Use the completed handout to complete the notes.

Watch this video about the Ratio test:

The Ratio Test

The *Ratio Test* is a test for absolute convergence.

Theorem: Let $\sum a_i$ be a series with nonzero terms.

1. $\sum a_i$ **converges** absolutely if $\boxed{\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| < 1}$

2. $\sum a_i$ **diverges** if $\boxed{\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| > 1}$

3. The *Ratio Test* is **inconclusive** if $\boxed{\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = 1}$

Watch this video of a problem similar to Exercise 1:

Exercise 1: Determine whether $\sum_{i=0}^{\infty} \frac{i^3 4^{i+1}}{5^i}$ **converges** or **diverges**.

$$\left|\frac{a_{i+1}}{a_i}\right| = \left|\frac{\frac{(i+1)^3 4^{i+2}}{5^{i+1}}}{\frac{i^3 4^{i+1}}{5^i}}\right| = \frac{(i+1)^3 4^{i+2}}{5^{i+1}} \cdot \frac{5^i}{i^3 4^{i+1}} = \frac{(i+1)^3}{i^3} \cdot \frac{5^i}{5^{i+1}} \cdot \frac{4^{i+2}}{4^{i+1}}$$

$$= \left(\frac{i+1}{i}\right)^3 \cdot \frac{1}{5} \cdot \frac{4}{1} = \frac{4}{5}\left(\frac{i+1}{i}\right)^3 \quad \text{Therefore, } \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\frac{4}{5}\left(\frac{i+1}{i}\right)^3$$

$$= \frac{4}{5}\lim_{i\to\infty}\left(\frac{i+1}{i}\right)^3 = \frac{4}{5}\left(\lim_{i\to\infty}\frac{i+1}{i}\right)^3 = \frac{4}{5}(1)^3 = \frac{4}{5} < 1 \Rightarrow \sum_{i=0}^{\infty}\frac{i^3 4^{i+1}}{5^i} \text{ converges (absolutely)}$$

Watch this video of a problem similar to Exercise 2:

Exercise 2: Determine whether $\sum_{i=1}^{\infty} \frac{i^i}{i!}$ **converges** or **diverges**.

$$\left|\frac{a_{i+1}}{a_i}\right| = \left|\frac{\frac{(i+1)^{i+1}}{(i+1)!}}{\frac{i^i}{i!}}\right| = \frac{(i+1)^{i+1}}{(i+1)!} \cdot \frac{i!}{i^i} = \frac{(i+1)^i(i+1)^1}{i!\,(i+1)} \cdot \frac{i!}{i^i} = \frac{(i+1)^i\cancel{(i+1)^1}}{\cancel{i!}\,\cancel{(i+1)}} \cdot \frac{\cancel{i!}}{i^i}$$

$$= \frac{(i+1)^i}{i^i} = \left(\frac{i+1}{i}\right)^i = \left(1+\frac{1}{i}\right)^i \quad \text{Therefore, } \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\left(1+\frac{1}{i}\right)^i = e > 1$$

See Handout 8.7
Page 4
Exercise 6

Thus, $\sum_{i=1}^{\infty}\frac{i^i}{i!}$ diverges.

Exercise 3: Determine whether $\sum_{i=0}^{\infty} \frac{(-1)^i \sqrt{i}}{i+1}$ **converges** or **diverges**.

$$\left|\frac{b_{i+1}}{b_i}\right| = \left|\frac{\frac{(-1)^{i+1}\sqrt{i+1}}{i+1+1}}{\frac{(-1)^i\sqrt{i}}{i+1}}\right| = \frac{\left|(-1)^{i+1}\right|\cdot\left|\sqrt{i+1}\right|}{\left|i+1+1\right|}\ \frac{\left|i+1\right|}{\left|(-1)^i\right|\cdot\left|\sqrt{i}\right|} = \boxed{\frac{1\cdot\sqrt{i+1}}{i+2}\cdot\frac{i+1}{1\cdot\sqrt{i}} = \sqrt{\frac{i+1}{i}}\cdot\frac{i+1}{i+2}}$$

Therefore, $\lim_{i\to\infty}\left|\frac{b_{i+1}}{b_i}\right| = \lim_{i\to\infty} \boxed{\sqrt{\frac{i+1}{i}}\cdot\frac{i+1}{i+2}} = \lim_{i\to\infty}\boxed{\sqrt{\frac{i+1}{i}}}\ \lim_{i\to\infty}\boxed{\frac{i+1}{i+2} = 1\cdot 1 = 1}$

Thus, the *Ratio Test* is $\boxed{\text{inconclusive}}$. We need to try a different test: the $\boxed{\text{Alternating Series Test}}$

$\sum_{i=0}^{\infty}\frac{(-1)^i\sqrt{i}}{i+1} = \sum_{i=1}^{\infty}(-1)^i a_i$ where $a_i = \boxed{\frac{\sqrt{i}}{i+1}}$

AST Condition 1: $\lim_{i\to\infty} a_i = 0$ Proof 1: Let $f(x) = \boxed{\frac{\sqrt{x}}{x+1}\ .\quad \lim_{x\to\infty}\frac{\sqrt{x}}{x+1} \to \frac{\infty}{\infty}\ \text{indeterminate}}$

$$\boxed{\overset{\text{L'Hôp.}}{=} \lim_{x\to\infty}\frac{(\sqrt{x})'}{(x+1)'} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{1} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}} = 0.\ \text{Therefore, } \lim_{x\to\infty}\frac{\sqrt{i}}{i+1} = 0 \text{ also.}}$$

AST Condition 2: $a_i \geq a_{i+1}$ Proof 2: Again, let $f(x) = \frac{\sqrt{x}}{x+1}$.

