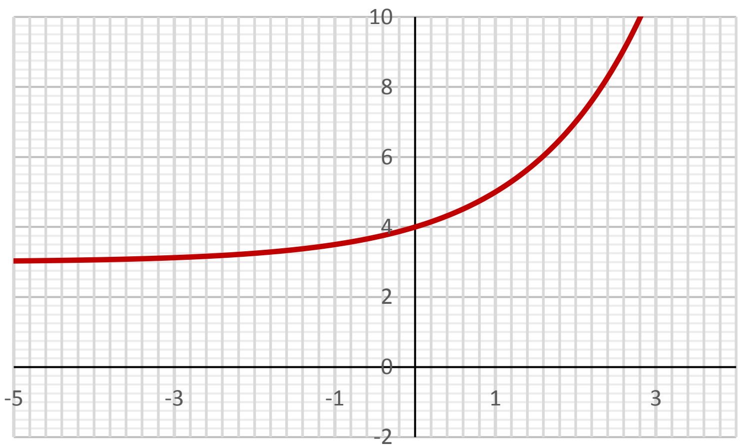
2)



Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

3)



Domain: $(-\infty,\infty)$

Range: $(3,\infty)$

ANSWERS to Practice Exercises: Section 5.3 (continued)

4) 0
5) -1
6) 0
7) 0
8) $-\frac{3}{4}$
9) $-\frac{5}{4}$
10) $-\frac{3}{2}$
11) 0
12) $-\frac{2}{3}$
13) 0
14) $\frac{5}{6}$
15) 0
16) -2
17) $-\frac{5}{6}$
18) 1
19) -1
20) $-\frac{4}{3}$
21) $-\frac{1}{4}$
22) $-\frac{3}{4}$
23) 0
24) $-\frac{3}{2}$
25) $\frac{2}{5}$
26) -1
27) $\frac{1}{4}$
28) $-\frac{1}{2}$
29) $\frac{1}{3}$
30) 0
31) -2
32) 3

Section 5.4: Logarithmic Functions and Equations

Objectives: Convert between logarithms and exponents and use that relationship to solve basic logarithmic equations.
Graph logarithmic functions.

The inverse of an exponential function is a function known as a logarithm. Logarithms are studied in detail in advanced algebra and science courses. Some places where logarithms arise in science are in measuring the *ph*-level of a chemical, measuring the intensity of an earthquake using the Richter scale, and measuring the intensity of a sound using decibels

DEFINITION OF THE LOGARITHMIC FUNCTION

Here, we will take an introductory look at how logarithms work. When working with radicals, we found that there were two ways to write radicals. The expression $\sqrt[n]{a^m}$ could be written as $a^{\frac{m}{n}}$. Each form has its advantages, thus we need to be comfortable using both the radical form and the rational exponent form.

Similarly, an exponent can be written in two forms, each with its own advantages. We are familiar with the first form, $b^y = x$, where b is the base, x can be thought of as our answer, and y is the exponent. The second way to write this is in logarithm form as $\log_b x = y$. The word "log" tells us that we are in this form. The parts of the equation all still mean the same thing: b is the base, x can be thought of as our answer, and y is the exponent.

LOGARITHMIC FUNCTION WITH BASE b

For $x > 0$, $b > 0$, and $b \neq 1$,

$$y = \log_b x \qquad \text{if and only if} \qquad x = b^y$$

The function given by $f(x) = \log_b x$ read as "log base b of x" is called the *logarithmic function with base* b.

Notice a logarithm is an exponent. Thus, logarithmic form will let us isolate an exponent. Using this idea, the equation $5^2 = 25$ could also be written in equivalent logarithmic form as $\log_5 25 = 2$. Both mean the same thing, both are still the same exponent problem, but each form has its own advantages. The most important thing to be comfortable doing with logarithms and exponents is to be able to switch back and forth between the two forms. This process is shown in the next few examples.

CONVERTING BETWEEN EXPONENTIAL AND LOGARITHMIC FORM

Example 1. In each part, write the exponential equation in its equivalent logarithmic form.

