3.3 Matrix Inverse and Matrix Equations

3.3.1 The Multiplicative Identity *I*

The number 1 is called the multiplicative identity for real numbers because any number x multiplied by 1 gives x as an answer (does not change the identity of x): $1 \cdot x = x \cdot 1 = x$. Whereas 1 is the multiplicative identity for real numbers, we designate the multiplicative identity for matrices by I. For example, the 2 x 2 identity matrix is as follows.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

I is a square matrix. It has 1s on the main diagonal and 0 entries elsewhere. I is to matrices as 1 is to real numbers. If we multiply a 2 x 2 matrix A by I, then we get the same matrix A for an answer. For example, suppose we multiply the 2 x 2 matrix below

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix}$$

by I. Verify that both $I \cdot A$ and $A \cdot I$ give A for an answer.

$$I \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + 0 \cdot (-3) & 1 \cdot 5 + 0 \cdot 4 \\ 0 \cdot 2 + 1 \cdot (-3) & 0 \cdot 5 + 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = A$$

$$A \cdot I = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 5 \cdot 0 & 2 \cdot 0 + 5 \cdot 1 \\ (-3) \cdot 1 + 4 \cdot 0 & (-3) \cdot 0 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = A$$

Therefore, $I \cdot A = A \cdot I = A$. The 3 x 3 and 4 x 4 identity matrices are shown below.

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_{4\times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.3.2 The Inverse of a Matrix

The multiplicative identity 1 plays a key role when solving equations. For example, solve the equation $4x = 12$.

$$\begin{aligned} 4x &= 12 \\ \frac{4x}{4} &= \frac{12}{4} && \text{Divide each side by 4.} \\ 1x &= 3 \\ x &= 3 && \text{Because } 1x = x. \end{aligned}$$

Now suppose A, X, and B are matrices. How would we solve the matrix equation

$$AX = B$$

for the matrix X? We cannot divide each side by the matrix A because *there is no division of matrices*!

To see what we do, let's return to basic algebra. How would we solve the equation *without division*? We would use multiplication. Instead of dividing each side by 4, we multiply each side by the **multiplication inverse** of 4, which is ¼. Look again at the solution of $4x = 12$.

$$\begin{aligned} 4x &= 12 & \\ \frac{1}{4}\cdot 4x &= \frac{1}{4}\cdot 12 & \text{Multiply each side by the multiplicative inverse of 4 or } 1/4. \\ 1x &= 3 & \text{Since } 1/4\cdot 4 = 1. \\ x &= 3 & \text{Because } 1x = x. \end{aligned}$$

The product of a number and its multiplicative inverse is 1. Verify on your graphing calculator that $\frac{1}{4}\cdot 4 = 1$. Also verify that $\frac{1}{4}\cdot 12 = 3$.

3.3.3 Notation for the Inverse of a Matrix

We introduce the notation A^{-1} because it is used when speaking about the inverse of a matrix. Just as $4^{-1} = \frac{1}{4}$ is the notation we use for the multiplicative inverse of 4, the multiplicative inverse of the matrix A is written A^{-1}. We read A^{-1} as "A inverse." Note that A^{-1}does not mean $\frac{1}{A}$. It simply denotes the inverse of A.

And just as $4^{-1}\cdot 4 = 1$, the product of the matrix A and its inverse A^{-1} is the identity matrix I.

$$A^{-1}\cdot A = I \qquad \text{and} \qquad A\cdot A^{-1} = I$$

Before we can verify these equations, we need a procedure for finding the inverse of a matrix.

3.3.4 Finding the Inverse of a Matrix A

Only square matrices have inverses. We will use the Gauss-Jordan Method to find the inverse A^{-1} of the matrix A, if it exists. We illustrate the process in the next two examples.

