2.5 Probability Using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions.

2.5.1 Examples

Example 2.5.1

A four digit PIN number is selected. What is the probability that there are no repeated digits?

There are ten possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of four digit PIN numbers with no repeated digits divided by the total number of four digit PIN numbers. This probability is $\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000}$ = 0.504.

Example 2.5.2

In horse racing, an exacta bet is one where the player tries to predict the top two finishers in a particular race in order. If there are 9 horses in a race, and a player decides to make an exacta bet at random, what is the probability that they win?

Since order matters for this situation, we'll use permutations. How many different exacta bets can be made? Since there are 9 horses and we must select 2 in order, we know there are 72 possible outcomes. That's the size of our sample space, so it will go in the denominator of the probability. Since only one of those outcomes is a winner, the numerator of the probability is 1. So, the probability of randomly selecting the winning exacta bet is $\frac{1}{72}$.

Try it Now 2.5.1

You are in a club with 10 people, 3 of whom are close friends of yours. If the officers of this club are chosen at random, what is the probability that you are named president and one of your friends is named vice president?

Try it Now 2.5.2
A bag contains slips of paper with letters written on them as follows: A, A, B, B, B, C, C, D, D, D, D, E. If you draw 3 slips, what is the probability that the letters will spell out (in order) the word BAD?

Example 2.5.3

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6$ = 12,271,512. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.00000000815.$$

Example 2.5.4

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6$ = 12,271,512. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5$ = 6 and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1$= 42. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \times {}_{42}C_1$ = 6 · 42 = 252. So the probability of winning the second prize is $\frac{{}_6C_5 \times {}_{42}C_1}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$

Try it Now 2.5.3
A multiple-choice question on an economics quiz contains ten questions with five possible answers each. Compute the probability of randomly guessing the answers and getting nine questions correct.

Example 2.5.5

Compute the probability of randomly drawing five cards from a deck and getting exactly one ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of five-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one ace and four other cards (none of them aces) from the deck. Since there are four aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one ace; since there are 48 non-aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \times {}_{48}C_4$ ways to choose one ace and four non-aces.

Putting this all together, we have $P(\text{one ace}) = \frac{{}_4C_1 \times {}_{48}C_4}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$.

Example 2.5.6

Compute the probability of randomly drawing five cards from a deck and getting exactly two aces.

The solution is similar to the previous example, except now we are choosing two aces out of four and three non-aces out of 48; the denominator remains the same:

$$P(\text{two aces}) = \frac{{}_4C_2 \times {}_{48}C_3}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399.$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 2.5.4

Compute the probability of randomly drawing five cards from a deck of cards and getting three aces and two kings.

Try it Now Answers

2.5.1. There are 10 people in the club, and 2 will be chosen to be officers. Since the order matters, there are ${}_{10}P_2 = 90$ different ways to select officers. Next, we must figure out how many outcomes are in our event. We'll use the Multiplication Rule for Counting to find that number. There is only 1 choice for president in our event, and there are 3 choices for vice president. So, there are 1×3=3 outcomes in the event. Thus, the probability that you will serve as president with one of your friends as vice president is $\frac{3}{90} = \frac{1}{30}$.

2.5.2. There are 12 slips of paper in the bag, and 3 will be drawn. So, there are ${}_{12}P_3 = 1320$ possible outcomes. Now, we'll compute the number of outcomes in our event. The first letter drawn must be a B, and there are 3 of those. Next must come an A (2 of those) and then a D (4 of those). Thus, there are $3 \times 2 \times 4 = 24$ outcomes in our event. So, the probability that the letters drawn spell out the word BAD is $\frac{24}{1320} = \frac{1}{55}$.

2.5.3. There are 5^{10} = 9,765,625 different ways the exam can be answered. There are ten possible locations for the one missed question and in each of those locations there are four wrong answers, so there are 40 ways the test could be answered with one wrong answer.

$$P(\text{9 answers correct}) = \frac{40}{9{,}765{,}625} \approx 0.000004096 \text{ chance}$$

2.5.4. $P(\text{three aces and two kings}) = \dfrac{({}_4C_3)({}_4C_2)}{{}_{52}C_5} = \dfrac{24}{2598960} \approx 0.0000092$

Section 2.5 Exercises

1. You own 16 CDs. You want to randomly arrange five of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

2. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.

3. In a lottery game, a player picks six numbers from 1 to 48. If five of the six numbers match those drawn, the player wins second prize. What is the probability of winning this prize?

4. In a lottery game, a player picks six numbers from 1 to 48. If four of the six numbers match those drawn, the player wins third prize. What is the probability of winning this prize?

5. Compute the probability that a five-card poker hand is dealt to you that contains all hearts.

6. Compute the probability that a five-card poker hand is dealt to you that contains four aces.

Section 2.5 Exercises – Answer Key

1) $\frac{1}{{}_{16}P_5} = \frac{1}{524160} \approx 0.000001908$

2) $\frac{{}_{14}C_{12}}{{}_{27}C_{12}} = \frac{91}{17383860} \approx 0.00000523$

3) $\frac{{}_6C_5 \times {}_{42}C_1}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$

4) $\frac{{}_6C_4 \times {}_{42}C_2}{{}_{48}C_6} = \frac{12915}{12271512} \approx 0.00105$

5) $\frac{{}_{13}C_5}{{}_{52}C_5} = \frac{1287}{2598960} \approx 0.0004952$

6) $\frac{{}_4C_4 \times {}_{48}C_1}{{}_{52}C_5} = \frac{48}{2598960} \approx 0.0000185$