# Class Notes

# for

# algebra-based

# PHYS 1

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## Section 0 – An Introduction

I think my happiest day professionally was in the late 1990s. After explaining a conceptual question to one of my Physics I students (a former lawyer), he responded with an emphatic "I don't believe you!" While I can not take credit for instilling his skepticism, it is the attitude I would hope all of my students, indeed <u>all</u> students, would possess. I have worked at some institutions where standards were maintained with the goal of providing excellent, but not necessarily pleasant, educational experiences for students, including one with a totally constructivist course, challenging from both sides of the lecture desk. I had heard the aphorism that learning occurs at the edge of student frustration, and so I set about making my courses just the right amount of frustrating. Luckily, I have been fortunate to have many patient and hard-working students along the way who have, mostly unwittingly, helped me to adjust and modify the presentation of material in this course.

What is the purpose of a PHYS I course? Most Physics professors would say that the purpose is to teach students Physics, whatever that means. Let's try to break that down:

1) 'Students who plan to become scientists and engineers need to know this stuff.' Well, you may need to know it for a couple of years while you learn your actual profession. My experience with some physicists who publish prolifically in their specialized fields but don't know Phys I material bears me out on this.
2) 'Students who plan to become health care professionals need to know this stuff.' Even less true. You may need to know some of this as you complete your biology courses, but you're going to forget it soon enough.
3) 'Students in general need to know how the world works.' Anecdotal evidence suggests that students revert to their original non-Newtonian way of thinking once the course ends.
4) 'Students should get a taste of what science is like.' Like most things worth doing, science is hard. Ideas must be discussed, calculations made, and theories tested against reality. It is not a spectator activity; a course comprising thirty minutes of demonstrations and ten minutes of lecture is not a Physics course.

And so, this brings us to the purpose everyone talks about but few bother to realize: critical thinking. This buzz word permeates mission statements for most colleges. I contend that, for beginners, the hard sciences, Physics and Chemistry in particular, are the best start on this path. It is not my intention to denigrate the humanities; in many perhaps non-obvious ways, this course is modelled on a traditional humanities writing course. We start from a premise, make and justify a series of logical arguments or steps until a conclusion is reached, and analyze that conclusion. We have an advantage over the humanities, though, in that our arguments can be made with steps that are quite clearly correct or falsifiable, and our conclusions can be checked against an objective reality. It is these skills that we hope students will carry on with them through life.

This is not a textbook. It was a set of class notes written up after each lecture in the fall of 2000 to augment the assigned textbook, rewritten and expanded in 2020. When I started these notes, algebra-based Physics textbooks were still written in a style that encouraged memorization over thinking. As a quick example, many books started their impulse-momentum sections by defining momentum, pointing out that it is conserved in some situations, and then moving on to examples. This approach avoids the effort of developing an idea from observation, confirming it with experimentation, and understanding how it fits in with all the other notions that have been discussed. These notes make a perhaps awkward attempt to introduce notions from guided discussions and demonstrations, each usually leading to a derivation and a useful result, which are of course only tentative until tested by lab experiences. In deference to my lawyer acquaintance, the idea here is to justify, within reason using special cases, any assertions made. Wider examples of less-special cases are left to more advanced courses.

So, with the exception of parts of Section 14, this course is run in a manner similar to a traditional high-school Geometry class with two column proofs. In Euclidean Geometry, there are five *axioms*, things which are assumed to be true. From these five notions, with the use of some definitions, all of Euclidean Geometry can be constructed. There is some leeway in choosing what the five axioms are, but a proper combination of five will do the job. In Physics also, we have chosen an open and consistent logical system; we have axioms, although we call them *laws*. Laws are ideas that we think are true because we have never seen them not to be true. This course is based on two such laws; everything else follows from those two.

These notes are intended for an algebra-based PHYS 101 course. Sections that extend beyond the norm for such a course are marked with an asterisk. Several alternative methods for solving problems normally treated with calculus are presented as demonstrations of ‘out of the box’ thinking. It is expected that students will still possess a textbook or at least a workbook of additional problems. With luck, notes for Physics 2 and Physics 3 will follow.

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I wish I could tell you how to succeed in Physics. I can make some suggestions based on what other people have told me.

1) Go to class. Sit in the front row if you can.
2) Ask a question (if your institution’s model allows it) if something is not clear. Do it immediately. Do not assume that you will be able to go home and figure it out on your own.
3) This course is NOT a jumble of unrelated facts. Especially in Physics I, everything is related to everything else through the two axioms. Try to see how. You will find many *derivations* in these notes which start with something you know and end with some new notion.
4) Most of your Physics time will be spent problem solving, whether for in-class examples, homework, practice, or exams. There is a generally accepted model for solving problems, as well as a generally accepted level of effort required. These include
   a) a figure. Figures tend to get complicated and eventually three-dimensional. You will want to color code the different quantities. In these notes, when necessary, forces are red, components of anything are blue, displacements are green, and other quantities some random color.
   b) a statement of what is known.

c) a statement of which principle of physics is being used to solve the problem.
d) a sufficient amount of written effort that illustrates the steps taken.
e) a result, including units.
f) any other information requested, such as "What is the physical significance of your result?"

5) Learning to solve physics problems is much like any other difficult endeavor, like playing the piano or competing in Sportsball. 'Don't practice until you get it right; practice until you never get it wrong.' You will need to start out easy and work your way up to more difficult situations. The difficulty of the problems assigned ramps up. In-class examples are generally fairly easy in order to illustrate a point. In-class assignments (the Exercises) are usually a bit more difficult. For reasons too complicated to discuss here, homework problems are of varied difficulty. You should continue to try more and more difficult practice problems from your workbook. Right before the exam, try the sample exam. Note that the sample exams provided here do not have a question for each topic; they are included to give you an idea of the types of questions that will appear. You should attempt as much as you can on your own, but…
6) Forming study groups is a useful way to test yourself by discussing the material with your classmates. Unless your class is run on a strict curve system, there is no disadvantage to you to help other students.
7) I'll add this little bit regarding exams. Most questions on an exam will be standard type problems, although they will not be rehashed homework problems. Often, there is a 'twist' to the question, but if the usual procedure is followed, the solution should still pop out. Some questions are derivations; for many students, this is the memorization part of the exam, although I contend that if you really know the material, you can simply do it and not waste that part of the brain. Some questions are essays. My experience is that students have the most trouble with the multiple choice questions. Often, there are ulterior purposes for asking these. Some are straight-forward, some are logic problems, some do not expect you to find the correct answer but instead to eliminate the incorrect ones. It's worth spending a moment to ask yourself,' what is this question really asking me?'

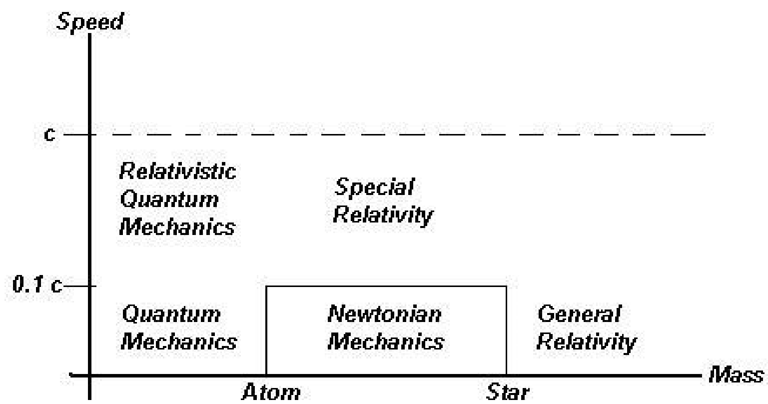
Best of luck!

D. Baum

## Section 1 - Background

## Introduction

In this semester, we will study what is now known as *Newtonian mechanics*. The laws we shall discuss are unfortunately only approximately true, special case limits of the actual laws of the universe. However, they are sufficiently correct to agree to high precision with reality so long as certain conditions are met. The diagram shows a rough breakdown of the approaches necessary to a given situation. So long as the speed of an object is less than about 10% of the speed of light, and the object is roughly larger than an atom but smaller than a star, we will probably be alright. In the future, as you study these other regimes of physics, you will see that each type will begin to agree with its neighbor as it approaches the boundary. For example, the equations of motion in special relativity and those for Newtonian physics look very different, but as the relativistic ones are applied to slower and slower objects, they begin to resemble the Newtonian relationships. This is known as the correspondence principle.

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### Dimensional Analysis

Since, unlike most fields of academic inquiry, the conclusions of physics must agree with objective reality, we must be prepared to make measurements of various physical properties. Modern physicists have determined that any physical quantity can be constructed from some combination of only seven basic, fundamental quantities or *dimensions*, the choice of which is somewhat arbitrary but currently standardized:

[Length]
[Mass]
[Time]
[Electrical Current]
[Number of Particles]
[Thermodynamic Temperature]
and
[Light Intensity].

When we talk about the dimension of a quantity, we don't mean dimension in the sense of the width, length, and height of a box. Each of these specific measurements is a [Length]. The dimension of the volume of a box is $[\text{Length}]^3$. Although we haven't defined it yet, you probably have an idea of the meaning of speed; the dimension of speed is [Length]/[Time].

So, for example, next semester you will encounter the *electric potential,* which has the dimensions of $[\text{Mass}][\text{Length}]^2/[\text{Current}][\text{Time}]^3$. Note that this construct is independent of the actual units used. For example, this quantity is often called the *voltage*, since the *volt* is the standard unit for electric potential, but of course other units could just as easily be used instead. The unit might change, but the dimension will remain the same.

DISCUSSION 1-1

*Dimensional analysis* can be a useful tool for gaining insight into the relationships among quantities that determine the behavior of a system. For example, can we make a prediction for the dependence of the *period* (P, the time to complete one cycle) of a simple pendulum without knowing much physics? On what parameters of the system could this depend? What are the dimensions of these quantities?

EXAMPLE 1-1

A list of such quantities would perhaps include the length $\ell$ of the string, the mass m of the bob, the amplitude of oscillation ($\theta_A$, the angle through which the bob swings), and perhaps the earth's gravity g, whatever that is.

period T = [Time]
mass m = [Mass]
string length $\ell$ = [Length]
amplitude $\theta_A$ = [1] (dimensionless, the radian is the ratio of two distances)
gravitational field strength g = $[\text{Length}]/[\text{Time}]^2$ (O.K., I had to give you this one).

Since we're looking for an expression for the period, whatever combination of parameters we decide on must have dimension of [Time]. Let's suppose that

$$\mathrm{P} \sim \mathrm{m^a}\, \mathrm{g^b}\, l^c\, \theta_{\mathrm{A}}^{\mathrm{d}}\ ,$$

where a, b, c, and d are powers of their respective variables and are to be determined. Then, looking at the dimensions,

$$[\mathrm{T}]^1 = [\mathrm{M}]^{\mathrm{a}}\left(\frac{[\mathrm{L}]}{[\mathrm{T}]^2}\right)^{\mathrm{b}}[\mathrm{L}]^{\mathrm{c}}(1)^{\mathrm{d}} = [\mathrm{M}]^{\mathrm{a}}[\mathrm{T}]^{-2\mathrm{b}}[\mathrm{L}]^{\mathrm{b+c}}(1)^{\mathrm{d}} \quad .$$

If we're going to have an equation, clearly both sides of the equation must have the same dimension. We see that there is no [Mass] on the left side, so a = 0. Continuing,

a = 0;

$1 = -2b \rightarrow b = -1/2;$
$0 = b + c \rightarrow b = -c = +1/2;$
d can not be determined.

The angle, being measured in dimensionless radians,[1] can't be determined. But, if we try a little experiment, we find that $\theta_A$ in fact has no effect on the period, so d = 0. Our final result is that we expect the period of a simple pendulum to go as

$$P \sim g^{-\frac{1}{2}} l^{\frac{1}{2}} = \sqrt{\frac{l}{g}} \quad .$$

The correct answer, as we'll see at the end of the course after much toil is

$$P = 2\pi \sqrt{\frac{l}{g}} \quad .$$

Since $2\pi$ is a dimensionless quantity, this method could not detect it. Even so, we got a good idea of how the period depends on the parameters of the system with relatively little effort.

EXERCISE 1-1

If we drop a marble from a height H above a table, it takes a certain amount of time to fall through distance H to the table. Work out roughly the relationship between the time t and the height H.

## Units

DISCUSSION 1-2

Which weighs more, a pound of rocks or a pound of feathers? Which weighs more, an ounce of gold or an ounce of potatos? Which weighs more, a pound of gold or a pound of potatos?

Making measurements requires that we develop units for the measurements, and standards for these units, so that we may all understand what the measurements mean. In the example above, an ounce of gold actually weighs more than an ounce of potatos, because gold, being a precious metal, is measured in troy ounces, which are larger than the avoirdupois ounces used for food. On the other hand, a pound of potatos weighs more than a pound of gold, because there are 16 avoirdupois ounces in an avoirdupois pound but only 12 troy ounces in a troy pound. So, not only do we need to define units, we need to define which particular system of units they are associated with.

[1] The radian is defined as the ratio of two distances.

In this class, we shall use the *système international*, also known as the *MKSA system* (for meter, kilogram, second, ampère).[2,3] You are probably much more familiar with the *U.S. Customary Units System*, which is a patchwork of bizarre quantities and units. Only three nations in the world have avoided an official change to the SI; in the U.S., the conversion was to have been accomplished by 1970. Metric road signs are in use on some federal highways in Ohio, Kentucky, Tennessee, Arizona, Vermont, New Hampshire, Maine, and New York (some New York signs are also in French!), and exits are numbered by km on Rte 1 in Delaware. Here is a partial list of units used to measure distance in the United States:

inch;
foot; 1 foot = 12 inches
yard; 1 yard = 3 feet
fathom; 1 fathom = 2 yards
rod; 1 rod = 16 2/3 ft
ell; 1 ell = 2 ft
mil; 1000 mils = 1 inch
furlong; 1 furlong = 220 yards
chain; 1 chain = 66 feet
link; 100 links = 1 chain
mile; 1 mile = 5280 feet = 1760 yards = 8 furlongs
league; 3 miles = 1 league
hand; 1 hand = 4 inches
span; 1 span = 9 inches
palm; 1 palm = 3 inches
finger; 1 finger = 7/8 inch
digit; 1 digit = 1/16 foot
shaftment; 1 shaftment = 6 inches

Do you know any others?

When we describe the distance from one point to another, we usually like to use units for which the number is of a reasonable size. What I mean is, if I describe the distance between my stapler and my computer, I would say, $2^1/_2$ feet, not $4.7 \times 10^{-4}$ miles. The distance between Catonsville and D.C is 39 miles, not 3120 chains. However, the conversion factors between units are quite unwieldy. The structure of the SI makes conversion between large and small units much more convenient. There is a small number of basic units, and all other units with the same dimension are some power of ten larger or smaller, usually specified with a Latin or Greek prefix:

giga = $10^9$
mega = $10^6$
kilo = $10^3$
milli = $10^{-3}$
micro = $10^{-6}$

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[2] There is more than one metric system, so we need to be specific.
[3] The metric system survives as one of the innovations of the First Republic (the calendar was not so lucky, but then how would we know when not to eat oysters?).

nano = $10^{-9}$
*et c.*

For example, the *meter* is the basic unit for length, and other units include the kilometer (1000 m), the milllimeter (1/1000 m), *et c*. So, I would express the distance from Catonsville to D.C. as 62 kilometers, not as 62,000 meters.