Example 3.3.1

Use the Gauss-Jordan Method to find the inverse of the 2 x 2 matrix A.

$$A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$$

Solution: We begin by forming the augmented matrix $(A|I)$ where A is on the left side of the vertical bar and the 2 x 2 identity matrix I is in the right side. Apply the three row operations to the left side of the bar until the identity matrix I is on the left side. We then get $(I|A^{-1})$. The inverse of A, if it exists, will be on the right side of the bar.

$$\underset{A \quad\quad I}{\left(\begin{array}{cc|cc} 5 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)} \xrightarrow[\text{to transform into}]{\text{apply the three row operations}} \underset{I \quad\quad A^{-1}}{\left(\begin{array}{cc|c} 1 & 0 & \text{inverse} \\ 0 & 1 & \text{matrix} \end{array}\right)}$$

The work is shown below.

$$\left(\begin{array}{cc|cc} 5 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow[\text{to Row 1 add } -2 \times \text{Row 2}]{\text{change 5 into a 1}} \left(\begin{array}{cc|cc} -1 & -10 & 1 & -2 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{Row 1 by } -1]{\text{multiply}} \left(\begin{array}{cc|cc} 1 & 10 & -1 & 2 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow[\text{to Row 2 add } -3 \times \text{Row 1}]{\text{change 3 into a 0}} \left(\begin{array}{cc|cc} 1 & 10 & -1 & 2 \\ 0 & -26 & 3 & -5 \end{array}\right)$$

$$\xrightarrow[\text{divide Row 2 by } -26]{\text{change } -26 \text{ into 1}} \left(\begin{array}{cc|cc} 1 & 10 & -1 & 2 \\ 0 & 1 & -\frac{3}{26} & \frac{5}{26} \end{array}\right) \xrightarrow[\text{to Row 1 add } -10 \times \text{Row 2}]{\text{change 10 into a 0}} \underset{I \quad\quad A^{-1}}{\left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{13} & \frac{1}{13} \\ 0 & 1 & -\frac{3}{26} & \frac{5}{26} \end{array}\right)}$$

Use your calculator to help you perform operations involving fractions.

From above, the inverse of the matrix $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$ is the matrix $A^{-1} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix}$. To check, multiply $A \cdot A^{-1}$ and show that the answer is the 2 x 2 identity matrix I.

$$A \cdot A^{-1} = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} = \begin{pmatrix} 5 \cdot \frac{2}{13} + (-2) \cdot -\frac{3}{26} & 5 \cdot \frac{1}{13} + (-2) \cdot \frac{5}{26} \\ 3 \cdot \frac{2}{13} + 4 \cdot -\frac{3}{26} & 3 \cdot \frac{1}{13} + 4 \cdot \frac{5}{26} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

You can verify that $A^{-1} \cdot A = I$.

Example 3.3.2

Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 3 & 1 & 5 \\ -1 & 1 & -2 \end{pmatrix}$.

Solution: Begin by writing the augmented matrix $(A|I)$. The matrix A is on the left side of the vertical bar and the 3 x 3 identity matrix I is on the right side.

$$\underbrace{\left(\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 3 & 1 & 5 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)}_{A \qquad\qquad I}$$

Apply the three row operations to reduce this matrix to the form $(I|A^{-1})$, where the 3 x 3 identity matrix I is on the left side of the vertical bar. Then the inverse of A, if it exists, will be on the right side.

$$\xrightarrow{R_1 \div 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 3 & 1 & 5 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} -3 \times R_1 + R_2 \to R_2 \\ R_3 + R_1 \to R_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -3/2 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1 \end{array}\right)$$

$$R_2 \leftrightarrow R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1 \\ 0 & 1 & -1 & -3/2 & 1 & 0 \end{array}\right)$$

$$-1 \times R_2 + R_3 \to R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & -1 \end{array}\right)$$

$$-1 \times R_3 \to R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$

$$-2 \times R_3 + R_1 \to R_1 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/2 & 2 & -2 \\ 0 & 1 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)$$

$$(I|A^{-1})$$

The left side of the vertical bar is the 3 x 3 identity matrix I and the right side is the inverse of the A matrix, which is

$$A = \begin{pmatrix} -\frac{7}{2} & 2 & -2 \\ \frac{1}{2} & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

You can verify that this is the inverse of A by showing that both $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are equal to I.

Example 3.3.3

Find the inverse A^{-1} of the matrix: $A = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$.