A. $2=10^x$ Identify base 10, answer 2, and exponent x

$\log_{10} 2 = x$ Our Answer

B. $m^3=5$ Identify base m, answer 5, and exponent 3

$\log_m 5 = 3$ Our Answer

C. $7^2=b$ Identify base 7, answer b, and exponent 2

$\log_7 b = 2$ Our Answer

D. $\left(\frac{2}{3}\right)^4=\frac{16}{81}$ Identify base $\frac{2}{3}$, answer $\frac{16}{81}$, and exponent 4

$\log_{\frac{2}{3}} \frac{16}{81} = 4$ Our Answer

Example 2. In each part, write the logarithmic equation in its equivalent exponential form.

A. $\log_4 16 = 2$ Identify base 4, answer 16, and exponent 2

$4^2=16$ Our Answer

B. $\log_3 x = 7$ Identify base 3, answer x, and exponent 7

$3^7=x$ Our Answer

C. $\log_9 3 = \frac{1}{2}$ Identify base 9, answer 3, and exponent $\frac{1}{2}$

$9^{\frac{1}{2}}=3$ Our Answer

EVALUATING LOGARITHMS

Once we are comfortable switching between logarithmic and exponential form, we are able to evaluate logarithmic expressions and solve exponential equations. We will first evaluate logarithmic expressions. An easy way to evaluate a logarithm is to set the logarithm equal to x and change it into an exponential equation.

Example 3. Evaluate.

Evaluate $\log_2 64$	Set logarithm equal to x
$\log_2 64 = x$	Change to exponential form
$2^x = 64$	Write using same bases, $64 = 2^6$
$2^x = 2^6$	Same bases, set exponents equal
$x = 6$	Our Answer

Example 4. Evaluate.

Evaluate $\log_{125} 5$	Set logarithm equal to x
$\log_{125} 5 = x$	Change to exponential form
$125^x = 5$	Write using same bases, $125 = 5^3$
$(5^3)^x = 5$	Multiply exponents
$5^{3x} = 5$	Same bases, set exponents equal ($5 = 5^1$)
$\frac{3x}{3} = \frac{1}{3}$	Solve Divide both sides by 3
$x = \frac{1}{3}$	Our Answer

Example 5. Evaluate.

Evaluate $\log_3 \frac{1}{27}$	Set logarithm equal to x
$\log_3 \frac{1}{27} = x$	Change to exponential form
$3^x = \frac{1}{27}$	Write using same bases, $\frac{1}{27} = 3^{-3}$
$3^x = 3^{-3}$	Same bases, set exponents equal
$x = -3$	Our Answer

SOLVING LOGARITHMIC EQUATIONS

Solving logarithmic equations is done in a very similar way, by changing the equation into exponential form and solving the resulting equation.

Example 6. Solve the equation.

$\log_5 x = 2$	Change to exponential form
$5^2 = x$	Evaluate exponent
$25 = x$	Our Solution

Example 7. Solve the equation.

$$\begin{aligned}
\log_2(3x+5) &= 4 && \text{Change to exponential form}\\
2^4 &= 3x+5 && \text{Evaluate exponent}\\
16 &= 3x+5 && \text{Solve}\\
\underline{-5 \quad} & \underline{\quad -5} && \text{Subtract } 5 \text{ from both sides}\\
\frac{11}{3} &= \frac{3x}{3} && \text{Divide both sides by } 3\\
\frac{11}{3} &= x && \text{Our Solution}
\end{aligned}$$

Example 8. Solve the equation.

$$\begin{aligned}
\log_x 8 &= 3 && \text{Change to exponential form}\\
x^3 &= 8 && \text{Cube root of both sides}\\
x &= 2 && \text{Our Solution}
\end{aligned}$$

COMMON LOGARITHM AND NATURAL LOGARITHM

There is one base of the logarithm that is used more often than any other base, mainly base 10. Similar to square roots and not writing the common index of 2 in the radical, we don't write the common base of 10 in the logarithm. So if we are working on a problem with no base written, we will always assume that the base is 10.