The definitions of each unit are also well specified, although many of the definitions have evolved. For example, the meter was initially defined in the 1790s as 1/10,000,000 of the distance from the equator to the North Pole along the meridian passing through Paris.[4] Since this is not an easy standard to use, it was redefined in 1889 as the distance between two scratches on a platinum-iridium bar, kept just outside Paris. Since taking a long trip to compare measurements with the bar is inconvenient, a number of other nations were provided with their own bars (ours is in Gaithersburg). As the necessity of making more precise measurements increased, the definition of the meter was changed so that anyone with the proper equipment could reproduce the standard; in 1960, the definition was changed to the distance covered by a 1,650,763.73 wavelengths of a particular orange emission line generated by $^{86}$Kr. Finally, the definition of the meter was changed again in 1983 to be the distance traveled by light in $^1/_{299,792,458}$ of a second.

Although it seems as if the progressive definitions of the metre are making it more difficult to compare our measurements to the standard, it is actually the reverse; by liberating the standard from a particular piece of matter and basing it more on the laws of the nature, which are universal, anyone with the appropriate equipment can reproduce the standard in the comfort of his own laboratory.

Students often find converting units difficult. The *factor label method* is useful and straightforward; one only need multiply by one.

EXAMPLE 1-2

Suppose that we wish to find out how many seconds X there are in 3 years:

$$\text{X seconds} = 3\text{ years} \quad .$$

Note that the units are different on each side, but that the dimensions are the same, [Time]. We'll multiply the right hand side of the equation by a quantity equal to one; we do that because multiplying a number by one does not change its value. The quantity we choose to multiply by is (12 months/1 yr). Since the numerator equals the denominator and since both have dimensions of [Time], the quotient equals one, and the right hand side is still equal to three years. We cancel the units and see that :

$$\text{X seconds} = 3\text{ years}\left(\frac{12\text{ months}}{1\text{ year}}\right) = 36\ \text{ months} \quad .$$

[4] The current 2°20′14.03″ meridian.

Continuing, a complete calculation would look like this:

$$\text{X seconds} = 3\text{ years}\left(\frac{12\text{ months}}{1\text{ year}}\right)\left(\frac{30\text{ days}}{1\text{ month}}\right)\left(\frac{24\text{ hours}}{1\text{ day}}\right)\left(\frac{60\text{ minutes}}{1\text{ hour}}\right)\left(\frac{60\text{ seconds}}{1\text{ hour}}\right)$$
$$= 9.33 \times 10^7\text{seconds}.$$

EXERCISE 1-2

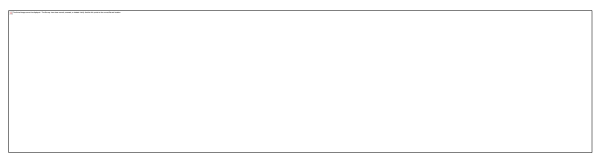
A meter is 100 centimeters. Find the volume in cubic centimeters of a box with a volume of one cubic meter.

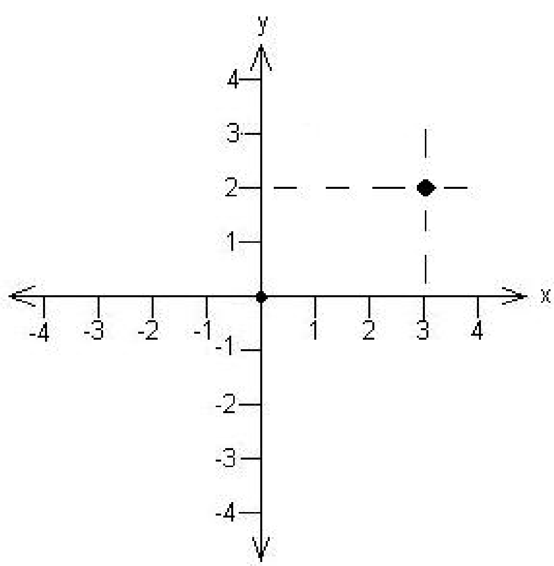
HOMEWORK 1-1

The interior of a typical ranch-style home may measure 50 ft x 24 ft x 8 ft. What is the volume of this home in cubic ft? Convert this result to cubic inches and to cubic centimeters.

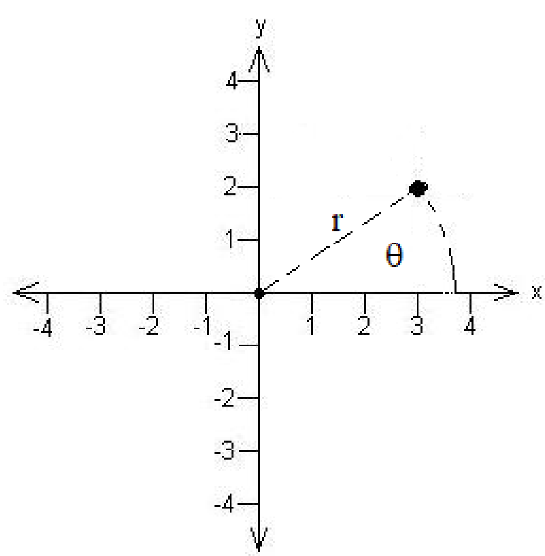
## Coördinate systems

As we shall soon see, we'll need a way of keeping track of the positions of objects, as well as other quantities. In one dimension, that's fairly easy; we use the equivalent of the 'number line' we learned back in third grade, with some arbitrary point chosen as the *origin* (and usually chosen to maximize our convenience).

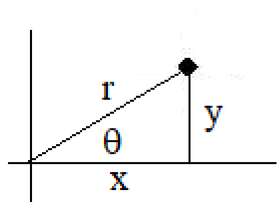





When we go to two dimensions, there are quite a number of systems, but the two most useful are the *rectilinear* or *Cartesian* system and the *polar* system. In the first, two 'number lines' are set up at right angles with the origins at the same spot and with equal unit spacing. We must however realize that these are not necessarily the x and y axes, but for now, let's say that they are.




The location of an object in two dimensions can be specified uniquely by reporting two numbers in an ordered pair in the form (a, b). The meaning is to start at the origin, move 'a' units in the x-direction and 'b' units in the y-direction; in this example, the location is (3, 2). The position can also be specified as a direction (usually reported as the angle measured counter-clockwise from the x-axis) and the distance from the origin, (r, θ). A negative angle is interpreted as being measured CW from the x axis. Conversion between these systems is possible through the use of the trig functions and the Pythagorean theorem:




$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \rightarrow y = r\ \sin\theta$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \rightarrow x = r\ \cos\theta$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \rightarrow \theta = \arctan\left(\frac{y}{x}\right)^{*}$$

$$\text{hypotenuse}^2 = \text{opposite}^2 + \text{adjacent}^2 \rightarrow r = +\sqrt{x^2 + y^2}$$

Note that r is never negative.

EXERCISE 1-3

Find r and theta for the point (3, 2) as shown in the figure above.

Now, we usually think of the lengths of the sides of a triangle as being positive numbers, which is why I introduced these relationships in the first quadrant. I assert, however, that with one small warning, these are valid in all four quadrants.

DISCUSSION 1-3

Keeping in mind that r is never negative, in which quadrants is x/r positive and where is it negative? Where is cos θ positive and where is it negative? Do these match up? What about y/r and sin θ?

Now, here is why there is an asterisk next to the arctan function. Get your calculator and find the arctangent of (2/3). Which quadrant is 33.7° in? Now find the arctangent of (-2/-3). In which quadrant should the answer be?

The problem is that your calculator does the division first, then the arctangent. It doesn't know the distinction between (-2/-3) and (2/3). Your calculator will always give you an angle between -90° and +90°; it's up to you to fix this each time. Here's my suggestion. If the angle is in fact in Quadrant I or IIII where x is positive, then the angle your calculator gives you is already correct, so you do nothing. On the other hand, if the angle is in II or III where x is negative, you must add 180°. So, the easiest test is to look at x. If x is positive, you're good. If x is negative, that's bad, and you need to fix it. I require this: if no correction is necessary, you must still indicate that you checked to see if one was necessary. I'll be happy with a √ Q on your paper.

EXERCISE 1-4

Find the polar coördinates for the cartesian location (-3, -1).

Find the cartesian coördinates for the polar location (4, 120º)

HOMEWORK 1-2

How far from the origin is a point located at (1 m, 4 m)?

## Scalars and Vectors

In this course, we deal with two types of quantities, *scalars* and *vectors*. There are other types of quantities, such as *tensors*, that thankfully we will not need to worry about. A scalar is a quantity that possesses only a size or *magnitude*. A vector possesses a magnitude and a direction.

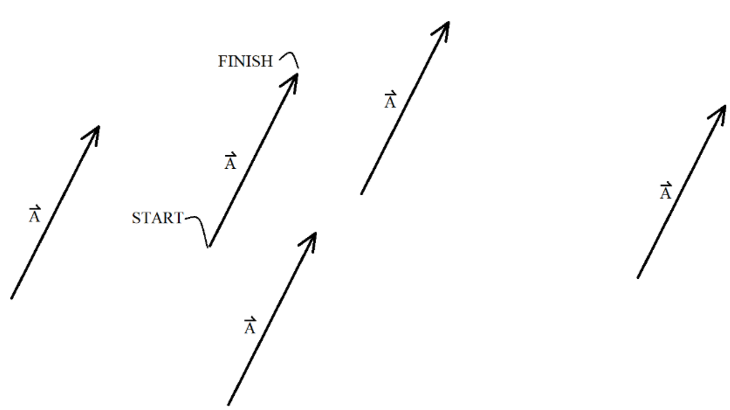
DISCUSSION 1-4

Consider the evening weather report. Which quantities are vectors and which are scalars?

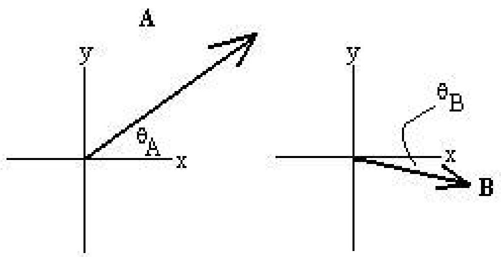
The notation for vectors is to use bold type or to place a half arrow above the symbol: **A** or $\vec{A}$. The magnitude only is written as A or less ambiguously as $|\vec{A}|$. During this course, we will sometimes drop the arrow and rely on your sense of context to know which quantities are vectors.

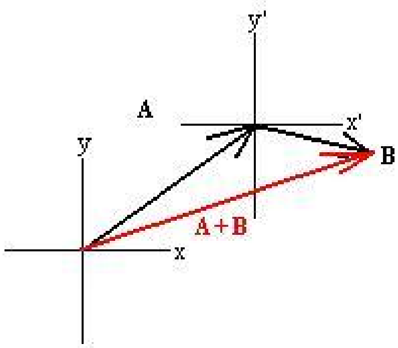
We often represent vectors with arrows drawn on for example a paper sheet. Arrows also have two properties we can make use of: they have direction and they have length. We can make the directions be the same, and make the length of the arrow be proportional to the magnitude of our vector.

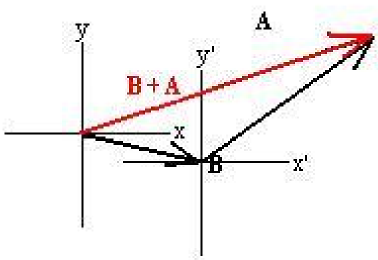
We want to investigate some properties of vectors. To do so, let's jump the gun a bit and introduce the vector *displacement*. The displacement represents the movement of an object. We can think of it as pointing from the starting position to the final position. This makes the visualization a bit easier at the start. Later, vectors will represent much more abstract quantities, such a momentum, magnetic fields, or isospin. We do need to be careful; once a vector is defined, it has only two properties, magnitude and direction. We can move the vector around as much as we wish so long as those two properties remain constant. For example, in the figure, vector $\vec{A}$ was constructed to represent the displacement from the START to the FINISH, but all of the other vectors drawn are just as validly vector $\vec{A}$.



We can visualize adding vectors in terms of displacements: $\vec{A} + \vec{B}$ says that we should start at our origin and travel A meters in a direction given by $\theta_A$, then from that intermediate destination, travel B meters in the direction given by $\theta_B$. Conceptually, this is known as the *tail-to-tip method of addition*.[5] The red vector is the sum, or *resultant*, of $\vec{A} + \vec{B}$. Now, look at the bottom diagram. If we were to perform the motion described by $\vec{B}$ first, then perform $\vec{A}$, we would wind up in the same place. That means that vector addition is *commutative*. The order of addition doesn't matter:

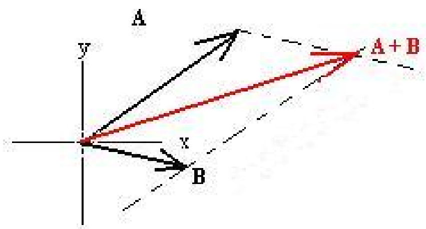



$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \ .$$

An alternate, but equivalent, method of addition is the parallelogram method. This helps explain the contention of commutativity; the two long sides are each A and the two short sides are each B. The resultant will be the diagonal of the parallelogram.




When more than two vectors are added graphically, we must do one at a time, so

$$\vec{A} + \vec{B} + \vec{C} + \vec{D} = \left(\left(\vec{A} + \vec{B}\right) + \vec{C}\right) + \vec{D}$$

Vector addition is also *associative*:

$$\vec{A} + \left(\vec{B} + \vec{C}\right) = \left(\vec{A} + \vec{B}\right) + \vec{C}$$

EXERCISE 1-5

Make an argument that vector addition is associative. Try a graphical solution with three vectors.

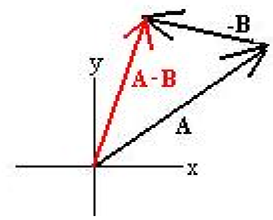
HOMEWORK 1-3

[5] Well, O.K. it's actually called the tip-to-tail method, but that makes no sense. Let's make it a thing.

Anne walks a certain distance due north, then turns due east and walks twice as far. At the end of her trip, she is 450 meters from her starting point, as the crow flies. What is the length of each leg of the trip? What is the direction of her displacement relative to north?