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}\cdot 1}{(x+1)^2} = \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}\cdot\frac{\frac{2\sqrt{x}}{1}}{\frac{2\sqrt{x}}{1}} = \frac{x+1-2x}{(x+1)^2\cdot 2\sqrt{x}} = \frac{1-x}{(x+1)^2\cdot 2\sqrt{x}}$$

If $x>1$, then $1-x<0$. $(x+1)^2>0$ and $2\sqrt{x}>0$ are true as well for $x>1$.

Thus, for $x>1$, $f'(x) = \frac{\ominus}{\oplus\cdot\oplus} = \ominus$

That is, $f'(x)<0$ for $x>1$. $\Rightarrow f(x)$ is decreasing for $x>1$

$\Rightarrow f(i) = a_i$ is a decreasing sequence for $i>1$

$\Rightarrow a_i \geq a_{i+1}$ for $i>1$

Both conditions of the Alternating Series Test have been met, so $\sum_{i=0}^{\infty}\frac{(-1)^i\sqrt{i}}{i+1}$ converges.

Watch this video about the Root test:

The Root Test

The *Root Test* works especially well for series involving n^{th} powers.

Theorem: Let $\sum a_i$ be a series with nonzero terms.

1. $\sum a_i$ **converges** absolutely if $\boxed{\lim_{i\to\infty} \sqrt[i]{|a_i|} < 1}$
2. $\sum a_i$ **diverges** if $\boxed{\lim_{i\to\infty} \sqrt[i]{|a_i|} > 1}$ *or* $\boxed{\lim_{i\to\infty} \sqrt[i]{|a_i|} = \infty}$
3. The *Root Test* is inconclusive if $\boxed{\lim_{i\to\infty} \sqrt[i]{|a_i|} = 1}$

Exercise 4a: Use the *Ratio Test* to determine whether $\sum_{i=1}^{\infty} \frac{e^{2i}}{i^i}$ **converges** or **diverges**.

$$\left|\frac{a_{i+1}}{a_i}\right| = \left|\frac{\frac{e^{2(i+1)}}{(i+1)^{i+1}}}{\frac{e^{2i}}{i^i}}\right| = \frac{e^{2i+2}}{(i+1)^{i+1}} \cdot \frac{i^i}{e^{2i}} = \frac{e^{2i}\cdot e^2}{e^{2i}} \cdot \frac{i^i}{(i+1)^i(i+1)} = e^2\left(\frac{i}{i+1}\right)^i \cdot \frac{1}{i+1}$$

$$\text{Therefore, } \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\left[e^2\left(\frac{i}{i+1}\right)^i\cdot\frac{1}{i+1}\right] = e^2\cdot\frac{1}{e}\cdot 0 = 0$$

By the Ratio Test, $\sum_{i=1}^{\infty} \frac{e^{2i}}{i^i}$ converges.

Watch this video of a problem similar to Exercise 4b:

Exercise 4b: Use the *Root Test* to determine whether $\sum_{i=1}^{\infty} \frac{e^{2i}}{i^i}$ **converges** or **diverges**.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|} = \sqrt[n]{\frac{(e^2)^n}{n^n}} = \frac{e^2}{n}$$

$$\text{Therefore, } \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{e^2}{n} = 0$$

By the Root Test, $\sum_{i=1}^{\infty} \frac{e^{2i}}{i^i}$ converges.

More Videos and Resources:

Video 1 Video 2 Video 3 Video 4

Textbook Exercises: Section 9.6

Problems: 1-6, 19, 23, 27, 29, 33, 35, 37, 43, 49, 51, 55, 57, 63, 67, 69, 87, 91

Textbook Exercises Videos: Section 9.6

Problem 9 Problem 39 Problem 59

Textbook PowerPoints Slides: Section 9.6

View a summary of the textbook reading in PowerPoint form.

Hyperlinks:

- Completed Handout: https://cwoer.ccbcmd.edu/math/math252/m252c9s6sol.pdf
- Video about the Ratio Test: http://www.larsoncalculus.com/etf6/content/calculus-videos/chapter-9/section-6/proofratiotest/
- Video similar to Exercise 1: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v02328b.html
- Video similar to Exercise 2: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v02328a.html
- Video the Root Test: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v01417a.html
- Video similar to Exercise 4b: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v01418a.html
- Video Resource 1: https://youtu.be/iy8mhbZTY7g
- Video Resource 2: https://youtu.be/gay5CEXnkhA
- Video Resource 3: https://youtu.be/av947KCWf2U
- Video Resource 4: http://college.cengage.com/mathematics/blackboard/shared/content/video_explanations/v02509a.html
- Textbook Exercises Video 9.6 Problem 23: https://youtu.be/n34XxF3qSWA
- Textbook Exercises Video 9.6 Problem 39: https://youtu.be/PgnsHZvi_po
- Textbook Exercises Video 9.6 Problem 59: https://youtu.be/6InSz1197ts
- Textbook Summary PowerPoint Section 9.6: https://cwoer.ccbcmd.edu/math/math252/Math252Section0906.pptx