Solution: Write the augmented matrix $(A|I)$ and use the three row operations to reduce it to the form $(I|A^{-1})$.

$$(A|I) = \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -3 & 6 & 0 & 1 \end{array}\right)$$

$$3 \times R_1 + R_2 \rightarrow R_2 \quad \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

Because each entry in the bottom row to the left of the vertical bar is 0, we stop. The 0s tell us that the matrix A has no inverse. It is not possible to install the 2 x 2 identity matrix on the left side of the vertical bar. The answer is “A has no inverse.” A matrix that does not have an inverse is called a **singular** matrix.

Try it Now 3.3.1

Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$.

3.3.5 Solving the Matrix Equation $A \cdot X = B$

Suppose we are given the matrix equation $A \cdot X = B$, where X is a matrix of variables. We will solve for X by first finding the inverse A^{-1} of the matrix A. Then we multiply each side of the equation by A^{-1}, *on the left side*. Why on the left? Recall, what we do to one side of an equation we must also do to the other side. Because matrix multiplication is not commutative, if we multiply on the left on one side, we must also multiply on the left on the other side.

$A \cdot X = B$

$A^{-1} \cdot A \cdot X = A^{-1} \cdot B$ Multiply each side of the equation by A^{-1} on the left side.

$I \cdot X = A^{-1} \cdot B$ Because A^{-1} and A are inverses, we have $A^{-1} \cdot A = I$.

$X = A^{-1} \cdot B$ The matrix I is the multiplicative identity, so $I \cdot X = X$.

Therefore, **the solution of the matrix equation** $A \cdot X = B$ is $X = A^{-1} \cdot B$.

Or simply put the product $A^{-1} \cdot B$ is the solution of the equation $A \cdot X = B$. We will use matrix inverses and this procedure to solve systems of equations.

3.3.6 Solving Systems of Equations Using the Matrix Inverse

Example 3.3.4

Use matrices to solve the system of two equations in two unknowns shown here.

$5x - 2y = 12$
$3x + 4y = 2$

Solution: Use the coefficients to make the matrix $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$. Use the variables to make the matrix $X = \begin{pmatrix} x \\ y \end{pmatrix}$. Use the constants to make the matrix $B = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$. We may then write the system of equations in (1) as a matrix equation in the form $A \cdot X = B$ as follows.

$$A \quad \cdot \quad X \quad = \quad B$$

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix} \end{array} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

Coefficient Matrix A
Variable Matrix X
Constants Matrix B

The coefficient matrix A contains like coefficients in the same column. The x-coefficients are in the first column and the y-coefficients are in the second column. The top to bottom order of the variables in matrix X must be in the same order as the column labels in matrix A.

Note, we may transform the matrix equation back to the system of equations by multiplying $A \cdot X$ on the left side of the matrix equation. We get $\begin{pmatrix} 5x - 2y \\ 3x + 4y \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$. Each side of the equation is a 2 x 1 matrix. Since they are equal corresponding entries are equal. Therefore, the 1^{st} row 1^{st} column of each matrix gives us the first equation $5x - 2y = 12$ and the 2^{nd} row 1^{st} column in each gives us the second equation $3x + 4y = 2$.

To solve the matrix equation, first find the inverse of matrix A. We already did this in Example 1. The inverse is $A^{-1} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix}$. To solve the matrix equation, multiply each side of the matrix equation by this inverse. The entire solution is below.

$$A \quad \cdot \quad X \quad = \quad B$$

$$\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

Multiply each side by the inverse A^{-1} on the left side.

$$\begin{array}{ccccccc} A^{-1} & \cdot & A & \cdot & X & = & A^{-1} & \cdot & B \\ \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} & \cdot & \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} & \cdot & \begin{pmatrix} 12 \\ 2 \end{pmatrix} \end{array}$$

Verify for yourself that $A^{-1} \cdot A = I$ on the left side, so that we get

$$\begin{array}{ccccccc} I & \cdot & X & = & A^{-1} & \cdot & B \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} & \cdot & \begin{pmatrix} 12 \\ 2 \end{pmatrix} \end{array}$$

Verify on the left side that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and verify on the right side that $\begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{3}{26} & \frac{5}{26} \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Finally we get $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Since corresponding entries are equal, our final answer is $x = 2$ and $y = -1$.