I have no idea how to subtract vectors, but I know a trick from grade school. When I learned to add, for example 5 + 2, I started at the origin of the number line and moved five to the right, then another two to the right. To subtract, say 5 – 2, I moved five to the right, then two to the left, that is, I did 5 + (-2). Let's try this:

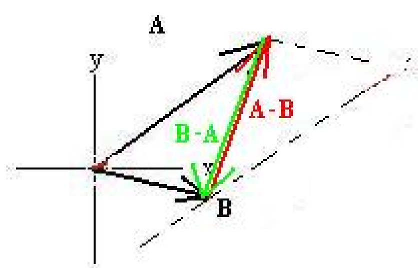
$$\vec{\mathrm{A}} - \vec{\mathrm{B}} = \vec{\mathrm{A}} + \left(-\vec{\mathrm{B}}\right).$$




The question is then, what is $-\vec{\mathrm{B}}$? I think we would want to require

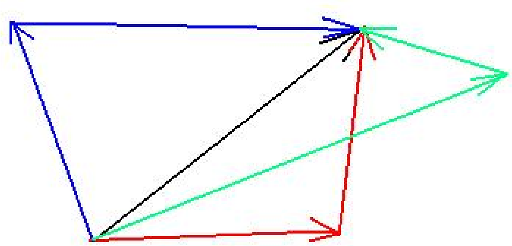
$$\vec{B} + \left(-\vec{B}\right) = \vec{0}\ ,$$[6]

That is, -B must have the same magnitude as B, but point in exactly the opposite direction

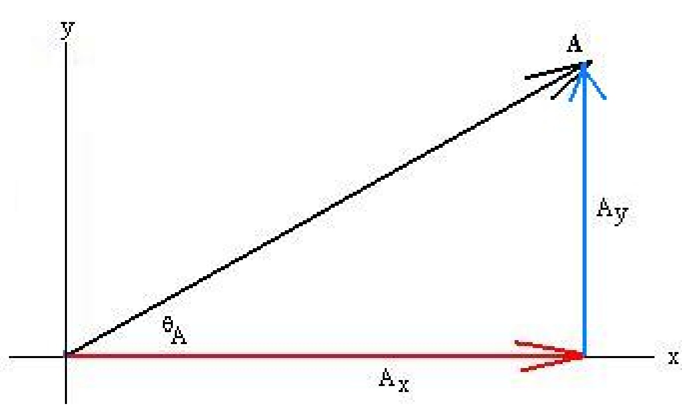



Comparison to the parallelogram method reveals that $\vec{\mathrm{A}} - \vec{\mathrm{B}}$ is the other diagonal of the parallelogram (as is $\vec{\mathrm{B}} - \vec{\mathrm{A}}$, the same diagonal but pointing in the other direction).

Once again, to add vectors graphically, one would take paper, ruler and protractor, choose a scale, and draw arrows to represent the vectors such that the length of each is proportional to the magnitude of the corresponding vector. To find the resultant, measure the length of the resultant with the ruler and back convert to find the magnitude, and use the protractor to find the direction.



Well, we really don't want you adding vectors with rulers and protractors for the rest of the semester. Let's investigate an analytic method. Now that we can add vectors, we can also see that any given vector (shown in black) can be written as the sum of two (or more) other vectors. In the diagram, you can see that the black vector is the sum of the two red vectors, but it is also the sum of the two green vectors as well as the sum of the two blue vectors. If that's true, we might as




[6] Technically speaking, this zero is also a vector, the *null vector*.

well choose two vectors that will be convenient for us. If we make the two vectors perpendicular, we might be able to use trig relationships to suss out some info.

$A_x$ is called the *x-component* of **A** and $A_y$ is the *y-component* of **A**, that is, how much the vector points in each direction. $A_x$ and $A_y$ are actually scalars, although they can be positive or negative or even zero. We convey the directional information through the use of the *unit vectors* $\hat{\imath}$ (x direction), $\hat{\jmath}$ (y direction), and $\hat{k}$ (z direction). Unit vectors have length one and are dimensionless (that information is carried in the components). Sticking with two dimensions for now, we can write that $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$. From trig, we see that $A_x = A\cos\theta_A$ and that $A_y = A\sin\theta_A$. Note that if we measure $\theta_A$ CCW from the x axis, that the signs of the trig functions correctly give the signs of the components.

EXAMPLE 1-3

Let $\vec{A}$ be 15 m at $\theta_A$= 120°, which is in the second quadrant. We find that

$$A_x = A\ \cos\theta_A = (15\text{ m})\cos 120^o = -7.5\text{ m}$$
$$A_y = A\ \sin\theta_A = (15\text{ m})\sin 120^o = +13\text{ m}$$

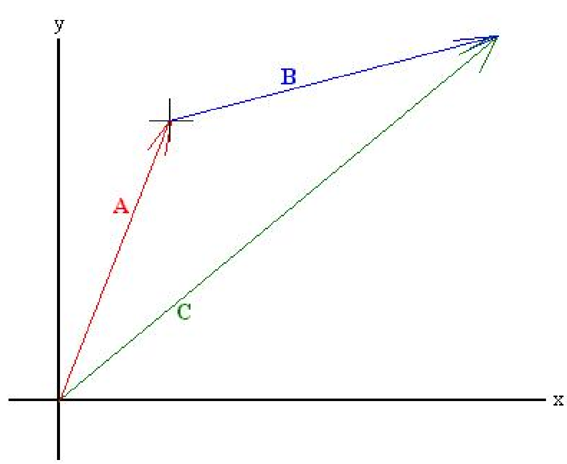
So, $\vec{A} = -7.5\,\hat{\imath} + 13\,\hat{\jmath}$ meters .

and the signs of these components match what we know about the direction of $\vec{A}$.

HOMEWORK 1-4

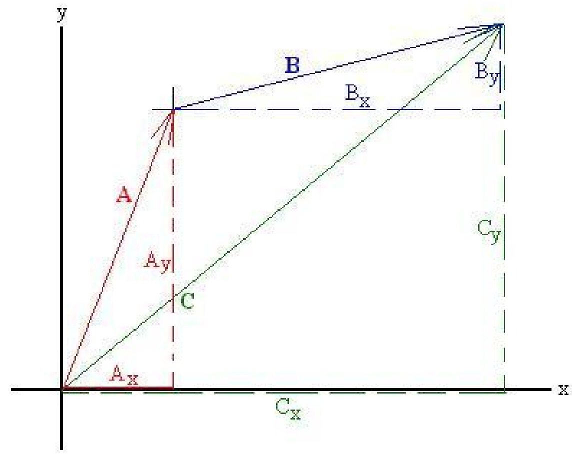
The direction of a vector is 127° measured from the x-axis, and its y-component is 12.0 units. Find the x-component of the vector and the magnitude of the vector.

Now, we have an alternate manner of adding vectors using the components. Let $\vec{C} = \vec{A} + \vec{B}$.



I hope it's clear that $C_x = A_x + B_x$ and $C_y = A_y + B_y$. We might say that **the components of the sum are the sums of the components**. Once we have the components of $\vec{C}$, we can convert them back to a magnitude and a direction angle.

EXAMPLE 1-4




Let $\vec{C} = \vec{A} + \vec{B}$. Find the magnitude and direction angle of $\vec{C}$.

$A = 7$ m $\theta_A = 35^o$
$B = 12$ m $\theta_B = 155^o$

First, we find the components of $\vec{A}$ and $\vec{B}$:

$$A_x = A\ \cos\theta_A = (7\text{ m})\cos 35^o$$
$$= +5.73\text{ m}$$
$$A_y = A\ \sin\theta_A = (7\text{ m})\sin 35^o$$
$$= +4.02\text{ m}$$
$$B_x = B\ \cos\theta_B = (12\text{ m})\cos 155^o = -10.88\text{ m}$$
$$B_y = B\ \sin\theta_B = (12\text{ m})\sin 155^o = +5.07\text{ m}\ .$$

Then we do with the components what we're asked to do with the vectors:

$$C_x = A_x + B_x = 5.73 + (\text{-}10.88) = \text{-}5.15\text{ m}$$
$$C_y = A_y + B_y = 4.02 + 5.07 = 9.09\text{ m}\ .$$

Then, we reconstitute the components of $\vec{C}$ back into a magnitude and direction:

$$C = +\sqrt{C_x^2 + C_y^2} = \sqrt{(\text{-}5.15)^2 + 9.09^2} = 10.45\text{ m}\ ,$$

$$\theta_C = \arctan\left(\frac{C_y}{C_x}\right)^* = \arctan\left(\frac{9.09}{\text{-}5.15}\right) = \arctan(\text{-}1.76) = \text{-}60.47^o\ .$$

Are we done? No, we need to check the quadrant of the angle to see if the calculator's answer is correct. In this case, it is not. Because $C_x<0$, we need to add $180^o$ to the result. So

$$\theta_C = -60.47 + 180 = 119.53^o\ .$$

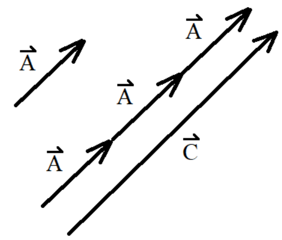
HOMEWORK 1-5

Vector $\vec{A}$ has magnitude 8.0 units at an angle of $60^o$ from the x-axis. Vector $\vec{B}$ has magnitude 6.0 at an angle of $-30^o$ from the x-axis. Find the magnitude and direction of vector $\vec{C} = \vec{A} + \vec{B}$.

## Vector Multiplication

There are a number of ways vectors can be multiplied; we'll deal with three.
The first type of multiplication is perhaps familiar from grade school. Let's multiply $\vec{A}$ by a scalar, 3, and call that $\vec{C}$:

$$\vec{C} = 3\,\vec{A}\ .$$



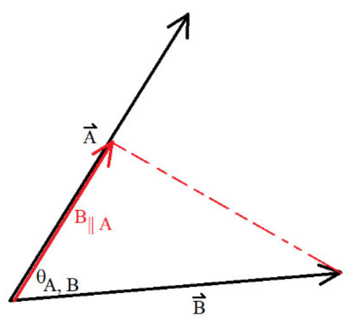

This type of multiplication is repeated addition.

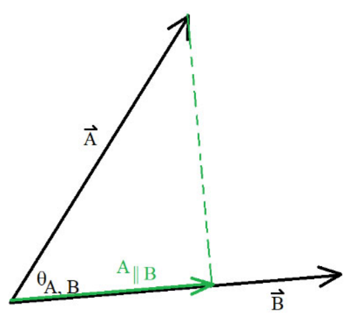
$$\vec{C} = \vec{A} + \vec{A} + \vec{A}\,.$$

We can see that $\vec{C}$ is in the same direction of $\vec{A}$ but with three times the magnitude. It should be easy to see that we could expand this notion to non-integer multiples as well. It's a little more complicated when the scalar is not a dimensionless number, but the notion is the same; the value and dimension of the magnitude will change, but the direction will remain the same.[7]

Next, we will define the *scalar product* (also called the *inner product* or the *dot product*) of two vectors $\vec{A}$ and $\vec{B}$ to be:

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta_{A,B}\ ,$$




that is, the magnitude of $\vec{A}$ times the magnitude of $\vec{B}$ times the cosine of the angle between them if they were placed tail to tail. The dot product is defined to be a scalar. One interpretation of this definition is that we are multiplying the magnitude of $\vec{A}$ by the component, or *projection*,[8] of $\vec{B}$ that lies in the direction of $\vec{A}$:

$$\vec{A}\cdot\vec{B} = A\,B_{||} = A\,(B\cos\theta_{A,B}) = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta_{A,B}\ ,$$

as shown in the figure on the left. Clearly, though, we could just as well think of it as the magnitude of $\vec{B}$ times the projection of $\vec{A}$ on $\vec{B}$:

$$\left|\vec{A}\right|\left|\vec{B}\right|\cos\theta_{A,B} = B\,(A\cos\theta_{A,B}) = B\,A_{||} = \vec{B}\cdot\vec{A}\ .$$

The dot product is therefor commutative.

[7] Looking way ahead, the momentum p of an object is given by the mass times the velocity v. Momentum and velocity are in the same direction, but they have very different dimensions.

[8] You can think of a projection as analogous to a shadow, the shadow that $\vec{B}$ casts on $\vec{A}$.

Keep in mind that there is nothing magical about the dot product. It is simply a shorthand way of writing a particular process; as the course progresses, we'll see that we are often interested in how much of one vector is in the direction of another.

DERIVATION 1-1*

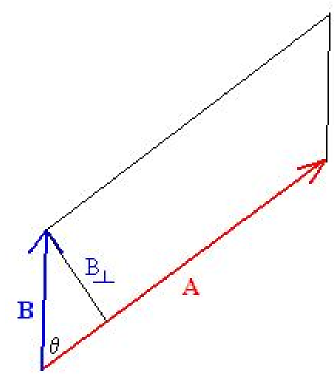
Alternatively, we can write the vectors $\vec{A}$ and $\vec{B}$ in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Remember that $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$. Then,

$$\begin{aligned}\vec{A}\cdot\vec{B} = {} & \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\cdot\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right) \\ & = A_xB_x\hat{\imath}\cdot\hat{\imath} + A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_xB_z\hat{\imath}\cdot\hat{k} + A_yB_x\hat{\imath}\cdot\hat{\jmath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\hat{\jmath}\cdot\hat{k} \\ & + A_zB_x\hat{\imath}\cdot\hat{k} + A_zB_y\hat{\jmath}\cdot\hat{k} + A_zB_z\hat{k}\cdot\hat{k} \\ & = A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

Another type of vector multiplication is the *vector product* or the *cross product,* which we define in two parts. We define the magnitude of the cross product to be

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta_{A,B} \ .$$

that is, we're taking the magnitude of A and multiplying by the component of B that is perpendicular to A. One interpretation of the cross product's magnitude is that it is the area of the parallelogram formed by the vectors **A** and **B** when they are placed tail to tail. Using an argument like the one for the dot product, we see that




$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{B}\times\vec{A}\right| \ .$$

However, there is a second part to the cross product, direction. We define the direction of $\vec{A}\times\vec{B}$ to be perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$. That leaves two possible directions, for example, in the diagram, into the page or out of the page. We define the direction sense using the *right-hand-rule* (RHR). Point your index finger of your right hand in the direction of $\vec{A}$ and your middle finger in the direction of $\vec{B}$; your right thumb then points in the direction of the cross product. You can then see that

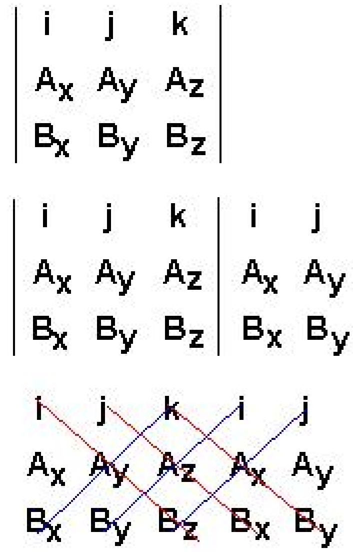
$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \ .$$

DERIVATION 1-2*

Alternatively, we can write the vectors $\vec{A}$ and $\vec{B}$ in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Remember that $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$ and $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, and $\hat{k}\times\hat{\imath} = \hat{\jmath}$. Then,

$$\vec{A}\times\vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\times\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$
$$= A_xB_x\hat{\imath}\times\hat{\imath} + A_xB_y\hat{\imath}\times\hat{\jmath} + A_xB_z\hat{\imath}\times\hat{k} + A_yB_x\hat{\jmath}\times\hat{\imath} + A_yB_y\hat{\jmath}\times\hat{\jmath} + A_yB_z\hat{\jmath}\times\hat{k}$$
$$+ A_zB_x\hat{k}\times\hat{\imath} + A_zB_y\hat{k}\times\hat{\jmath} + A_zB_z\hat{k}\times\hat{k}$$
$$= \left(A_yB_z - A_zB_y\right)\hat{\imath} + \left(A_zB_x - A_xB_z\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}\,.$$

There is a quick way of remembering how to do this. Arrange the components into a table as seen in the top figure. Rewrite the first two columns at the right of the table, as shown in the middle figure. Lastly, multiply the quantities along each diagonal as shown. If the diagonal is to the down and to the right (red), add the product and if it's to the left (blue), subtract.