The other important log is the "natural", or "log base e", denoted as "$\ln(x)$" and usually pronounced as "ell-enn-of-x". (Note: That's "ell-enn", not "one-enn" or "eye-enn"!) Just as the number e arises naturally in math and the sciences, so also does the natural log, which is why you need to be familiar with it.

The number e is a constant approximately equal to 2.72. The number e is a constant similar in idea to π in that it goes on forever without repeat or pattern. Just as π naturally occurs in several geometry applications, e appears in many exponential applications which we will see in the next section.

Example 9. Solve the equation.

$$\begin{aligned}
\log x &= -2 && \text{Rewrite as exponent, } 10 \text{ is base}\\
10^{-2} &= x && \text{Evaluate. Remember a negative exponent gives a fraction}\\
\frac{1}{100} &= x && \text{Our Solution}
\end{aligned}$$

Example 10. Solve the equation.

$\ln x = 3$ Rewrite as exponent, e is base

$e^3 = x$ Our Solution

So far, this lesson has introduced the idea of logarithms, changing between logarithms and exponents, evaluating logarithms, and solving basic logarithmic equations. In an advanced algebra course, logarithms will be studied in much greater detail.

GRAPHING LOGARITHMIC FUNCTIONS

It was mentioned in the previous section that logarithmic functions are inverses of exponential functions. To get an understanding of the graph of $f(x) = \log_b x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

The domain of $f(x) = \log_b x$ consists of all positive real numbers, $(0, \infty)$.

The range of $f(x) = \log_b x$ consists of all real numbers, $(-\infty, \infty)$.

The graph below shows the exponential function $f(x) = 2^x$ and its inverse function $g(x) = \log_2 x$.

Example 11. Graph $f(x) = \log_3 x$.

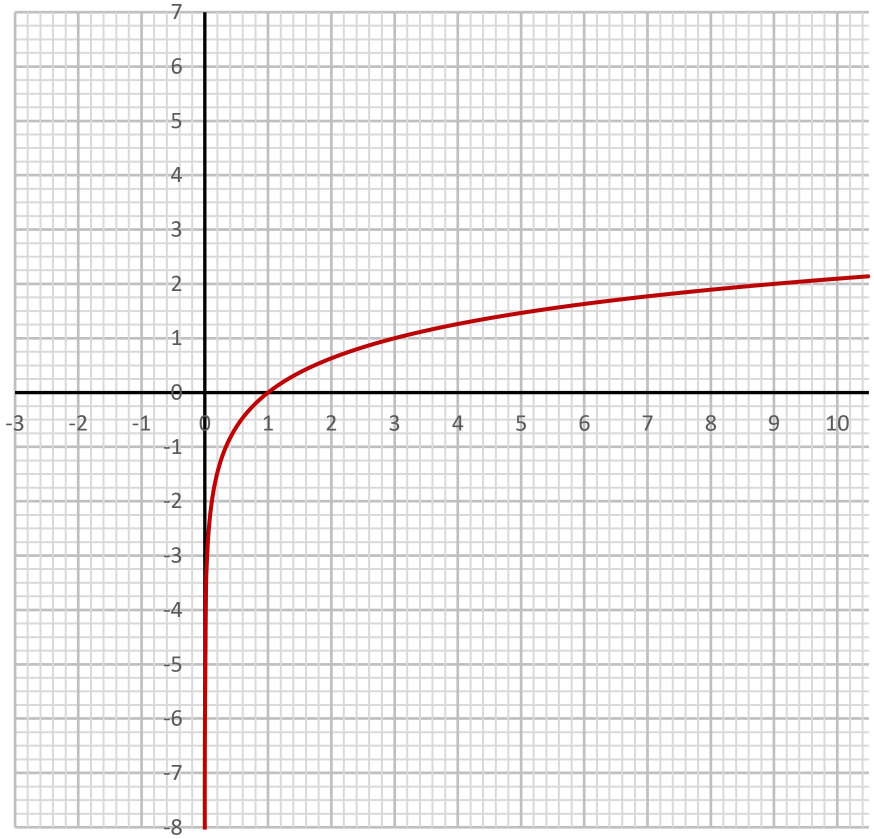
Use the definition from earlier in the section:

For $x > 0$, $b > 0$, and $b \neq 1$, $y = \log_b x$ if and only if $x = b^y$.