Try it Now 3.3.2

Use matrices to solve the system of equations:

$5x + 3y = 18$

$3x + 2y = 11.$

3.3.7 Solving Larger Systems of Equations

We will use the procedure illustrated in the previous examples to solve larger systems of equations.

Example 3.3.5

Use matrices to solve the following system:

$5x + 2y - 3z = 10$

$2x - z = -4$

$-x + 3y + 2z = 5.$

Solution:

First, build the coefficient matrix A, the variable matrix X, and the constants matrix B.

$$A = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{pmatrix} 5 & 2 & -3 \\ 2 & 0 & -1 \\ -1 & 3 & 2 \end{pmatrix} \end{array} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B = \begin{pmatrix} 10 \\ -4 \\ 5 \end{pmatrix}$$

Second, write the matrix equation for this system in the form $A \cdot X = B$.

$$\underset{A}{\begin{pmatrix} 5 & 2 & -3 \\ 2 & 0 & -1 \\ -1 & 3 & 2 \end{pmatrix}} \cdot \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 10 \\ -4 \\ 5 \end{pmatrix}}$$

Third, find the inverse of A using either the Gauss-Jordan Method, or your scientific calculator. We get

$$A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{13}{9} & \frac{2}{9} \\ \frac{1}{3} & -\frac{7}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{17}{9} & \frac{4}{9} \end{pmatrix}$$

Fourth, calculate the solution of the system, which is given by $X = A^{-1} \cdot B$.

$$\underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{A^{-1}}{\begin{pmatrix} -\frac{1}{3} & \frac{13}{9} & \frac{2}{9} \\ \frac{1}{3} & -\frac{7}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{17}{9} & \frac{4}{9} \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} 10 \\ -4 \\ 5 \end{pmatrix}} = \begin{pmatrix} -8 \\ 7 \\ -12 \end{pmatrix}$$

$$\text{Or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ 7 \\ -12 \end{pmatrix}$$

Therefore, $x = -8$, $y = 7$, and $z = -12$.

You can check your answer by substituting these values into each of the original three equations (and can use the calculator to speed up calculations).

To use matrices to solve a system of equations, we must first write each equation in standard form, similar to the approach in the previous section. That is, write the variables on the left side of the equals sign and the constants on the right side.

Example 3.3.6

Leslie plans to retire in ten years. She has $100,000 in savings. She plans to invest this money in relatively safe areas. She wants to earn interest income of $7,000 each year. To diversify, she plans to invest in four areas:

Stocks earning 10% annually
A rated bonds earning 7% annually
AAA rated bonds earning 5.5% annually
CDs earing 3% annually

The amount she invests in stocks will equal the total invested in AAA bonds and CDs. Also, she wants to invest $5,000 more in AAA bonds than in A bonds. How much should Leslie invest in each of the four areas?

Solution: *First, build a table* summarizing all of the given information. Since we want to invest a portion of the $100,000 in four different areas, we will need four variables.

Investment Type	**Amount to Invest**	**Annual Rate**	**Annual Interest Earned**
Stocks	x	10%	$0.10 \cdot x$
A bonds	y	7%	$0.07 \cdot y$
AAA bonds	z	5.5%	$0.055 \cdot z$
CDs	w	3%	$0.03 \cdot w$
Totals	$100,000	-	$7,000

amount in stocks = amount in AAA bounds + amount in CDs
invest $5,000 more in AAA bonds than in A bonds

Note, *x* dollars of the $100,000 total is invested in stocks. Since stocks earn 10% per year, the annual interest earned from stocks of 10% of *x* or $0.10 \cdot x$ dollars.