HOMEWORK 1-6*

Given that

$$\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k} \quad \text{and}\ \vec{B} = -\hat{\imath} + 3\hat{\jmath} + 2\hat{k} \quad ,$$

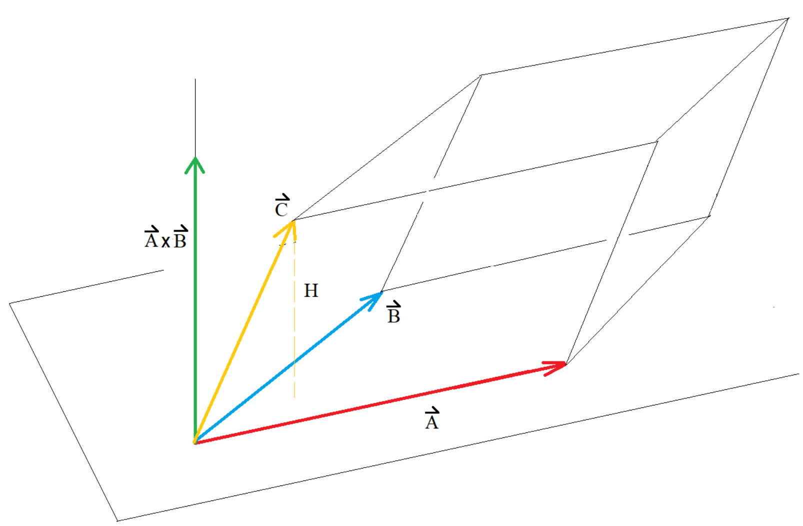
find $\vec{A}\cdot\vec{B}$ and $\vec{A}\times\vec{B}$ .

It is also sometimes useful to combine successive multiplications. Consider the *scalar triple product*. We'll be using this for one problem only, but this seems like the appropriate time to introduce it. Consider three vectors, not all in the same plane. The scalar triple product has an interesting useful property:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times A\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right)\ .$$

DERIVATION 1-3*

The three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, when paced tail to tail to tail, are the edges of a parallelepiped solid. As discussed above, the magnitude of the cross product of $\vec{A}$ and $\vec{B}$ gives the area of the parallelogram-shaped base of the solid. The volume V of the solid will be the base area times the height, H:




$$V = H\left|\vec{A}\times\vec{B}\right|\ \ .$$

The height H is the projection of $\vec{C}$ on $\vec{A} \times \vec{B}$, so

$$V = \vec{C} \cdot (\vec{A} \times \vec{B}) \quad .$$

Now, we do need to be a little careful, in that $\vec{C}$ should be on the same side of the AB plane as $\vec{A} \times \vec{B}$; if not, then we get the negative of the volume instead.

Now, imagine that we were to roll the solid onto its BC face. The volume would be

$$V = \vec{A} \cdot (\vec{B} \times \vec{C}) \quad .$$

Rolling it over again onto its AC side,

$$V = \vec{B} \cdot (\vec{C} \times \vec{A}) \quad .$$

Since rolling the solid over doesn't change its volume, we have a useful relationship:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times A) = \vec{C} \cdot (\vec{A} \times \vec{B}) \quad .$$

Lastly, let's consider the *vector triple product*, $\vec{A} \times (\vec{B} \times \vec{C})$. I assert that

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \quad .$$

DERIVATION 1-4*

The straightforward path is to write each vector in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, then perform the operations required on each side of the equation. Let's try to see if we can do it in a less tedious way.[9]

The vector $\vec{B} \times \vec{C}$ is of course perpendicular to the plane containing both $\vec{B}$ and $\vec{C}$. When we cross $\vec{A}$ with that vector, the result is perpendicular to $\vec{B} \times \vec{C}$, which means it lies back in the B-C plane. Therefore, we can write the triple product in terms of some additive combination of $\vec{B}$ and $\vec{C}$:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \alpha \vec{B} + \beta \vec{C} \quad ,$$

where alpha and beta are real numbers. The triple cross product must for this same reason also be perpendicular to $\vec{A}$, so

$$\vec{A} \cdot (\alpha \vec{B} + \beta \vec{C}) = 0 \, ,$$

$$\alpha \vec{A} \cdot \vec{B} = -\beta \vec{A} \cdot \vec{C} \quad .$$

[9] Ercelebi, Atilla, "A×(B×C).pdf," accessed 12/4/2020, www.fen.bilkent.edu.tr/~ercelebi/Ax(BxC).pdf.

This requires that

$$\alpha = \gamma\, \vec{A} \cdot \vec{C} \quad \text{and} \quad \beta = -\,\gamma\, \vec{A} \cdot \vec{B} \;\; ,$$

with gamma some presently unknown number that will cancel out upon substitution back into the previous equation.[10] This relationship should be correct for <u>any</u> vectors, so let's see if we can determine gamma by applying these relationships to a specific set of vectors, î, ĵ, and k̂:

$$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \alpha\, \hat{\imath} + \beta\, \vec{\jmath} \;\; ,$$

$$\hat{\imath} \times \hat{k} = (\gamma\, \hat{\imath} \cdot \hat{\jmath})\, \hat{\imath} + (-\,\gamma\, \hat{\imath} \cdot \hat{\imath})\, \vec{\jmath} \;\; ,$$

$$-\hat{\jmath} = (0)\, \hat{\imath} + (-\,\gamma)\, \vec{\jmath} \;\; ,$$

$$\gamma = 1 \;\; .$$

Now we have that

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \alpha\, \vec{B} + \beta\, \vec{C} = \left(\vec{A} \cdot \vec{C}\right) \vec{B} - \left(\vec{A} \cdot \vec{B}\right) \vec{C} \;\; .$$

---

EXERCISE 1-1 Solution

What quantities might affect the time and what are their respective dimensions? Well, we have

height H = [Length]
time t = [Time]
mass m = [Mass]
gravitational field strength g = [Length]/[Time]$^2$

We might guess that

$$t \sim H^a m^b g^c \;\; .$$

$$[T]^1 = [L]^a [M]^b \left(\frac{[L]}{[T]^2}\right)^c = [L]^{a+c} [M]^b [T]^{-2c} \;\; .$$

Then,

0 = a+c;
b = 0;

[10] In other words, α = A·C, β = A·C is not the only possible solution; α = 6.7 A·C, β = 6.7 A·B would fit as well. We need an unambiguous solution.

1 = -2c → c = -1/2;
a = -c = +1/2.

$$t \sim H^{1/2} g^{-1/2} = \sqrt{\frac{H}{g}}\ .$$

The correct relationship, as we will see in the next section, is

$$t = \sqrt{\frac{2H}{g}}\ .$$

EXERCISE 1-2 Solution

$$\mathrm{X\ cm^3} = \mathrm{1m^3}\left(\frac{100\ \mathrm{cm}}{1\ \mathrm{m}}\right)\left(\frac{100\ \mathrm{cm}}{1\ \mathrm{m}}\right)\left(\frac{100\ \mathrm{cm}}{1\ \mathrm{m}}\right) = 10^6\mathrm{cm^3}\ .$$

Note that you must cancel each of the three meters in the original value.

EXERCISE 1-3 Solution

$$\theta = \arctan\left(\frac{y}{x}\right)^* = \arctan\left(\frac{2}{3}\right) = 33.7^o$$

$$r = +\sqrt{x^2 + y^2} = +\sqrt{3^2 + 2^2} = 3.61$$

EXERCISE 1-4 Solution

x = -1, y = -3 (There were no units.)

$$r = +\sqrt{x^2 + y^2} = +\sqrt{(-3)^2 + (-1)^2} = \sqrt{10} = 3.16$$

Be sure to square the negative signs!

$$\theta = \arctan\left(\frac{-3}{-1}\right)^* = \arctan(3)^* = 71.6^o$$

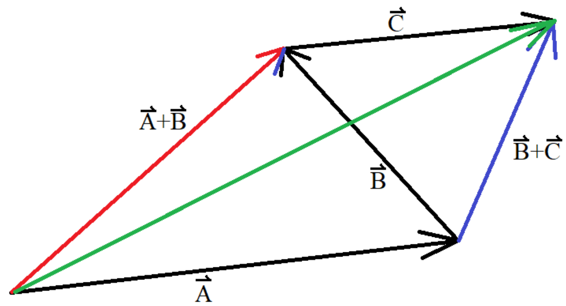
But, x is negative, so we need to add 180$^o$ to get 251.6$^o$ as the correct answer.

For the second part of the exercise, we have r = 4, θ = 120$^o$. In this direction, there's no ambiguity.

$$x = r\ \cos\theta = 4\ \cos 120^0 = -2$$

$$y = r\ \sin\theta = 4\ \sin 120^o = 3.46$$

EXERCISE 1-5 Solution




This is a demonstration, not a proof:

The green vector is the sum of $\vec{A}$, $\vec{B}$, and $\vec{C}$, and can be written as $(\vec{A}+\vec{B}) + \vec{C}$, or as $\vec{A}+ (\vec{B}+\vec{C})$.

## Section 2 - Kinematics in One Dimension

*Kinematics* is the study of the motion (same root as *cinema*) of an object, without regard to the causes of that motion. Most of this section involves defining a number of terms.

## Displacement and Distance

We'll need first of all to be able to define the *location* of an object. In one dimension, we can consider the 'numberline' axis discussed in the last section and use the variable 'x' to label the position relative to the origin in meters, so that statements such as 'x = +3 m' and 'x = -7.46 m' mean that the object is 3 m from the origin in the positive direction (not necessarily to the right of it, though!) and 7.46 m from the origin in the negative direction, respectively.

We also need to be able to describe the change in location of an object. We define the *displacement* $\Delta x$ of an object which started at initial position $x_i$ and ended up at final position $x_f$ as

$$\Delta x = x_f - x_i \quad .$$

Note that, according to this definition, the displacement depends only on where the object started and ended, not on the path taken.

DISCUSSION 2-1

> Suppose that an object starts out at $x_i$ = 3 m and ends at $x_f$ = 5m, and makes that trip smoothly and without reversing direction. What is the displacement? Now, suppose instead that the object travels from x = 3 m to x = 15 m, then back to x = - 8 m, then on to x = 5 m. What is the displacement in that case?
>
> Suppose instead that the object moved from x = 5 m to x = 3 m. What then would be the displacement? Is the displacement a scalar or vector quantity? [1]
>
> Suppose that Object 1 moves from x = 5 m to x = 9 m, while Object 2 moves from x = 7 m to x = 11 m. Which object had the larger displacement?

In the first two cases, the displacements are the same at +2 m. As was mentioned, the path is not relevant to the displacement. In the third case, the displacement is -2 m. Since there is a difference between the motion 2 meters to the right and motion 2 meters to the left, displacement must be a vector. Finally, both Objects 1 and 2 have the same displacements. In the next few sections, we will let the sign of a vector indicate its direction.

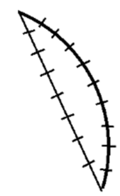
*Distance* (s) is the term we use for the length of the path taken. As a rough analogy, think of the distance as the number of steps one takes getting from A to B.

---

[1] So, if displacement is a vector, it follows that position must technically also be a vector. But how can an object be described as being at x = 5 m in any particular direction? I think we must consider position as being relative to the origin, somewhat awkwardly, as a virtual displacement.

DISCUSSION 2-2

What relationship exists between the distance and the magnitude of the displacement? Suppose that I walk the 7 meters from the desk to the back of the room. What is the magnitude of my displacement? What is my distance? Now, however, I walk from the front of the room to the rear, then back to the front, then return to the rear. What is the magnitude of my overall displacement? What is my overall distance? Can you explain why these results are different?



No matter what I do, any motion adds to my distance travelled (I'm taking steps). On the other hand, if walking toward the rear of the room is positive displacement, walking toward the front is negative displacement that cancels the first part of my trip, and eventually I return to the front from where I started for a total displacement of zero. Similarly, if I travel along a circular arc path, the distance and the magnitude of my displacement will be different. So long as the direction of motion doesn't change, the distance is the same as the magnitude of the displacement.

EXERCISE 2-1

Suppose that an object starts out at $x_i = 3$ m and ends at $x_f = 5$ m, and makes that trip smoothly and without reversing direction. What is the distance? Now, suppose instead that the object travels from x = 3 m to x = 15 m, then back to x = - 8 m, then on to x = 5 m. What is the distance in that case?

## Velocity and Speed

Often, we want to know how quickly an object gets from one location to another. If we say that the object is at position $x_i$ at time $t_i$, and arrives at position $x_f$ at time $t_f$, then we can define the *average velocity* to be the displacement *per* unit time, or

$$v_{AVE} = \frac{x_f - x_i}{t_f - t_i} \ .$$

DISCUSSION 2-2

Is the average velocity a vector or a scalar?

EXAMPLE 2-1

Find the average velocity in each of the cases below. You can assume that the motions start at t = 0 seconds.

Suppose that an object starts out at $x_i = 3$ m and ends at $x_f = 5$ m, and makes that trip smoothly and without reversing direction in 3 seconds.

Suppose instead that the object travels from x = 3 m to x = 15 m, then back to x = - 8 m, then on to x = 5 m, all in 3 seconds.

For the first case,

$$\vec{v}_{AVE} = \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i} = \frac{5 - 3}{3 - 0} = 0.67\ m/s\,.$$

For the second,

$$\vec{v}_{AVE} = \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i} = \frac{5 - 3}{3 - 0} = 0.67\ m/s\,.$$

These results may seem a bit strange in that a person doing the first motion could do so in a leisurely manner, while someone performing the second would be zipping back and forth in a superhuman way. Words that may seem to mean the same thing in everyday speech can mean very different things in Physics, according to how we define them. For example, …

*Average speed* is defined as the distance traveled *per* unit time:

$$\text{average speed} = \frac{s}{\Delta t}\,.$$

There is no special symbol for the average speed.

EXAMPLE 2-2

Suppose that an object starts out at $x_i = 3$ m and ends at $x_f = 5$ m, and makes that trip smoothly and without reversing direction in 3 seconds.