Rewrite $f(x) = \log_3 x$ in exponential form as $3^{f(x)} = x$ or $3^y = x$.

Evaluate $3^y = x$ for selected *y*-values $y = -2, -1, 0, 1, 2$ and graph.

x	$y = f(x)$
$3^{-2} = \frac{1}{9}$	-2
$3^{-1} = \frac{1}{3}$	-1
$3^0 = 1$	0
$3^1 = 3$	1
$3^2 = 9$	2



Written in interval notation, he domain of $f(x) = \log_3 x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Practice Exercises
Section 5.4: Logarithmic Functions and Equations

Rewrite each equation in exponential form.

1. $\log_9 81 = 2$
2. $\ln a = -16$
3. $\log_7 \frac{1}{49} = -2$
4. $\log 10,000 = 4$
5. $\log_{13} 169 = 2$
6. $\ln 1 = 0$

Rewrite each equation in logarithmic form.

7. $8^0 = 1$
8. $17^{-2} = \frac{1}{289}$
9. $15^2 = 225$
10. $144^{\frac{1}{2}} = 12$
11. $64^{\frac{1}{6}} = 2$
12. $19^2 = 361$

Evaluate each expression.

13. $\log_{125} 5$
14. $\log_5 125$
15. $\log_{343} \frac{1}{7}$
16. $\log_7 1$
17. $\log_4 16$
18. $\log_4 \frac{1}{64}$
19. $\log_6 36$
20. $\log_{36} 6$
21. $\log_2 64$
22. $\log_3 243$

Solve each equation.

23. $\log_5 x = 1$
24. $\log_8 k = 3$
25. $\log_2 x = -2$
26. $\log n = 3$

The Practice Exercises are continued on the next page.

Practice Exercises: Section 5.4 (continued)

Solve each equation.

27. $\log_{11} k = 2$
28. $\log_4 p = 4$
29. $\log_9 (n+9) = 4$
30. $\log_{11}(x-4) = -1$
31. $\log_5(-3m) = 3$
32. $\log_2(-8r) = 1$
33. $\log_{11}(x+5) = -1$
34. $\log_7(-3n) = 4$
35. $\log_4(6b+4) = 0$
36. $\log_{11}(10v+1) = -1$
37. $\log_5(-10x+4) = 4$
38. $\log_9(7-6x) = -2$
39. $\log_2(10-5a) = 3$
40. $\log_8(3k-1) = 1$

Graph each logarithmic function.