Second, build equations using the given information. Two columns in the table give us equations. The "Amount to Invest" column gives us one equation. Since the amounts invested in each of the four areas must add up to $100,000, we have

$$x + y + z + w = 100{,}000$$

The "Annual Interest Earned" column gives us a second equation. Since the sum of the interests earned in each of the four investments must total $7,000, we have

$$0.10x + 0.07y + 0.055z + 0.03w = 7{,}000$$

The next two equations come from conditions Leslie imposed on her investments. She said the following.

$$\begin{array}{ccccc} amount\ in\ stocks & = & amount\ in\ AAA\ bonds & + & amount\ in\ CDs \\ x & = & z & + & w \end{array}$$

or $x = z + w$. To use this equation in a matrix we must first write it in standard form (variables on the left side and constants on the right side). To do this subtract *z* from each side and subtract *w* from each side. We get

$$x - z - w = 0$$

The last equation comes from Leslie's final condition. She said the following.

invest $5,000 more in AAA bonds than in A bonds

In other words, z (amount to invest in AAA bonds) is $5,000 more than y (amount to invest in A bonds). So if we subtract $5,000 from z we get y.

$$z - 5{,}000 = y$$

To put this equation in standard form, add $5,000 to each side and subtract y from each side. We get

$$-y + z = 5{,}000$$

In summary, we have the following system of four equations in four unknowns.

$x + y + z + w = 100{,}000$
$0.10x + 0.07y + 0.055z + 0.03w = 7{,}000$
$x - z - w = 0$
$-y + z = 5000$

We will solve this system using matrix equations. *Third, write a matrix equation in the form* $A \cdot X = B$. Recall A is the matrix of coefficients. The first column is x coefficients, the second column is y coefficients, etc. The matrix of variables is X and the matrix of constants is B.

$$\begin{array}{ccccc} A & \cdot & X & = & B \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0.10 & 0.07 & 0.055 & 0.03 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} & = & \begin{pmatrix} 100000 \\ 7000 \\ 0 \\ 5000 \end{pmatrix} \end{array}$$

Find the inverse of A matrix using either the Gauss-Jordan Method or the calculator. We get

$$A^{-1} = \begin{pmatrix} \frac{19}{12} & -\frac{50}{3} & \frac{13}{12} & \frac{5}{12} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & -\frac{5}{6} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & \frac{1}{6} \\ \frac{15}{4} & -50 & \frac{5}{4} & \frac{1}{4} \end{pmatrix}$$

The solution of the system is $X = A^{-1} \cdot B$.

$$\begin{array}{ccccc} X & = & A^{-1} & \cdot & B \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{19}{12} & -\frac{50}{3} & \frac{13}{12} & \frac{5}{12} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & -\frac{5}{6} \\ -\frac{13}{6} & \frac{100}{3} & -\frac{7}{6} & \frac{1}{6} \\ \frac{15}{4} & -50 & \frac{5}{4} & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 100000 \\ 7000 \\ 0 \\ 5000 \end{pmatrix} = \begin{pmatrix} 43750 \\ 12500 \\ 17500 \\ 26250 \end{pmatrix}$$

The solution is $x = 43750$, $y = 12500$, $z = 17500$, and $w = 26250$.

Leslie should invest $43,750 in stocks, $12,500 in A rated bonds, $17,500 in AAA rated bonds, and $26,250 in CDs in order to make $7,000 in interest earnings annually.

To check, return to the original word problem and make sure the answers satisfy all the conditions. For example, the sum of the four amounts $43,750 + $12,500 + $17,500 + $25,250 = $100,000, the total she had to invest. Take time and check the other conditions yourself.

If the answer $X = A^{-1} \cdot B$ in Example 6 contained a negative number, then there would be no practical solution to Leslie's investment plan. We cannot invest a negative amount. Leslie would have to change at least one of her conditions and start over again.

Try it Now Answers

3.3.1.

$$(A|I) = \left(\begin{array}{cc|cc} 5 & -2 & 1 & 0 \\ -7 & 3 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ -7 & 3 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{7}{5} & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{7}{5} & 5 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & 7 & 5 \end{array}\right) = (I|A^{-1})$$

So, therefore the inverse matrix is $A^{-1} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$.

3.3.2. Convert the system

$$5x + 3y = 18$$
$$3x + 2y = 11$$

into matrices.

$$A \cdot X = B$$
$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 11 \end{pmatrix}$$

Now find the inverse of matrix A.

$$A^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

Calculate the solution of the system which is given by $X = A^{-1} \cdot B$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 11 \end{pmatrix} = \begin{pmatrix} (2 \cdot 18) + (-3 \cdot 11) \\ (-3 \cdot 18) + (5 \cdot 11) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Therefore,$x = 3$ and $y = 1$.