Suppose instead that the object travels from x = 3 m to x = 15 m, then back to x = - 8 m, then on to x = 5 m, all in 3 seconds.

We've previously found the distances for these cases, so

for the first case,

$$\text{average speed} = \frac{s}{\Delta t} = \frac{2}{3} = 0.67\ m/s\,,$$

and for the second,

$$\text{average speed} = \frac{s}{\Delta t} = \frac{48}{3} = 16\ m/s\,.\backslash$$

These results may be more in line with your expectations.

HOMEWORK 2-1

Joe runs the length of a 10 m room in 4 seconds, then immediately turns and walks back in 8 seconds. Find his average velocity

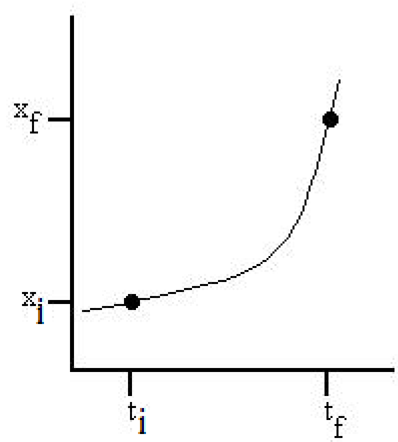
a) for the trip down the room.
b) for the trip back.
c) for the entire trip.

HOMEWORK 2-2

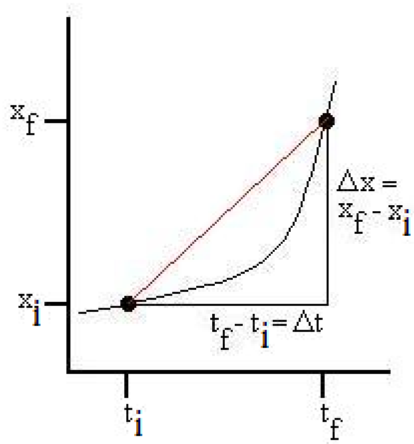
You may remember the fable of the tortoise and the hare. The hare runs at 3 m/s while the tortoise 'runs' at 0.2 m/s. They start the race at the same time, but the hare decides to take a two minute nap along the way. In the end, the tortoise beats the hare by 0.3 meters.

a) How long was the track?
b) How much time did the tortoise use to complete the race?

The average velocity discussed above is considered over an interval of time. How can we find the *instantaneous velocity*, the velocity at an instant of time? Consider the following graph, which shows the position of an object as a function of time, x(t). How would the average velocity between $t_i$ and $t_f$ be represented on this graph?




$\vec{v}_{AVE} = \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i} =$ rise over run = slope of the line connecting the two points.

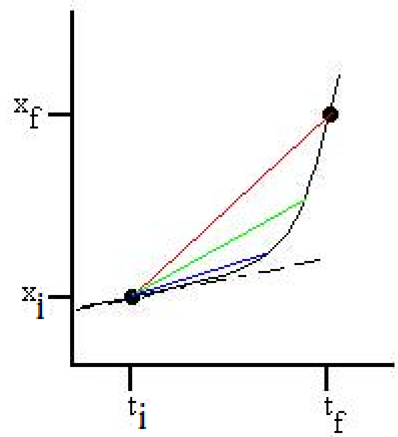



How can we find the velocity at time $t_i$? Let's decrease the time interval. As the interval becomes smaller, the average velocity approaches the instantaneous velocity, or graphically, the slope representing the average velocity approaches the slope of the line tangent to the x(t) curve at the point at which we wish to know v(t). Mathematically, we write this as

$$\vec{v}_{INSTANT} = \lim_{\Delta t \to 0} \frac{\Delta\vec{x}}{\Delta t} \quad .$$

We can do the same with the *instantaneous speed*:

$$\text{instantaneous speed} = \lim_{\Delta t \to 0} \frac{s}{\Delta t} \ .$$




We then see that, for infinitely short time intervals, an object doesn't have time to reverse direction, and so the argument we gave above leads us to say that the instantaneous speed is the same as the magnitude of the instantaneous velocity.

LOOKING AHEAD

This process of taking a limit of a quantity as the time interval goes to zero is very common in this course. Sometimes, I will use an abbreviation for this process. So, suppose quantity Q is a function of time and we would like to refer to its instantaneous time rate of change (even if perhaps we can't actually calculate it). Let

$$\mathrm{ITRC}(Q) = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t}$$

as a form of shorthand notation.

## Acceleration and Jerk

Sometimes, we want to know how quickly the velocity is changing. We define the *average acceleration* as the change in velocity *per* unit time:

$$\vec{a}_{AVE} = \frac{\Delta \vec{v}}{\Delta t} \ ,$$

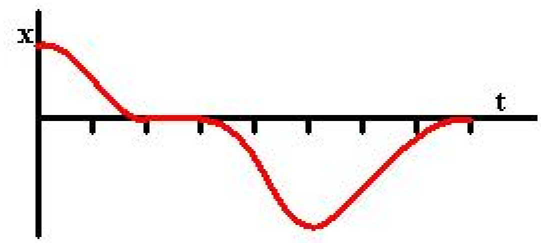
and the *instantaneous acceleration* as

$$\vec{a}_{INSTANT} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \mathrm{ITRC}(\vec{v}) \ .$$

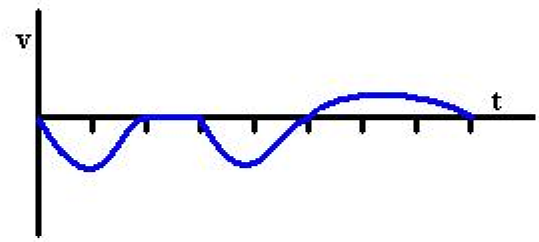
The analysis is the same for a as it was for v, so the effort will not be repeated here. Suffice it to state that the instantaneous acceleration can be found graphically by finding the slope of the line tangent to the v(t) curve. In one dimension (only!), the direction of the acceleration can be found this way: if the object is speeding up, the acceleration is in the same direction as the velocity and if it is slowing, the acceleration is in the opposite direction as the velocity.

EXAMPLE 2-3

Given a graph of the position x of an object as a function of time, <u>sketch</u> the velocity *v*. time curve.




There are a number of special points we can look for to guide us. Whenever the x(t) curve is horizontal, the velocity is zero. In this example, the curve is flat at t = 0, between 2 and 3, at 5, and at 8. Then, whenever the x(t) curve is moving downward, the velocity is negative (and vice versa), and the magnitude of the velocity is proportional to the steepness of the x(t) curve.




EXERCISE 2-2

From the v(t) curve shown, sketch the a(t) curve. Remember that since this is being done by eye, the accuracy of each successive curve will be reduced.

We can continue the process indefinitely. For example, the *jerk* is defined as

$$\vec{\mathrm{j}}_{\mathrm{AVE}} = \frac{\Delta\vec{\mathrm{a}}}{\Delta\mathrm{t}} \quad ; \quad \vec{\mathrm{j}}_{\mathrm{INST}} = \lim_{\Delta\mathrm{t}\to 0}\frac{\Delta\vec{\mathrm{a}}}{\Delta\mathrm{t}} = \mathrm{ITRC}(\vec{\mathrm{a}}) = \text{slope of acceleration graph,}$$

and so on with the *kick* and then the *lurch*:

$$\vec{\mathrm{k}}_{\mathrm{AVE}} = \frac{\Delta\vec{\mathrm{j}}}{\Delta\mathrm{t}} \quad ; \quad \vec{\mathrm{k}}_{\mathrm{INST}} = \lim_{\Delta\mathrm{t}\to 0}\frac{\Delta\vec{\mathrm{j}}}{\Delta\mathrm{t}} = \mathrm{ITRC}(\vec{\mathrm{j}}) = \text{slope of jerk graph ,}$$

$$\vec{\mathrm{l}}_{\mathrm{AVE}} = \frac{\Delta\mathrm{k}}{\Delta\mathrm{t}} \quad ; \quad \vec{\mathrm{l}}_{\mathrm{INST}} = \lim_{\Delta\mathrm{t}\to 0}\frac{\Delta\vec{\mathrm{k}}}{\Delta\mathrm{t}} = \mathrm{ITRC}\left(\vec{\mathrm{k}}\right) = \text{slope of kick graph .}$$

This process can continue as far as you like or need, although there seem not to be specific terms for the rest of these quantities. The acceleration, jerk, kick, and lurch are all vector quantities.

## Kinematic Equations

Let's use these definitions to derive some possibly useful relationships. Generally in Physics, but particularly in this course, we examine special cases. In this section, we'll restrict ourselves to situations where the object's acceleration is constant. In terms of notation, we'll make the following simplifications:

- all final quantities ($t_f$, $x_f$, $v_f$) can be replaced with the more general corresponding variables (t, x, v). This perhaps has a psychological aspect, in that we don't necessarily want to limit the end points of our problems to specific values;
- the problem starts at $t_i = 0$, so that we can just drop that term. If necessary, we can replace any t in the results with $t_f - t_i$;
- the arrow over vector quantities will be dropped. The direction of any vector will be given by the sign of the value inserted into the equation for solution.

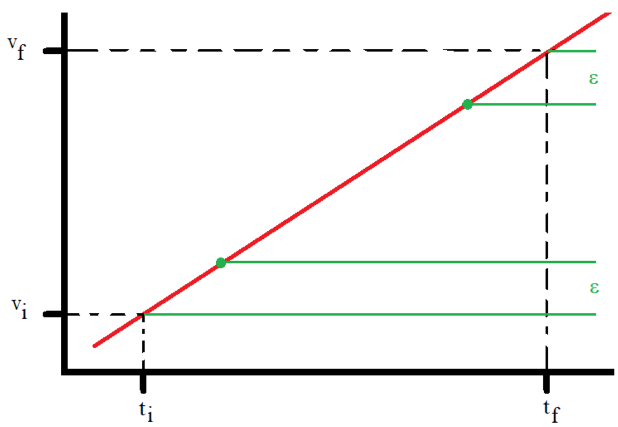
Let's start with the definition of the acceleration (since the acceleration is constant, that value is also the average value): [2]

$$\vec{a}_{AVE} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\vec{v} - \vec{v}_i}{t}$$

which re-arranges to

$$\vec{v} = \vec{v}_i + \vec{a}t \quad \text{(KEq 1)} \ .$$

We shall refer to this relationship as *kinematic equation Nr 1*.[3]




The next relationship is not developed from the definitions above. What would a graph of velocity look like if the acceleration is constant? Since the acceleration is the slope of the v(t) curve, constant slope means that the curve is a line. I've drawn the line with a positive slope, although it could just as well have a negative or even zero slope. We want to determine an expression for the average velocity that is independent of the definition. We need to average the infinitely many values the velocity has in the interval $t_i$ to $t_f$. We can do it without calculus if we're a little clever. First, average just the two endpoints to get

$$\frac{\vec{v}_i + \vec{v}_f}{2} \ .$$

Now, average these two points, one above $v_i$ by an amount ε and the other below $v_f$ by the same amount ε to get

$$\frac{(\vec{v}_i + \vec{\varepsilon}) + (\vec{v}_f - \vec{\varepsilon})}{2} = \frac{\vec{v}_i + \vec{v}_f}{2} \ .$$

So ε can have any value and result is the same average value for any given pair of symmetrically

[2] For example, if every student earns an 85 on an exam, what is the average exam grade?
[3] Although we said this will be valid only for constant acceleration, it is in fact valid if a is the time-averaged acceleration.

placed points. Since the overall average is the average of the pairs' averages, it should be clear that the overall average will also be

$$\vec{v}_{AVE} = \frac{\vec{v}_i + \vec{v}_f}{2} \quad \text{(KEq 2)}\,.$$

This is *kinematic equation Nr 2*. Note that this argument only works for lines, that is, when the acceleration is constant.

DERIVATION 2-1

We have two expressions for the average velocity, the definition and the one we just derived. The two expressions must be equal, so long as the condition of constant acceleration is met.

$$\vec{v}_{AVE} = \frac{\vec{v}_i + \vec{v}}{2} = \frac{\vec{x} - \vec{x}_i}{t}$$

Now, we'll substitute KEq 1 in for v:

$$\frac{\vec{v}_i + (\vec{v}_i + \vec{a}t)}{2} = \frac{\vec{x} - \vec{x}_i}{t}\quad.$$

A bit of math,

$$(2\vec{v}_i + \vec{a}t)t = 2(\vec{x} - \vec{x}_i)$$

$$2\vec{v}_i t + \vec{a}t^2 = 2(\vec{x} - \vec{x}_i)$$

$$\vec{v}_i t + \tfrac{1}{2}\vec{a}t^2 = \vec{x} - \vec{x}_i$$

and we have that

$$\vec{x} = \vec{x}_i + \vec{v}_i t + \tfrac{1}{2}\vec{a}t^2 \quad \text{(KEq 3)}\,.$$

DERIVATION 2-2

Let's start with the same two expressions for the average velocity as in the previous derivation.

$$\vec{v}_{AVE} = \frac{\vec{v}_i + \vec{v}}{2} = \frac{\vec{x} - \vec{x}_i}{t}$$

Multiply both sides by 2:

$$\vec{v}_i + \vec{v} = 2\frac{\vec{x} - \vec{x}_i}{t} \quad .$$

Here's the tricky part. We're going to take the dot product of each side with $\vec{a}t$; on the right, we'll use that exact term, but on the left, we'll use something equivalent to it (from KEq 1), $\vec{v} - \vec{v}_i$:

$$(\vec{v} - \vec{v}_i) \cdot (\vec{v}_i + \vec{v}) = 2\frac{\vec{x} - \vec{x}_i}{t} \cdot \vec{a}t$$

$$\vec{v} \cdot \vec{v} - \vec{v}_i \cdot \vec{v}_i = 2\vec{a} \cdot (\vec{x} - \vec{x}_i)$$

$$v^2 - v_i^2 = 2\vec{a} \cdot (\vec{x} - \vec{x}_i)$$

$$v^2 = v_i^2 + 2\vec{a} \cdot (\vec{x} - \vec{x}_i) \quad \text{(KEq 4).}$$ [4]

Now, we have four *kinematic equations* that are valid in the special case of constant acceleration:

$$v = v_i + at \quad \text{(KEq 1)} \;\; ;$$

$$v_{AVE} = \frac{v_i + v_f}{2} \quad \text{(KEq 2)} \;\; ;$$

$$x = x_i + v_i t + \tfrac{1}{2}at^2 \quad \text{(KEq 3)} \;\; ;$$

$$v^2 = v_i^2 + 2a(x - x_i) \quad \text{(KEq 4)} \;\; .$$

Various combinations and perturbations of these should allow for solving most problems. Here, however, is a warning: do not rely on the equations by themselves to solve problems. The equations are in a sense tools, but it still requires the brain to direct their use. Keep in mind that various textbooks and websites may use different kinematic equations; you must start your solutions with one or more of these specific four or a definition given here in this course and work from there.

