4.3 Linear Programming – The Simplex Method

World View Note: George Dantzig invented the field of linear programming and it revolutionized the way government and private enterprise conducted business. In 1947, he invented the simplex method to efficiently find the optimal solution for linear programming problems.

The **simplex method** is an alternate method to graphing that can be used to solve linear programming problems—particularly those with more than two variables. We first list the algorithm for the simplex method, and then we examine a few examples.

1. Setup the problem. That is, write the objectives functions and constraints.
2. Convert the inequalities into equations. This is done by adding one *slack variable* to each inequality. Set the objective function equal to zero.
3. Construct the initial simplex tableau. Write the objective function as the bottom row.
4. We will pivot, as done in the Gauss Jordan method. To determine the pivot column, identify the most negative entry in the bottom row.
5. To identify the pivot row, calculate the quotients of the far right column and those in the pivot column (excluding the bottom row for the objective function). The smallest positive quotient identifies the pivot row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element. A quotient that is zero, negative, or undefined is not considered.
6. Perform pivoting to make this pivot element a one and all other entries in this column zero. This is done the same way as we did with the Gauss-Jordan method.
7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.
8. Extract the solution. The optimal solution is in the bottom right corner. Variables whose columns did not produce a single 1 and remaining zeroes are assumed to be zero. For columns that have a 1 and all other entries 0, the value of the variable is in the far right column.

Example 4.3.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours per week. She has determined that for every hour she works at Job I, she needs 2 hours or preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution:

In solving this problem, we will use the algorithm list above.

(Step 1) Setup the problem. That is, write the objective functions and constraints. Therefore, let x represent the number of hours per week Niki will work at Job I and y be the number of hours per week Niki will work at Job II. We are trying to maximize income, so we will label the variable to be maximized as I.

$$\begin{aligned} &\text{Maximize } I = 40x + 30y \text{ subject to} \\ &x + y \leq 12 \\ &2x + 1y \leq 16 \\ &x \geq 0 \\ &y \geq 0 \end{aligned}$$

(Step 2) Convert the inequalities into equations. This is done by adding one slack variable for each inequality. For example, to convert the inequality $x + y \leq 12$ into an equation, we add a non-negative variable (usually *s*, *t*, or *u*) and we get:

$$x + y + s = 12$$

Here the variable *s* picks up the slack, and it represents the amount by which $x + y$ falls short of 12. (For example, if Niki works fewer than 12 hours, say 10, then *s* is 2.) Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

$$2x + y + t = 16$$

We will set the objective function equal to 0 by subtracting 40*x* and 30*y* from both sides. We get:

$$I - 40x - 30y = 0$$

(Step 3) We are now ready to setup the initial simplex tableau. The objective function is represented in the bottom row. The initial simplex tableau follows.

x	*y*	*s*	*t*	*I*	
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

(Step 4) The most negative entry in the bottom row identifies the pivot column.

x	*y*	*s*	*t*	*I*	
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0
↑					

-40 is the most negative entry, so it identifies the pivot column.

(Step 5) Calculate the quotients of the final column and pivot column. The smallest positive quotient identifies the pivot row. Quotients that are zero, negative, or undefined are not considered.

x	*y*	*s*	*t*	*I*		
1	1	1	0	0	12	$12 \div 1 = 12$
2	1	0	1	0	16	$\leftarrow 16 \div 2 = 8$
-40	-30	0	0	1	0	
↑						

The smallest positive quotient is 8, so this identifies the pivot row. Therefore, the pivot entry is 2.

(Step 6) Begin pivoting as done in the Gauss-Jordan method. The pivot entry becomes 1 and all entries above and below become 0. We will start by dividing R2 by 2 so the pivot entry becomes 1.

x	*y*	*s*	*t*	*I*	
1	1	1	0	0	12
1	1/2	0	1/2	0	8
-40	-30	0	0	1	0

To get zeroes above and below the pivot entry, we will replace R1 with R1 – R2 and we will replace R3 with R3 + 40R2.

x	*y*	*s*	*t*	*I*	
0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

(Step 7) Since a negative number still exists in the bottom row, we repeat steps 4-6.