41. $f(x) = \log_2 x$
42. $f(x) = \log_3(x-1)$

ANSWERS to Practice Exercises
Section 5.4: Logarithmic Functions and Equations

1) $9^2 = 81$

2) $e^{-16} = a$

3) $7^{-2} = \frac{1}{49}$

4) $10^4 = 10,000$

5) $13^2 = 169$

6) $e^0 = 1$

---

7) $\log_8 1 = 0$

8) $\log_{17} \frac{1}{289} = -2$

9) $\log_{15} 225 = 2$

10) $\log_{144} 12 = \frac{1}{2}$

11) $\log_{64} 2 = \frac{1}{6}$

12) $\log_{19} 361 = 2$

---

13) $\frac{1}{3}$

14) 3

15) $-\frac{1}{3}$

16) 0

17) 2

18) -3

19) 2

20) $\frac{1}{2}$

21) 6

22) 5

---

23) 5

24) 512

25) $\frac{1}{4}$

26) 1000

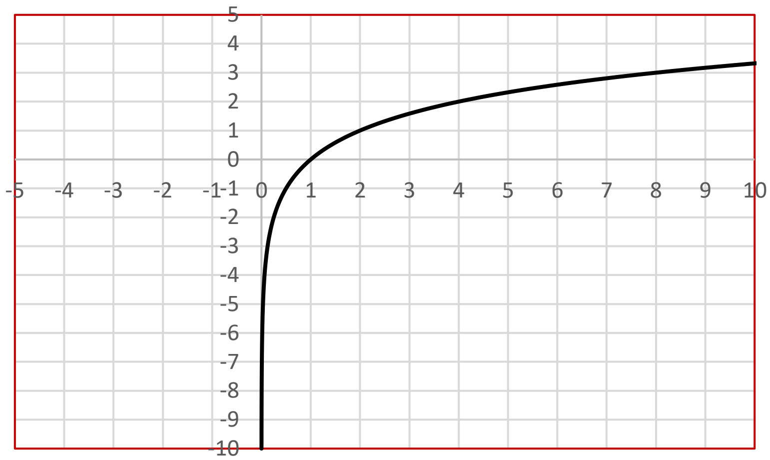
The Answers to Practice Exercises are continued on the next page.

ANSWERS to Practice Exercises: Section 5.4 (continued)

27) 121
28) 256
29) 6552
30) $\frac{45}{11}$
31) $-\frac{125}{3}$
32) $-\frac{1}{4}$
33) $-\frac{54}{11}$
34) $-\frac{2401}{3}$
35) $-\frac{1}{2}$
36) $-\frac{1}{11}$
37) $-\frac{621}{10}$
38) $\frac{283}{243}$
39) $\frac{2}{5}$
40) 3

---

41)



42)

Section 5.5: Compound Interest

Objective: Calculate final account balances using the formulas for compound and continuous interest.

COMPOUND INTEREST

One application of exponential functions involves compound interest. When money is invested in an account (or borrowed as a loan), a certain amount is added to the balance. This money added to the balance is called the *interest*. Once that interest is added to the balance, it will earn more interest during the next compounding period. This idea of earning interest on interest is called compound interest.

As an example, suppose you invest \$100 at 10% interest compounded annually:

After one year you will earn \$10 in interest, giving you a new balance of \$110.

The next year you will earn 10% of \$110 or \$11, giving you a new balance of \$121.

The third year you will earn 10% of \$121 or \$12.10, giving you a new balance of \$133.10.

This pattern will continue each year until you close the account.

There are several ways interest can be paid. The first way, as described above, is compounded annually. In this model, the interest is paid once per year. But interest can be compounded more often. Some common compounding periods include semi-annually (twice per year), quarterly (four times per year, such as quarterly taxes), monthly (12 times per year, such as a savings account), weekly (52 times per year), or even daily (365 times per year, such as some student loans). When interest is compounded in any of these ways, we can calculate the balance after any amount of time using the following formula:

COMPOUND INTEREST FORMULA

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = final amount
P = principal (starting balance)
r = annual interest rate (as a decimal)
n = number of compounding periods per year
t = time (in years)

Example 1.

If you take a car loan for $25,000 with an annual interest rate of 6.5% compounded quarterly with no payments required for the first five years, what will your balance be at the end of those five years?

Identify information given:

$P = 25{,}000$, $r = 0.065$, $n = 4$, $t = 5$

$$A = 25000\left(1+\frac{0.065}{4}\right)^{4\cdot 5}$$ Plug each value into formula, evaluate parentheses

$$A = 25000(1.01625)^{4\cdot 5}$$ Multiply exponents

$$A = 25000(1.01625)^{20}$$ Evaluate exponent

$$A = 25000(1.38041977\ldots)$$ Multiply

$$A = 34510.49$$

$34, 510.49 Our Answer

Notice that we did not round any of the numbers until the final answer (which we rounded to the nearest cent).

We can also find a missing part of the equation by using our techniques for solving equations.

Example 2.

What principal amount will grow to $3000 if invested at 6.5% compounded weekly for 4 years?