As we start to do examples and homeworks, I suggest strongly that you stick to the model presented here. I think the steps shown form the best path to avoid making errors or omissions. If I knew a better way, I'd show you that. In Section 1, I made a point that Physics is not just plugging numbers into equations. However, to get started, that's basically how we're going to roll. The process is to make a sketch to help visualize the situation; this can be used to indicate the origin and which direction is positive for whomever is reading your solution. Then, we make an inventory of quantities we know, quantities we think we know, and quantities we want to know. Then, if we're lucky, there will be a kinematic equation that has only those quantities in it. If not, we may need

[4] There is a slightly quicker way to do this that unfortunately doesn't preserve the vector nature of the quantities.

to use two equations. Identify which equation is to be used. And again, as mentioned above, we're going to drop the arrows above the vector quantities for convenience.

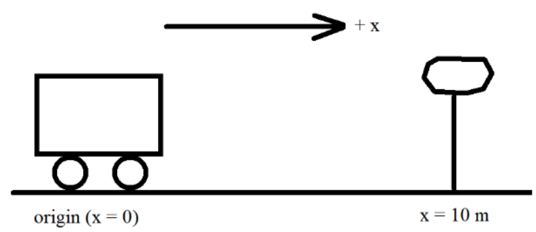
It is better if you manipulate the equations symbolically, inserting the numerical values just before the end. There are two reasons for this. The first is that you will start to see how the various quantities interact with one another as they appear over and over again throughout the course. The second is more practical. Suppose your boss tells you that the car was moving at 15 m/s, and you insert the numbers at the top of a long and tedious calculation. When he returns to tell you the initial speed as actually 12 m/s, what will you then need to do? Better to be able to just insert the new value into the penultimate step.

EXAMPLE 2-3

A distracted driver traveling at 15 m/s notices a stop sign when he is 10 m from the stop line. If the car decelerates at 6 m/s$^2$, how quickly is the car moving as it passes the stop line?

Let's write down the quantities which we know either implicitly or explicitly, as well as what we want to figure out; I call this the *inventory*.

Let positive x be in the direction the car is moving and the origin be where the driver first applies the brakes.




$x_i = 0$ m
$x_f = 10$ m
$v_i = 15$ m/s
$v_f = ?$ ←
$a = -6$ m/s$^2$ (a *deceleration* of 6 m/s$^2$ is an acceleration of - 6 m/s$^2$, since a velocity becoming less positive is the same as one becoming more negative).
$t = ?$

Since the kinematic equations are really all the same relationships presented in slightly different forms, we can look for one which contains all of the quantities above. Sometimes this works, sometimes not; in this case we're lucky:

$$v^2 = v_i^2 + 2a(x - x_i)\,. \quad {}^5$$

In fact, not much algebraic manipulation is necessary:

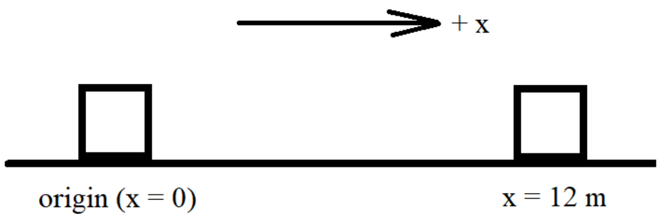
$$v = \sqrt{v_i^2 + 2a(x - x_i)} = \sqrt{15^2 + 2(-6)(10 - 0)} = \boxed{+10.3\ \mathrm{m/s}}\,.$$

[5] Since we're in one dimension, we can drop the dot product if we assign the proper sign to the vector values. If the displacement and acceleration are in the same direction (++ or --), then the dot product is positive as expected. If they are in opposite directions (+- or -+), then the dot product is correctly negative)

I've chosen the positive root, because I know that the car is travelling in the +x-direction.

We should spend a few moments discussing why the equation gave us two possible answers. The kinematic equations are valid if the acceleration is constant, in this case at -6 m/s$^2$, for all time. That is, it assumes that the car is starting off infinitely ago at x = negative infinity, travelling with a speed of plus infinity, slowing and slowing until it arrived at the origin at t=0 while traveling at 15 m/s. It then slowed to a stop. While the car in our problem may stop and remain stopped, the equation thinks that the car begins to move in the negative x-direction, passing the sign at -10.3 m/s, passing the origin at -15 m/s, and continuing back to negative infinity, arriving there with infinite speed at the end of eternity. It falls on us to make sense of whatever results the equations give us.

EXAMPLE 2-4




A box moving along the x-axis initially has a velocity of +8 m/s. It experiences a constant acceleration as it travels 12 meters, at which point it has a velocity of +26 m/s. What was the acceleration and for how much time did the box travel?

$x_i = 0$ m
$x_f = 12$ m
$v_i = +8$ m/s
$v_f = +25$ m/s
a = ? ←
t = ? ←

I'm pretty sure there is no one equation that will give us both quantities. Let's try KEq 4 to find the acceleration:

$$v^2 = v_i^2 + 2a(x - x_i)$$

$$a = \frac{v^2 - v_i^2}{2(x - x_i)} = \frac{26^2 - 8^2}{2(12 - 0)} = +25.5\,\frac{m}{s^2}\ .$$

Now, we know more, so we have more options. We could use KEq 3, although that would require solving a quadratic equation, or we can use KEq 1, which is much easier:

$$v = v_i + at$$

$$t = \frac{v - v_i}{a} = \frac{25 - 8}{25.5} = 0.67\text{ s}\ .$$

EXAMPLE 2-5

A different box moves along the x-axis, initially with velocity 6 m/s with an acceleration of +2 $m/s^2$. How long does it take to slide 8 meters?

$x_i$ = 0 m
$x_f$ = 8 m
$v_i$ = +6 m/s
$v_f$ = ?
a = +2 $m/s^2$
t = ? ←

Looks like we need to use KEq 3:

$$x = x_i + v_i t + \frac{1}{2}at^2 \quad .$$

Since this is a quadratic, this is one of the few exceptions as to when to insert values. Put the numbers in and re-arrange the equation to the standard $ax^2 + bx + c = 0$ format and make use of the well-known quadratic solution formula.

$$8 = 0 + 6t + \frac{1}{2}(2)t^2$$

$$t^2 + 6t - 8 = 0 \quad ; \quad a = 1, b = 6, \text{and } c = -8$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-8)}}{2(1)} = +1.12 \text{ sec or } -7.12 \text{ sec} \ .$$

Once again, we have received two solutions. In this case, we realize that the box arrives at its destination after it left its starting point, so the correct answer is +1.12 seconds.

HOMEWORK 2-3

A car with an initial velocity of +6 m/s accelerated for 4 seconds, by which time its velocity is +21 m/s. What was the car's acceleration and what is its displacement?

DISCUSSION 2-3

What should we do when the acceleration is not constant? Suppose that it is constant over some time interval, then changes abruptly to a different constant value.

EXERCISE 2-3

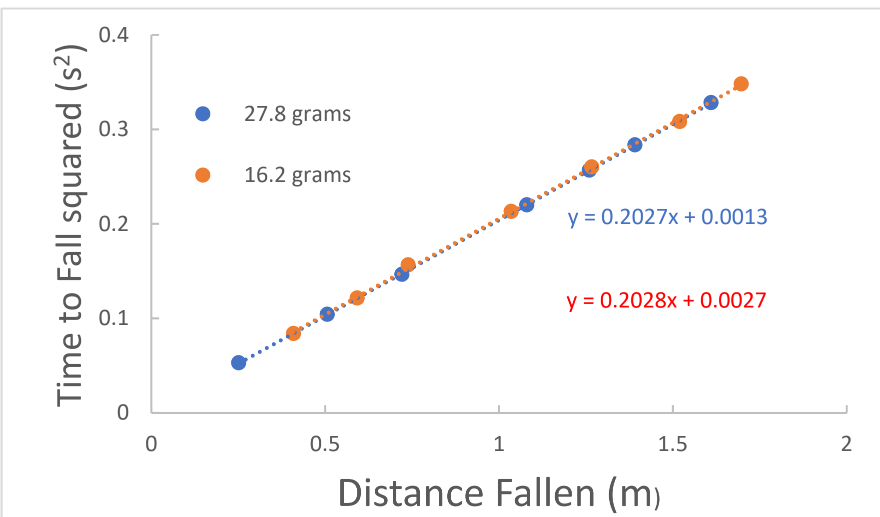
A car starts from rest at t = 0 and accelerates at +4 $m/s^2$ for 5 seconds. It then continues to accelerate at +6 m/s for an additional 3 seconds. How far has the car travelled in those 8 seconds?

HOMEWORK 2-4

> A speeder passes a parked black and white at 40 m/s and continues on, obliviously maintaining a constant velocity down the straight, level road. At that instant, the cop car starts from rest with a uniform acceleration of 5 m/s$^2$. How much time passes before the cop catches up to the speeder? How far does the cop car travel in that time? How quickly is the cop car going when it catches up with the speeder?

## Acceleration due to Gravity

In the very special case of an object moving freely near the surface of the earth under no other influence except the earth's gravity, the acceleration of the object will be some value near 9.8 m/s$^2$ downward (it does vary from place to place). You will verify this in a laboratory exercise. Your text probably refers to this quantity as the *acceleration due to gravity*, *g*. I would prefer that for now you use the symbol $a_g$, reserving g for the strength of the gravitational field, which is not the same thing, as we shall discuss in Section 5. Except in lab, we will be rounding this number to 10 m/s$^2$, since we are not trying to send a probe to Mars or anything complicated like that. The graph shows the results of a double experiment where metal balls of different masses were dropped from known altitudes H and the times to fall to the floor were measured. Let's spend some time on this.




First, we are assuming that he acceleration due to gravity is a constant, which may or may not be true. We can consider this assumption to be our *hypothesis*. If that is true, then we should be able to make use of the kinematic equations to make a testable prediction. For example, kinematic equation 3 is

$$x = x_i + v_i t + \frac{1}{2}at^2 \ .$$

If the balls start from rest at the origin, this reduces to

$$H = 0 + 0 + \frac{1}{2}a_g t^2 \quad \rightarrow \quad H = \frac{1}{2}a_g t^2 \ .$$

The parameter we control is the altitude, H; this is the *independent variable* and as such it should be placed on the horizontal axis. The *dependent variable* is the time, and it goes on the vertical axis. Here, our proposed relationship doesn't imply a linear relationship between H and t, but rather one between H and t$^2$:

$$t^2 = \frac{2}{a_g} H \ .$$

So, we won't plot the time against the altitude, but rather the <u>square</u> of the time against the altitude. The proportional relationship between $t^2$ and H is demonstrated by the data forming a line that passes through the origin.

Next, we'd lie to extract some information from the line. A line can completely defined by just two quantities, usually the slope and the y-intercept, although the x-intercept can replace one of the other two. Comparing the 'theoretical' relationship with the fit to experiment,

$$t^2 = \frac{2}{a_g} H$$
$$y = (\text{slope})x + (\text{intercept}) \ ,$$

we see that the slope should equal 2 divided by the acceleration. Solving, we obtain

| Mass | Acceleration due to gravity |
|---|---|
| 27.8 grams | 9.867 $m/s^2$ |
| 16.2 grams | 9.862 $m/s^2$ |

both results being less than 1% from the accepted value for Catonsville.

Let's review:

1. We assumed that $a_g$ is a constant.
2. Based on that assumption, we used out theory to predict the relationship between the altitude of release and time to fall.
3. We linearized the relationship by plotting $t^2$ instead of t *v*. H.
4. We did least squares best fits to determine the slopes and intercepts of the lines.
5. By comparing the 'theory' and the equation for a line, we verified that there is no missing constant term in the theory equation, and that the acceleration is indeed constant.
6. We determined the value of the acceleration for each ball.
7. We showed that the acceleration is independent of the mass of the object dropped.

EXAMPLE 2-6

Let's drop a water balloon onto the sidewalk from the top of a 20 m tall building. How quickly will the balloon be moving at the bottom and how long will it take to arrive there?

We have some choices to make. You can make any point you like the origin, but there are two obvious choices that would probably make the solution mathematically easier to solve: the top of the building and the foot of the building. Similarly, you can make up be positive, or down be positive. Let up be positive and the origin be the top of the building. In addition, the word 'drop' tells us something about the initial velocity. Then,

$x_i = 0$ m
$x_f$ = -20 m
$v_i = 0$ m/s
$v_f$ = ? ←
a = -10 m/s$^2$
t = ? ←

Let's check to see if there is a single kinematic equation that will give us the answer, and there is, KEq 4:

$$v^2 = v_i^2 + 2a(x - x_i) \quad ,$$

$$v = \sqrt{v_i^2 + 2a(x - x_i)} = \sqrt{0^2 + 2(-10)(-20 - 0)} = \sqrt{400} = -20 \text{ m/s} \quad .$$

We must take the negative root here, because the balloon is moving in the negative direction at the end of the problem. The equation doesn't know to do that; we need to keep an eye out. As for the time, the shortest method is to make use of the final velocity above and use KEq 1:

$$v = v_i + at$$

$$t = \frac{v - v_i}{a} = \frac{-20 - 0}{-10} = 2 \text{ s} \quad .$$

Alternatively, we might have used KEq 3:

$$x = x_i + v_i t + \frac{1}{2} a t^2 \quad ,$$

$$-20 = 0 + 0t + \frac{1}{2}(-10)t^2 \quad ,$$

which, while technically a quadratic equation, is easy to solve.

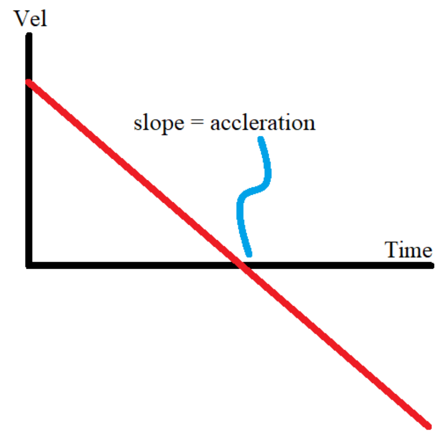
$$t^2 = \frac{2(-20)}{-10} = 4 \quad \rightarrow \quad t = +2 \text{ seconds}$$

We take the positive root because the balloon hits the ground after it's dropped.

DISCUSSION 2-4

Consider an object thrown straight upward. What is the acceleration on the way up? What is the acceleration on the way down? What is the acceleration right at the very top?

Well, let's consider the object's velocity. On the way up, it's slowing, so the acceleration is in the opposite direction, or downward. On the way down, the object is speeding up, so the acceleration is also downward. Is the acceleration zero at the very top? It is common to assume that it is zero, but that confuses that velocity with the acceleration. We can look at the graph and see that the slope of the v(t) graph when v = 0 is still -9.8 m/s$^2$. Or, think of the velocity the instant before the peak (positive) and the instant just after the peak (negative); the acceleration measures the change in velocity, which became more negative in that time interval.