-10 is the most negative number in the bottom row, so it identifies the pivot column. Calculating the quotients of the final column and pivot column gives 8 and 16. Since 8 is the smallest positive quotient, it identifies the pivot row.

x	*y*	*s*	*t*	*I*		
0	1/2	1	-1/2	0	4	$\leftarrow 4 \div \frac{1}{2} = 8$
1	1/2	0	1/2	0	8	$8 \div \frac{1}{2} = 16$
0	-10	0	20	1	320	
	↑					

Therefore, 1/2 is the pivot entry. We need to convert this to 1 and all entries below to 0.

x	*y*	*s*	*t*	*I*	

0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

To change the pivot entry to 1, we will multiply R1 by 2/1, or simply 2.

x	y	s	t	I	
0	1	2	-1	0	8
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

To get zeroes below the pivot entry, we will replace R2 with R2 – $\frac{1}{2}$R1 and we will replace R3 with R3 + 10R1.

x	y	s	t	I	
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400

We no longer have negative entries in the final row! Now step 7 is complete. If a negative number was present in the bottom row, we would repeat steps 4-7 again.

(Step 8) Extract the solution. Look for columns that have a 1 and all other entries 0. Our solution is $y = 8$, $x = 4$, and $I = 400$. All other variables whose columns did not produce a single 1 and remaining zeroes are assumed to be zero. Therefore, $s = 0$ and $t = 0$. This means that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to $400.

Example 4.3.2

Solve using the simplex method.

Maximize $Z = x + 2y + z$ subject to $x + y \leq 16$

$$x + 3z \leq 36$$
$$5y + z \leq 100$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

Solution:

(Step 1) This problem is already setup for us.

(Step 2) Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

$$x + y + s = 16$$
$$x + 3z + t = 36$$

$$5y + z + u = 100$$

Next, we set the objective function equal to 0.

$$Z - x - 2y - z = 0$$

(Step 3) We are now ready to setup the initial simplex tableau. The objective function is represented in the bottom row. The initial simplex tableau follows.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
0	5	1	0	0	1	0	100
-1	-2	-1	0	0	0	1	0

(Step 4) The most negative entry in the bottom row identifies the pivot column.

x	y	z	s	t	u	Z	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
0	5	1	0	0	1	0	100
-1	-2	-1	0	0	0	1	0
	↑						

-2 is the most negative entry, so it identifies the pivot column.

(Step 5) Calculate the quotients of the final column and pivot column. The smallest positive quotient identifies the pivot row. Quotients that are zero, negative, or undefined are not considered.

x	y	z	s	t	u	Z		
1	1	0	1	0	0	0	16	←16/1=16
1	0	3	0	1	0	0	36	36/0 undefined
0	5	1	0	0	1	0	100	100/5=20
-1	-2	-1	0	0	0	1	0	
	↑							

The smallest positive quotient is 16, so this identifies the pivot row. Therefore, the pivot entry is 1.

(Step 6) Begin pivoting as done in the Gauss-Jordan method. The pivot entry becomes 1 and all entries above and below become 0. Since the pivot entry is already 1, we work on making the entries below 0.

We will replace R3 with R3 – 5R1, and we will replace R4 with R4 + 2R1.

x	*y*	*z*	*s*	*t*	*u*	*Z*	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
-5	0	1	-5	0	1	0	20
1	0	-1	2	0	0	1	32

(Step 7) Since a negative number still exists in the bottom row, we repeat steps 4-6.

-1 is the most negative number (and only negative number) in the bottom row, so it identifies the pivot column. Calculating the quotients of the final column and pivot column gives 12 and 20. Since 12 is the smallest positive quotient, it identifies the pivot row.

x	*y*	*z*	*s*	*t*	*u*	*Z*		
1	1	0	1	0	0	0	16	16/0 undefined
1	0	3	0	1	0	0	36	← 36/3=12
-5	0	1	-5	0	1	0	20	20/1=20
1	0	-1	2	0	0	1	32	
		↑						

Therefore, 3 is the pivot entry. We need to convert this to 1 and all entries below to 0.