Identify information given:

$A = 3000$, $r = 0.065$, $n = 52$, $t = 4$

$$3000 = P\left(1+\frac{0.065}{52}\right)^{52\cdot 4}$$ Evaluate parentheses

$$3000 = P(1.00125)^{52\cdot 4}$$ Multiply exponents

$$3000 = P(1.00125)^{208}$$ Evaluate exponent

$$\frac{3000}{1.296719528\ldots} = \frac{P(1.296719528\ldots)}{1.296719528\ldots}$$ Divide each side by $1.296719528\ldots$

$$2313.53 = P$$ Solution for P

$2313.53 Our Answer

It is interesting to compare the same investments made at several different types of compounding periods. The next few examples do just that.

Example 3.

If \$4000 is invested in an account paying 3% interest compounded monthly, what is the balance after 7 years?

Identify information given:

$P = 4000$, $r = 0.03$, $n = 12$, $t = 7$

$A = 4000\left(1 + \frac{0.03}{12}\right)^{12\cdot 7}$ Plug each value into formula, evaluate parentheses

$A = 4000(1.0025)^{12\cdot 7}$ Multiply exponents

$A = 4000(1.0025)^{84}$ Evaluate exponent

$A = 4000(1.2333548)$ Multiply

$A = 4933.42$

\$4933.42 Our Answer

To investigate what happens to the balance if the compounding happens more often, we will consider the same problem but with interest compounded daily.

Example 4.

If \$4000 is invested in an account paying 3% interest compounded daily, what is the balance after 7 years?

Identify information given:

$P = 4000$, $r = 0.03$, $n = 365$, $t = 7$

$A = 4000\left(1 + \frac{0.03}{365}\right)^{365\cdot 7}$ Plug each value into formula, evaluate parentheses

$A = 4000(1.00008219\ldots)^{365\cdot 7}$ Multiply exponent

$A = 4000(1.00008219\ldots)^{2555}$ Evaluate exponent

$A = 4000(1.23366741\ldots)$ Multiply

$A = 4934.67$

\$4934.67 Our Answer

While this difference in amounts in Examples 3 and 4 is not very large, it is still a bit higher when the interest is compounded more often. The table below shows the result for the same problem but with different compounding periods.

Compounding	Balance
Annually	$4919.50
Semi-Annually	$4927.02
Quarterly	$4930.85
Monthly	$4933.42
Weekly	$4934.41
Daily	$4934.67

As the table illustrates, the more often interest is compounded, the higher the final balance because we are calculating interest on interest. So once interest is added into the account, it will start earning interest for the next compounding period and thus giving a higher final balance.

COUNTINUOUSLY COMPOUNDED INTEREST

The next question one might consider is what is the maximum number of compounding periods possible? We actually have a way to calculate interest compounded an infinite number of times a year. This is when the interest is compounded *continuously*. When we see the word continuously we will know that we cannot use the first formula. Instead we will use the following formula:

CONTINUOUSLY COMPOUNDED INTEREST FORMULA

$$A = Pe^{rt}$$

A = final amount
P = principal (starting balance)
e = a constant approximately equal to 2.71828183…
r = annual interest rate (as a decimal)
t = time (in years)

The number e is a constant similar in idea to the number π in that it goes on forever without repeat or pattern. Just as π naturally occurs in several geometry applications, e appears in many exponential applications, continuous interest being one of them. If you have a scientific calculator you probably have an e^x button (often using the 2nd or shift key, then hit the ln button) that will be useful in calculating interest compounded continuously.

Example 5.

If \$4000 is invested in an account paying 3% interest compounded continuously, what is the balance after 7 years?

	Identify information given:
	$P = 4000$, $r = 0.03$, $t = 7$
$A = 4000e^{0.03 \cdot 7}$	Plug each value into formula, multiply exponent
$A = 4000e^{0.21}$	Evaluate $e^{0.21}$
$A = 4000(1.23367806\ldots)$	Multiply
$A = 4934.71$	
\$4934.71	Our Answer

Consider the following example, illustrating how powerful compound interest can be.

Example 6.