EXERCISE 2-4

A ball is thrown from the street such that it rises past a 25m high window ledge at 12 m/s. Find

a) the velocity with which it was launched,
b) the maximum altitude above the street it reaches,
c) how long ago it was thrown, and
d) the time until it returns to the ground.

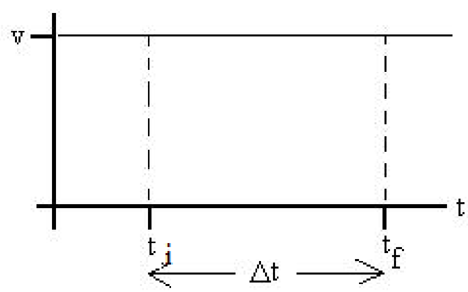
HOMEWORK 2-5

A rocket is launched straight upward from rest with an acceleration of 40 m/s$^2$ for 5 seconds, at which time it runs out of fuel. How high will it rise? HINT: what is the rocket doing at the moment it runs out of fuel?

## A Different Graphical Interpretation

DISCUSSION 2-5

Consider a special case of an object moving with constant velocity in one dimension. The graph of this motion is shown. We've already defined the average velocity (or in this particular case just the plain old velocity, since it's constant) as



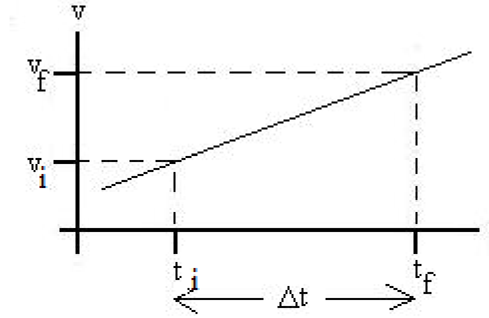

$$v = \frac{\Delta x}{\Delta t} \ .$$

From this relationship, we see that the displacement is given by

$$\Delta x = v\,\Delta t \ .$$

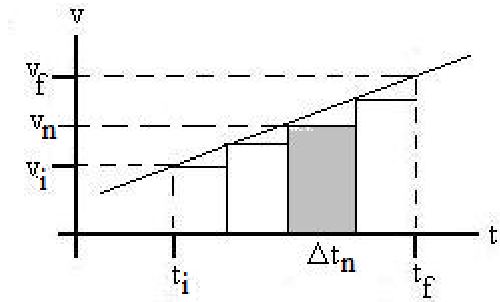
How is this quantity represented on the graph?

What about other cases? Let's try constant but non-zero acceleration, *i.e.*, the velocity is represented by a straight but not horizontal line. Since the velocity is not constant, we can not use the trick above, but we can use a craftier one: Let's break the time interval up into many very small time intervals, $\Delta t_n$ (n is just an index, *i.e.*, if n = 23, we're talking about the 23rd such interval), so that the velocity is almost constant over each. Then the displacement over each interval, $\Delta x_n$, is $v_n \Delta t_n$, and the total displacement should be




$$\Delta x = \sum_n \Delta x_n = \sum_n v_n \, \Delta t_n \quad .$$

Graphically, this is the sum of the areas of each of the little rectangles in the figure.




Of course, in this example, we are always underestimating the displacement, because we're multiplying each $\Delta t$ by the lower than average initial velocity of each interval. So, we want to make as many intervals as possible, each over as small a time interval as possible, to reduce this error. As we let the number of intervals go towards infinity, we can see that the little triangles atop each rectangle get smaller and smaller,[6] and that the total rectangle area we are counting tends toward the total area under the line. So, we see that the total displacement will be represented on such a graph as the area under the curve.

The original shape under this curve was a trapezoid, the area of which is

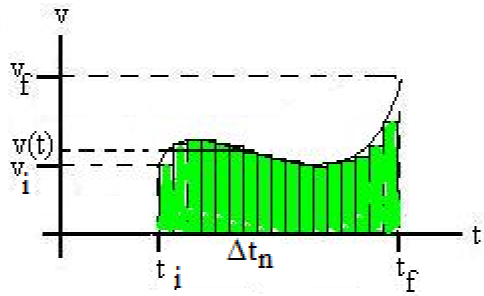
$$\frac{h_1 + h_2}{2} b \quad .$$

Substituting values results in

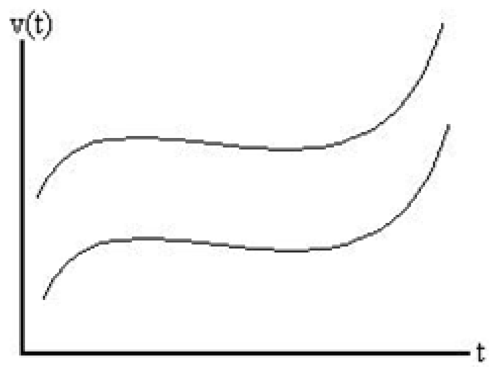
$$\frac{v_i + v_f}{2} \Delta t = v_{AVE} \, \Delta t = \Delta x \quad .$$

[6] As we increase the number of intervals by a factor, G, the number of triangles increases by G, but the base and height of each triangle are reduced by factor G and their areas each by factor $G^2$. The error caused by the missing triangles then goes as $G/G^2 = 1/G$. As $G \rightarrow \infty$, the missing area goes to zero.

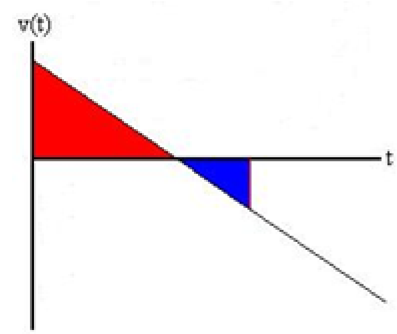
So, for an object with constant acceleration, the area under the velocity *v.* time curve represents the displacement. Then, we can generalize this result for any shape of curve: the area under a velocity *v.* time curve represents the displacement.



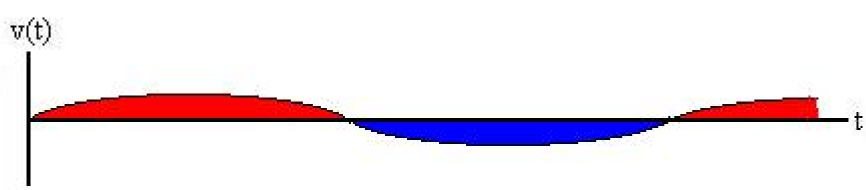

Also, we can use exactly the same argument to assert that the area under the acceleration *v.* time curve is the change in velocity. That bears repeating: we can't get the velocity from the curve, only the change in velocity, in the same way that we got the displacement, the change in position, for the velocity *v.* time curve, not the object's position itself.




Here is a quick explanation. Consider the two velocity *v.* time curves shown. Each of the two curves will generate the same acceleration curve, since the slopes of the two are the same for each value of time, t. So, given a particular acceleration curve, it would be impossible to determine which of an infinite number of velocity curves it was derived from.

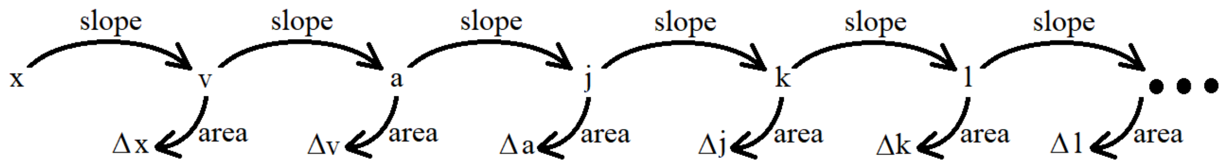



Let's look at another situation. In this case, the velocity starts out positive, but there is a negative acceleration (slope of the line). Eventually, the velocity becomes zero and the object comes momentarily to rest, having traveled through a displacement represented by the area under the curve (the red area). As time progresses, we see that the velocity becomes negative, the object reverses direction, and we would expect that the displacement from the starting position will decrease. The way to make this consistent with our interpretation of the area is that any area under the time axis must be considered negative. Indeed, the object may well arrive back at its starting point, for a total displacement of zero.




Consider a mass oscillating on a spring. The velocity *v.* time curve is shown below. Once *per* cycle, the object returns to its starting point for a displacement of zero. At that time, the area under the curve must equal zero.

So, to review this section, we can in principle find the ITRC of a quantity by examining the slope of that quantity's time graph, and we can find the change in the preceding quantity by looking at the area under its time curve.




HOMEWORK 2-6

> The velocity *v*. time graph of an object is approximated by a triangle that starts at v = 0 at t = 0, rises to a maximum of v = 7 m/s at t = 4 sec, then returns to zero at t = 13 sec. How far did the object travel? HINT: sketch the graph first.

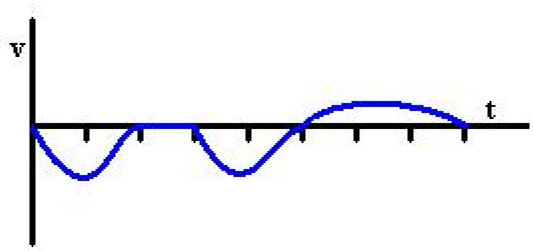
---

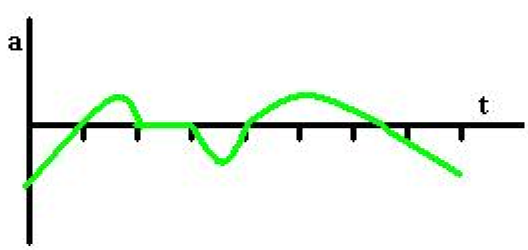
EXERCISE 2-1 Solution

In the first case, since the trip is made without reversing direction, the distance will be the same as the magnitude of the displacement, or 2 meters. Or, if you prefer, we've taken 2 m worth of steps. In the second case, each segment of the trip is one way, so we can count segment by segment.

3→15    12 m
15→ -8    23 m
-8→ 5    13 m
48 meters in total

EXERCISE 2-2 Solution




EXERCISE 2-3 Solution

We treat this as two separate problems, where to location of the car at the end of the first part becomes the initial location for the second part, and the final velocity from the first part becomes the initial velocity for the second part. So, for the first part,

$x_i = 0$ m
$x_f = ?$ ←
$v_i = 0$ m/s
$v_f = ?$ ←
$a = +4$ m/s$^2$
$t = 5$ sec

KEq 3:

$$x = x_i + v_i t + \frac{1}{2}at^2$$

$$x = 0 + (0)t + \frac{1}{2}4(5)^2 = 50 \text{ m}$$

KEq 1:

$$v = v_i + at = 0 + 4(5) = 20 \text{ m/s}.$$

For Part Two:

$x_i = 5$ m
$x_f = ?$ ←
$v_i = 20$ m/s
$v_f = ?$ ←
$a = +6$ m/s$^2$
$t = 3$ sec

KEq 3:

$$x = x_i + v_i t + \frac{1}{2}at^2$$

$$x = 50 + (20)3 + \frac{1}{2}6(3)^2 = 137 \text{ m}$$

EXERCISE 2-4 Solution

The first thing is that this is really three problems. Every kinematic problem has a starting point and an ending point. For Parts a and c, the problem begins at the ground and ends at the window ledge with the ball rising. Let's put the origin at the foot of the building and make upwards be positive. Then, the inventory looks like this:

$x_i = 0$ m
$x_f = 25$ m
$v_i = ?$ ←

$v_f$ = +12 m/s
a = - 10 m/s$^2$
t = ? ←

KEq 4:

$$v^2 = v_i^2 + 2a(x - x_i)$$

$$v_i^2 = v^2 - 2a(x - x_i)$$

$$v_i = \sqrt{v^2 - 2a(x - x_i)} = \sqrt{12^2 - 2(-10)(25 - 0)} = +25.4 \text{ m/s}$$

We take the positive root because the ball was obviously thrown upward.

KEq 1:

$$v = v_i + at$$

$$t = \frac{v - v_i}{a} = \frac{12 - 25.4}{-10} = 1.34 \text{ seconds} .$$

Part b is a different problem. It can start either at ground level or at the window ledge, but it definitely ends at the highest altitude, where, for a moment, the ball stops. Let's start at the ledge.

$x_i$ = 25 m
$x_f$ = ? ←
$v_i$ = +12 m/s
$v_f$ = 0 m/s (the ball stops)
a = - 10 m/s$^2$
t = ?

KEq 4:

$$v^2 = v_i^2 + 2a(x - x_i)$$

$$x = x_i + \frac{v^2 - v_i^2}{2a} = 25 + \frac{0^2 - 12^2}{2(-10)} = 32.2 \text{ m} .$$

Keep in mind that you could have used KEq 1 to find the time, then KEq 3 to find the altitude. There are often several paths to the answer.

Part d is yet another problem, starting at the window ledge and ending when the ball hits the ground.

$x_i$ = 25 m
$x_f$ = 0 m (lands on the ground)

$v_i$ = +12 m/s
$v_f$ = ?
a = - 10 m/s$^2$
t = ? ←

KEq 3

$$x = x_i + v_i t + \frac{1}{2} a t^2$$

Since we're solving for the time, let's go ahead and insert values.

$$0 = 25 + 12t + \frac{1}{2}(-10)t^2$$

$$5t^2 - 12t - 25 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(5)(-25)}}{2(5)} = +3.74\,\text{sec or} - 1.34\,\text{sec}\ .$$

These are the times that the ball is on the ground, and we want the one that is in the future, that is, at 3.74 seconds. Note also that we have solved Part c again, since the ball was launched from the ground 1.34 seconds before reaching the ledge.

## SECTION THREE – KINEMATICS IN TWO DIMENSIONS

In the last section, we discussed the kinematics of a point mass in one dimension. Again, kinematics describes the motion of an object without regard to the cause of that motion. In this section, we shall examine two special cases of two dimensional motion: projectile motion and uniform circular motion.

We need a way of keeping track of the motion of a particle. Luckily, we discussed this back in Section One, where we defined the *position vector* $\vec{\mathbf{r}}$ as

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} \ .$$

The displacement is then

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = (x_f\,\hat{\imath} + y_f\,\hat{\jmath}) - (x_i\,\hat{\imath} + y_i\,\hat{\jmath}) = (x_f - x_i\,)\,\hat{\imath} + (y_f - y_i\,)\,\hat{\jmath} = \Delta x\,\hat{\imath} + \Delta y\,\hat{\jmath} \ .$$

so that the displacement is the vector sum of the individual displacements in the x and y directions (no surprise there). The average velocity is

$$\vec{v}_{AVE} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta(x\,\hat{\imath} + y\,\hat{\jmath})}{\Delta t} = \frac{\Delta x\,\hat{\imath} + \Delta y\,\hat{\jmath}}{\Delta t} = \frac{\Delta x}{\Delta t}\,\hat{\imath} + \frac{\Delta y}{\Delta t}\,\hat{\jmath} = v_{x\,AVE}\,\hat{\imath} + v_{y\,AVE}\,\hat{\jmath} \ .$$

The instantaneous velocity **v** is defined as before as

$$\vec{v}_{INST} = \lim_{\Delta t \to 0} \vec{v}_{AVE} = \lim_{\Delta t \to 0} v_{x\,AVE}\,\hat{\imath} + \lim_{\Delta t \to 0} v_{y\,AVE}\,\hat{\jmath} = v_{x\,INST}\,\hat{\imath} + v_{y\,INST}\,\hat{\jmath} \ .$$

And, of course, because the acceleration is to the velocity as the velocity is to the position, we can immediately write that

$$\vec{a}_{AVE} = a_{x\,AVE}\,\hat{\imath} + a_{y\,AVE}\,\hat{\jmath} \quad \text{and} \quad \vec{a}_{INST} = a_{x\,INST}\,\hat{\imath} + a_{y\,INST}\,\hat{\jmath} \ .$$

DISCUSSION

> Consider a ball whirled around on the end of a string at constant speed. Is the velocity of the ball constant? Is its acceleration?