x	*y*	*z*	*s*	*t*	*u*	*Z*	
1	1	0	1	0	0	0	16
1	0	3	0	1	0	0	36
-5	0	1	-5	0	1	0	20
1	0	-1	2	0	0	1	32

To change the pivot entry to 1, we will multiply R1 by 1/3 (or divide by 3).

x	*y*	*z*	*s*	*t*	*u*	*Z*	
1	1	0	1	0	0	0	16
1/3	0	1	0	1/3	0	0	12
-5	0	1	-5	0	1	0	20
1	0	-1	2	0	0	1	32

We already have 0 above the pivot entry. To get zeroes below the pivot entry, we will replace R3 with R3 –R2 and we will replace R4 with R4 + R2.

x	*y*	*z*	*s*	*t*	*u*	*Z*	
1	1	0	1	0	0	0	16
1/3	0	1	0	1/3	0	0	12
-16/3	0	0	-5	-1/3	1	0	8

4/3	0	0	2	1/3	0	1	44

We no longer have negative entries in the final row! Now step 7 is complete. If a negative number was present in the bottom row, we would repeat steps 4-7 again.

(Step 8) Extract the solution. Look for columns that have a 1 and all other entries 0. This gives $y = 16, z = 12, u = 8$, and $Z = 44$. All other variables whose columns did not produce a single 1 and remaining zeroes are assumed to be zero. Therefore, $x = 0, s = 0$, and $t = 0$. For our model's final solution, we are only concerned about the original variables (not the slack variables).Therefore, Z attains a maximum value of 44 when $x = 0, y = 16$, and $z = 12$.

Try it Now 4.3.1

Solve using the simplex method.

Maximize $\boldsymbol{R = 2x + 3y + z}$ subject to

$$\begin{aligned} &\boldsymbol{x + y + z \leq 40} \\ &\boldsymbol{2x + y - z \leq 10} \\ &\boldsymbol{x - y \leq 10} \\ &\boldsymbol{x \geq 0} \\ &\boldsymbol{y \geq 0} \\ &\boldsymbol{z \geq 0} \end{aligned}$$

Try it Now Answer

4.3.1. A maximum of $R = 90$ is attained when $\boldsymbol{x = 0, y = 25, z = 15.}$

Section 4.3 Exercises

Solve using the simplex method.

1. Maximize $R = 2x + 4y$ subject to $x + y \leq 6$
$x + 3y \leq 12$
$x \geq 0$
$y \geq 0$

2. Maximize $R = 5x + 3y$ subject to $x + y \leq 12$
$2x + y \leq 16$
$x \geq 0$
$y \geq 0$

3. Maximize $Z = 5x + 8y$ subject to $x + 2y \leq 30$
$3x + y \leq 30$
$x \geq 0$
$y \geq 0$

4. Maximize $T = 2x + y + z$ subject to $x + y + 2z \leq 8$
$2x + z \leq 14$
$x \geq 0$
$y \geq 0$
$z \geq 0$

5. Maximize $Z = x + 2y + 3z$ subject to $x + y + z \leq 12$
$2x + y + 3z \leq 18$
$x \geq 0$
$y \geq 0$
$z \geq 0$

6. Maximize $P = x + 2y + z$ subject to $x - 3y + z \leq 3$
$y + z \leq 10$
$x \geq 0$
$y \geq 0$
$z \geq 0$

7. A factory manufactures chairs, tables, and bookcases each requiring he use of three operations: cutting, assembling, and finishing. The first operation (cutting) can be used at most 600 hours, the second (assembling) at most 500 hours, and the third (finishing) at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing. A table requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing. A bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of

finishing. If the profit is $20 per chair, $30 per table, and $25 per bookcase, how many of each item should be manufactured to maximize profit?

Section 4.3 Exercises – Answer Key

1. $R = 18, x = 3, y = 3$
2. $R = 44, x = 4, y = 8$
3. $Z = 126, x = 6, y = 12$
4. $T = 15, x = 7, y = 1, z = 0$
5. $Z = 27, x = 0, y = 9, z = 3$
6. $P = 53, x = 33, y = 10, z = 0$
7. Maximum profit of $8,500 is attained when 0 chairs, 200 tables and 100 bookcases are manufactured.