If you invest \$6.16 in an account paying 12% interest compounded continuously for 100 years, and that is all you have to leave your children as an inheritance, what will the final balance be that they will receive?

	Identify information given:
	$P = 6.16$, $r = 0.12$, $t = 100$
$A = 6.16e^{0.12 \cdot 100}$	Plug each value into formula, multiply exponent
$A = 6.16e^{12}$	Evaluate e^{12}
$A = 6.16(162,754.79)$	Multiply
$A = 1,002,569.52$	
\$1,002,569.52	Our Answer

In 100 years that one time investment of \$6.16 investment grew to over one million dollars. That's the power of compound interest!

Practice Exercises
Section 5.5: Compound Interest

Find the balance when:

1) \$500 is invested at 4% compounded annually for 10 years.
2) \$600 is invested at 6% compounded annually for 6 years.
3) \$750 is invested at 3% compounded continuously for 8 years.
4) \$1500 is invested at 4% compounded semiannually for 7 years.
5) \$900 is invested at 6% compounded monthly for 5 years.
6) \$950 is invested at 4% compounded continuously for 12 years.
7) \$2000 is invested at 5% compounded quarterly for 6 years.
8) \$2250 is invested at 4% compounded daily for 9 years.
9) \$3500 is invested at 6% compounded continuously for 12 years.

Answer each question.

10) What principal will amount to \$2000 if invested at 4% interest compounded semiannually for 5 years?

11) What principal will amount to \$3500 if invested at 4% interest compounded quarterly for 5 years?

12) What principal will amount to \$3000 if invested at 3% interest compounded semiannually for 10 years?

13) What principal will amount to \$2500 if invested at 5% interest compounded semiannually for 7.5 years?

14) What principal will amount to \$1750 if invested at 3% interest compounded quarterly for 5 years?

15) A thousand dollars is left in a bank savings account drawing 7% interest, compounded quarterly for 10 years. What is the balance at the end of that time?

The Practice Exercises are continued on the next page.

Practice Exercises: Section 5.5 (continued)

Answer each question.

16) A thousand dollars is left in a credit union drawing 7% compounded monthly. What is the balance at the end of 10 years?

17) $1750 is invested in an account earning 13.5% interest compounded monthly for a 2 year period. What is the balance at the end of 2 years?

18) You lend out $5500 at 10% compounded monthly. If the debt is repaid in 18 months, what is the total owed at the time of repayment?

19) You borrow $25000 at 12.25% interest compounded monthly. If you are unable to make any payments the first year, how much do you owe, excluding penalties?

20) An 8.5% account earns continuous interest. If $2500 is deposited for 5 years, what is the total accumulated?

21) You lend $100 at 10% continuous interest. If you are repaid 2 months later, what is owed?

ANSWERS to Practice Exercises
Section 5.5: Compound Interest

1) $ 740.12

2) $ 851.11

3) $ 953.44

4) $ 1979.22

5) $ 1213.97

6) $ 1535.27

7) $ 2694.70

8) $ 3224.93

9) $ 7190.52

10) $ 1640.70

11) $ 2868.41

12) $ 2227.41

13) $ 1726.16

14) $ 1507.08

15) $ 2001.60

ANSWERS to Practice Exercises: Section 5.5 (continued)