Section 3.3 Exercises

In problems, 1-4, use matrix multiplication to decide if the matrices are inverses of each other (check to see if their product is the identity matrix I).

1. $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

In problems 5-10, find the inverse of each matrix, if it exists. Express all entries in fraction form.

5. $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

6. $\begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix}$

7. $\begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

9. $\begin{pmatrix} 1 & -2 & 3 & 0 \\ -2 & 2 & -2 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$

10. $\begin{pmatrix} -2 & -1 & 3 & 1 \\ 1 & 2 & -2 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}$

In problems 11-21, solve each system of equations by finding the inverse of the coefficient matrix.

11.

$2x + 3y = 10$
$x - y = -5$

12.

$3x - 5y = -1$
$2x + 3y = 12$

13.

$4x + 5y = 9$
$3x - 2y = -2$

14.

$5x - 3y = 14$
$3x + 4y = 6$

15.

$2x + y = 3$
$-4x + 2y = 1$

16.

$-x - 4y = 1$
$3x - 12y = 5$

17.

$4x + 5y = -2$
$x + y + z = -1$
$y - 3z = 3$

18.

$x + 2y + 3z = 2$
$x + 4y + 3z = -8$
$y - z = -4$

19. $2x - 3y + 4z = 5$
$3x + y - 2z = -3$
$4x + 2y + 3z = 1$

20. **Bond Investment**: An investment company recommends that a client invest in AAA, AA, and A rated bonds. The average annual yield on AAA bonds is 6%, on AA bonds 7%, and on A bonds 10%. The client tells the company she wants to invest twice as much in AAA bonds as in A bonds. How much should be invested in each type of bond under the following conditions?

    a) The total investment is $50,000 and the investor wants an annual income (that is, earned interest) of $3,620.
    b) The total investment is $15,000 yielding an annual income of $1,075.
    c) The total investment is $21,475 yielding $1,630 annual income. Is this workable? Explain. What happens if we replace $1,630 by $1,550?

Section 3.3 Exercises – Answer Key

1. Yes
2. No
3. No
4. Yes
5. $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$
6. $\begin{pmatrix} -5 & 2 \\ -3 & 1 \end{pmatrix}$
7. $\begin{pmatrix} \frac{2}{11} & -\frac{1}{22} & -\frac{7}{11} \\ \frac{2}{11} & -\frac{1}{22} & \frac{4}{11} \\ \frac{1}{11} & \frac{5}{22} & \frac{2}{11} \end{pmatrix}$
8. $\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{9}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$
9. $\begin{pmatrix} 3 & 1 & 3 & -2 \\ 4 & 2 & 3 & -1 \\ 2 & 1 & 1 & 0 \\ -2 & -1 & -2 & 2 \end{pmatrix}$
10. $\begin{pmatrix} -\frac{4}{5} & -\frac{6}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{3}{10} & \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{5} & -\frac{7}{10} & -\frac{3}{10} & \frac{11}{10} \end{pmatrix}$
11. $x = -1, y = 4$
12. $x = 3, y = 2$
13. $x = \frac{8}{23}, y = \frac{35}{23}$
14. $x = \frac{74}{29}, y = -\frac{12}{29}$

15. $x = \frac{5}{8}, y = \frac{7}{4}$

16. $x = \frac{1}{3}, y = -\frac{1}{3}$

17. $x = -8, y = 6, z = 1$

18. $x = 15, y = -5, z = -1$

19. $x = -\frac{14}{73}, y = -\frac{39}{73}, z = \frac{69}{73}$

20.
   a. $24,000 in AAA bonds, $14,000 in AA bonds, $12,000 in A bonds.
   b. $5,000 in AAA bonds, $7,500 in AA bonds, $2,500 in A bonds.
   c. No, it is not possible to have a total investment of $21,475 yielding $1,630 annual income because the amounts would have to be $25,350 in AAA bonds, -$16,550 in AA bonds, and $12,675 in A bonds. It is not possible to invest a negative amount of money. It is possible if the annual income is $1,550. The amounts would have to be $9,350 in AAA bonds, $7,450 in AA bonds, and $4,675 in A bonds.