16) $ 2009.66

17) $ 2288.98

18) $ 6386.12

19) $ 28240.43

20) $ 3823.98

21) $ 101.68

Review: Chapter 5

1. Evaluate $f(x)=x^2-3x+2$ for $f(-3)$.

2. Evaluate $g(x)=x^2-3$ for $g(x+3)$.

3. What is the domain of $f(x)=\sqrt{x+4}$?

4. What is the domain of $g(x)=\dfrac{x+4}{2x-14}$?

5. State if the following relation represents a function: $\{(0,1),(2,-5),(3,1),(5,0)\}$

6. If $f(x)=2x-5$ and $g(x)=x+2$, determine $(f+g)(-7)$.

7. If $g(n)=n-2$ and $f(n)=4n+3$, determine $(g\cdot f)(5)$.

8. If $f(x)=x+3$ and $g(x)=4x+2$, determine $(f-g)(x)$.

9. Solve: $4^{x+2}=256$.

10. Solve: $\left(\dfrac{1}{3}\right)^x=81$.

11. Rewrite in exponential form: $\log_3\dfrac{1}{9}=-2$

12. Rewrite in exponential form: $\log 100=2$

13. Rewrite in logarithmic form: $5^2=25$

14. Rewrite in logarithmic form: $4^{-3}=\dfrac{1}{64}$

15. Evaluate $\log_2 64$.

16. Evaluate $\log_6\dfrac{1}{216}$.

17. Evaluate $\ln\sqrt[5]{e^2}$.

18. Solve $\log_3 x = 0$.

19. Solve $\log_2 k = -3$.

20. $\log_4(6b - 8) = 1$.

21. Graph $f(x) = 3^x$

22. Graph $f(x) = \log_4 x$.

23. How much should be invested at 3.5%, compounded monthly, if you want \$3500 in 4 years? Round answer to the nearest cent.

24. If \$250 is invested at 8%, compounded continuously, what will the value be in 10 years? Round answer to the nearest cent.

ANSWERS to Review: Chapter 5

1) 20

2) x^2+6x+6

3) $\{x \mid x \geq -4\}$ or $[-4,\infty)$

4) $\{x \mid x \neq 7\}$ or $(-\infty,7)\cup(7,\infty)$

5) yes

6) -24

7) 69

8) $-3x+1$

9) $x=2$

10) $x=-4$

11) $3^{-2}=\dfrac{1}{9}$

12) $10^2=100$

13) $\log_5 25=2$

14) $\log_4 \dfrac{1}{64}=-3$

15) 6

16) -3

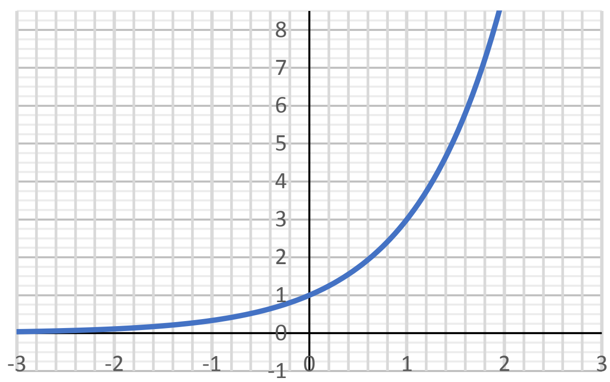
17) $\dfrac{2}{5}$

18) $x=1$

19) $k=\frac{1}{8}$

20) $b=2$

21)



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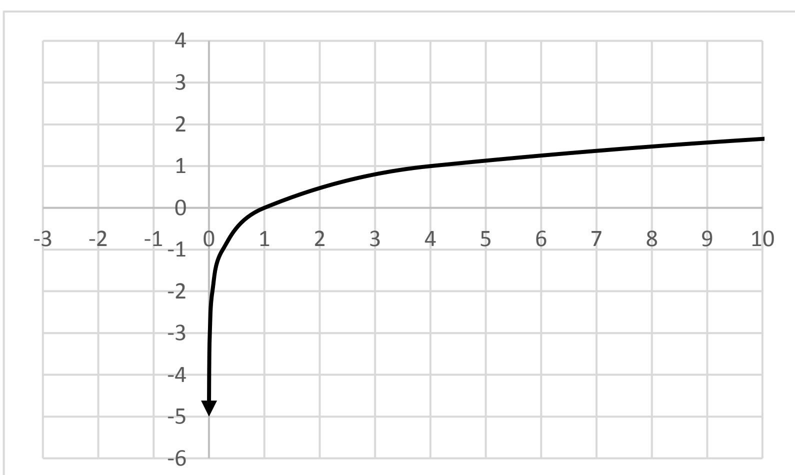

23) \$3043.37

24) \$556.39