---

### PROJECTILE MOTION IN TWO DIMENSIONS

Projectile motion describes objects that are thrown, dropped, launched, tossed, pitched, hurled, catapulted, or chucked near the surface of a planet. Such objects are said to be in *free fall*. We shall assume the following for now:

> The planet's gravitational field is uniform, *i.e.*, constant in direction and magnitude. Once an object is launched, the only agency acting on it is gravity; therefore its acceleration is a constant $a_g$ downward.

This assumption leads us to suspect that the horizontal and vertical motions of an object are independent. We confirmed to some degree of satisfaction by observing a demonstration.

DEMONSTRATION 3-1

VIDEO

First, two balls were released from rest at the same time and allowed to fall toward the table; they arrived at the same time. Then, one ball was dropped while the other was launched horizontally from the same height at the same time; once again, they arrived at the same instant. This led us to an interesting conclusion, namely that the motions of the object in the horizontal and vertical direction will be independent of one another, thereby making a two-dimensional problem in fact two one-dimensional problems. Of course, there are many situations where this is not true. For example, if we were to account for *drag*, or as you seem to know it, *air resistance*, this assumption could be false.

So, following our assumptions, we have two sets of kinematic equations, which I am simply copying from Section Two,

| | |
|---|---|
| $v_{xf} = v_{xi} + a_x t$ | $v_{yf} = v_{yi} + a_y t$ |
| $v_{x\,AVE} = \dfrac{v_{xf} + v_{xi}}{2}$ | $v_{y\,AVE} = \dfrac{v_{yf} + v_{yi}}{2}$ |
| $x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$ | $y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$ |
| $v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$ | $v_{yf}^2 = v_{yi}^2 + 2a_y(y_f - y_i)$ |

with the time as the obvious connection between the two motions.

Before we start on examples, let me talk briefly about what I call Rule Number One,[1] which says that in problems in which there is acceleration, you should make one of the coördinate axes in the direction of the acceleration and the other perpendicular to that. The reason for this is to avoid breaking the acceleration into components, an action that generally makes the mathematics of solving a problem much more difficult. For projectile problems, this probably seems very natural; make horizontal the x-axis and vertical the y-axis. Remember, though, as the semester moves along, that the situation may change.

[1] Strictly speaking, it's a Really Strong Suggestion.

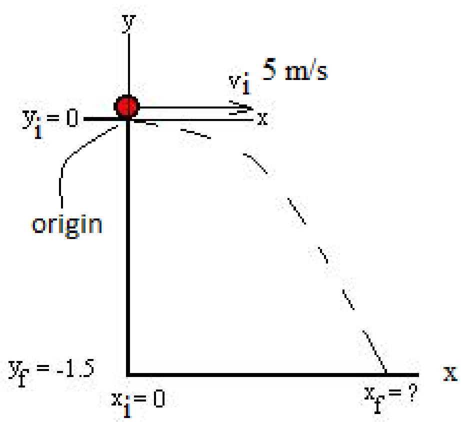
Here follows an example that you should use as the model for solving most projectile problems.

ADMONITION

When we do projectile problems, we remember that the problem runs from the moment <u>just after the ball</u> leaves the table to <u>just before it hits the floor</u>. If the problem asks for the final velocity, do not assume it is zero because the ball hit the floor and presumably stopped! During the collision with the floor, there was another agency besides gravity acting on the ball, and so the acceleration was not constant and the kinematic equations are not valid.

EXAMPLE 3-1

A ball is rolled horizontally off a table 1.2 m in height at 5 m/s. How far from the base of the table will the ball strike the floor?




First, draw a figure to help visualize the situation, including a system of axes with an origin. Your choice of origin can be arbitrary, but in this problem, there are two obvious locations: the top edge of the table and the foot of the table. The top is a slightly better choice for reasons you are invited to work out on your own. But don't get hung up on it, the bottom will work out O.K., too. The axes are chosen to be x horizontal and y vertical, according to Rule Number One above. All these things are labeled in the diagram so that whoever is grading your paper can easily tell what you are doing. Here, I've added in a few other pieces of information as well.

Next, make your inventory of what you know, what you think you know, and what you want to know. This is pretty standard for every problem. We're interested in what's happening in the x-direction, so let's start there. I use question marks for quantities I don't know and arrows for the ones I don't know but want to know.

$x_i = 0$ m

$x_f = ?\ \leftarrow$

$v_{xi} = +5$ m/s
$v_{xf} = +5$ m/s (why?)
$a_x = 0$ m/s$^2$ (the acceleration is downward, and not at all horizontal, once the ball is in free fall)
$t = ?$

As we did in the last section, we'll try to find a kinematic equation, or a combination, that will give us what we want to know. Is there one?

Since there is not enough information on the x-side, we need to look to the y-side. Here I will give you what I call an 80% Rule.[2] Generally, it's the time that is common to both sides of the problem, so I will find the time for the y-side, if possible, then bring it back over to the x-side, at which point I may have enough information there to solve. Since the time features prominently in KEq. 3, I'll probably use that on <u>both</u> sides.

$y_i = 0$ m
$y_f = -1.2$ m (upward is positive and the ball moved downward from the origin)
$v_{yi} = 0$ m/s (the ball was travelling horizontally and not at all vertically as it left the table)
$v_{yf} = ?$
$a_y = -10$ m/s$^2$ (we chose upward to be positive)
$t = ?$

Next, we state which principle of Physics we are using, in this case, KEq. 3:

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

As discussed in Section Two, it's best to try to manipulate the symbols to secure a general abstract solution, but since we want to learn the time, KEq. 3 will become a quadratic equation in t, which is the exception to our rule. Inserting the numbers and re-arranging to the standard format leaves us with

$$(5)t^2 + (0)t + (-1.2) = 0\,,$$

which, it turns out, we can solve directly:

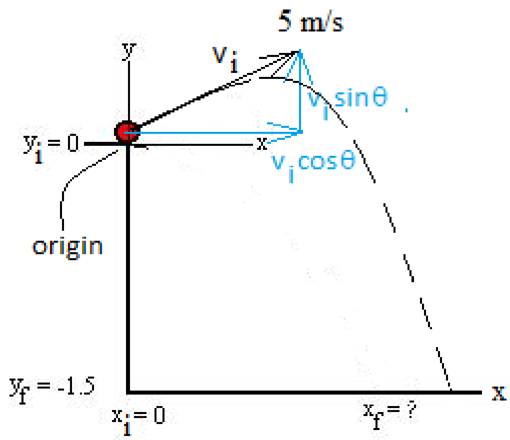
$$t = \pm\sqrt{\frac{1.2}{5}} = \pm\,0.49 \text{ seconds}\ .$$

Since the ball hits the floor after it leaves the table, the time must be positive, so t = 0.49 seconds. We'll take this back to the x-side to find $x_f$.

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 5(0.49) + 0(0.49^2) = 2.45 \text{ m}\ .$$

EXAMPLE 3-2

[2] The 80% number is obviously made up, I merely mean that this will work a large *per centage* of the time and it's how I myself would start such a problem. If it doesn't work, then try something else.




Take the same ball as above and launch it from the edge of the table at 5 m/s at a 30º angle above the horizontal. How far from the foot of the table will it land?

What is the only way in which this problem is different from the previous example? How will you deal with that difference?

Use the same origin and coördinate system as above, because, well, why not?

$x_i = 0$ m

$x_f = ?$ ←

$v_{xi} = v_i \cos\theta = 5 \cos(30^\circ) = +4.33$ m/s (find the x-component of the initial velocity)

$v_{xf} = +4.33$ m/s

$a_x = 0$ m/s$^2$

$t = ?$

Well, we don't have any more information about the x-motion this time around than we did the last, so our plan should be the same as for the previous example. Let's move on to inventory for the y-side:

$y_i = 0$ m

$y_f = -1.2$ m (upward is positive and the ball moved downward from the origin)

$v_{yi} = v_i \sin\theta = 5 \sin(30^\circ) = +2.5$ m/s

$v_{yf} = ?$

$a_y = -10$ m/s$^2$ (we chose upward to be positive)

$t = ?$

KEq. 3:

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

This will be quadratic, so insert the numbers and re-arrange:

$$(5)t^2 + (-2.5)t + (-1.2) = 0\,,$$

$$t = \frac{-(-2.5) \pm \sqrt{(-2.5)^2 - 4(5)(-1.2)}}{2(5)} = -0.3 \text{ OR } +0.8 \text{ seconds}\ .$$

Taking the positive time back to KEq.3 in the x-side:

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 4.33(0.8) + 0(0.8^2) = 3.46 \text{ m}\ .$$

HOMEWORK 3-1

A ball is thrown horizontally from the top of a 26 m tall building and hits the ground 12 meters from the base of the building. With what initial speed was the ball thrown?

EXERCISE 3-1

A classic problem involves a hunter on the ground trying to shoot a monkey at the top of a tree. The hunter is 30 m away for the base of the tree, the tree is 40 m high, and the speed of the arrow, once off the bow, is 35 m/s. Not having taken Physics, the hunter aims directly at the monkey and shoots. The monkey, however, sees the hunter shoot, and figures the quickest escape is simply to fall immediately from the tree towards the ground. Show that, in spite of this, the hunter hits the monkey after all. You should ignore the hunter's height, that is, the arrow starts at ground level.

EXAMPLE 3-3

Let's try an example where we don't use the 80% Rule. An object is thrown horizontally from the top of a building of height H and hits the ground below four seconds later at a 45° angle. How tall is the building and with what speed was it launched? How far from the base of the building did the object land?

You should draw the figure for this. Let's put the origin at the base of the building and make positive x be horizontal to the right and positive y be upward. What do we know?

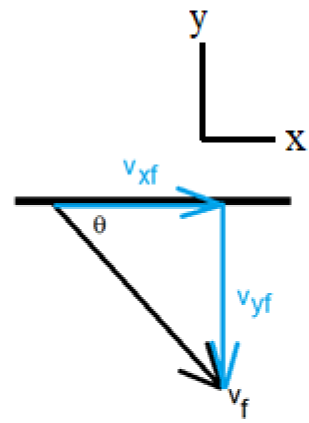
| | |
|---|---|
| $x_i = 0$ m | $y_i = ? \leftarrow$ (this is the height H) |
| $x_f = ? \leftarrow$ | $y_f = 0$ |
| $v_{xi} = ? \leftarrow$ | $v_{yi} = 0$ m/s (launched horizontally) |
| $v_{xf} = v_{xi}$ | $v_{yf} = ?$ |
| $a_x = 0$ m/s$^2$ | $a_y = -10$ m/s$^2$ |
| $t = 4$ seconds | |

Finding the height of the building is straightforward with KEq. 3:

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$H = y_i = y_f - v_{yi}t - \frac{1}{2}a_y t^2 = 0 - 0(4) - (-5)(4^2) = 80\text{ m} \ .$$

Now for the x values. Looking at the x-side, we see that even knowing the time does us no good. So, what else links the two sides? We know something about the final velocity components. The angle in the diagram is -45° (below the x-axis) and the ratio of the final velocity components is




$$\frac{v_{yf}}{v_{xf}} = \tan\theta = \tan(-45^o) = -1 \quad \rightarrow \quad v_{xf} = -v_{yf} .$$

Since the final x velocity is the same as the initial, KEq. 1 tells us that

$$v_{xi} = v_{xf} = -v_{yf} = -(v_{yi} + a_y t) = -(0 + (-10)(4)) = 40 \text{ m/s} .$$

Lastly, $x_f$ is given by KEq. 3 as

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 40(4) + 0 = 160 \text{ m} .$$

DISCUSSION

Students, if asked, often guess that the object lands 80 meters from the base of the building; after all, it hit the ground at a 45° angle, and the building is 80 m tall. This would seem to imply that the object followed a straight line from the top of the building after having made an abrupt change of direction immediately after launch. In a moment, we'll discuss the path actually taken by the object.

EXERCISE 3-2

Repeat Example 3-3 if the object had hit instead at an angle 53 degrees below the horizontal.

## Shape of a Projectile's Path

There are some interesting ideas circulating about the shape of the path (the *trajectory*) taken by a thrown object. As mentioned, some students assume that the object of the previous example follows a straight line path from the top of the building to the ground. On the other hand, cartoon physics says that a coyote running horizontally off a cliff continues horizontally, until he realizes his predicament, then falls straight downward. Let's try to determine the actual type of path a projectile will take through space near the surface of the earth, that is, we want y as a function of x.

DERIVATION 3-1

Start once again with the kinematic equations; we'll use our 80% Rule. Call the starting point the origin, and let upward be +y and horizontal direction of motion be +x. Then,[3]

| | |
|---|---|
| $x_i = 0$ | $y_i = 0$ |
| $x_f = ?$ | $y_f = ?$ |
| $v_{xi} = v_o \cos\theta_o$ | $v_{yi} = v_o \sin\theta_o$ |
| $v_{xf} = ?$ | $v_{yf} = ?$ |
| $a_x = 0$ | $a_y = a_g$ |
| $t = ?$ | |

This time, we'll start with the x-side and find the time:

$$x = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$x = 0 + v_o\cos\theta_o t + 0 \quad \rightarrow \quad t = \frac{x}{v_o\cos\theta_o} \; .$$

Now to the y-side:

$$y = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

$$y = 0 + v_o\sin\theta_o \frac{x}{v_o\cos\theta_o} + \tfrac{1}{2}a_g\left(\frac{x}{v_o\cos\theta_o}\right)^2 = (\tan\theta_o)\, x + \left(\frac{a_g}{2v_o^2\cos^2\theta_o}\right) x^2 \; .$$

This looks messy, but that's O.K., because we don't care at the moment about most of it. For any given launch of an object, $v_o$ and $\theta_o$ are fixed. We can't go back and change their values midway through the trip. Let's replace the tangent term with a generic positive constant, A. Then, lump all the constants in the $x^2$ term together and call them negative constant B (remember that $a_g$ is negative here):

$$y(x) = Ax + Bx^2 \; .$$

You should I hope recognize this form of curve; it is an example of a *parabola*, specifically one 'open down' and symmetric around a vertical axis.

So, so long as we meet the conditions outlined at the beginning of this section, any object thrown near a planet's surface should follow a parabolic path.

---

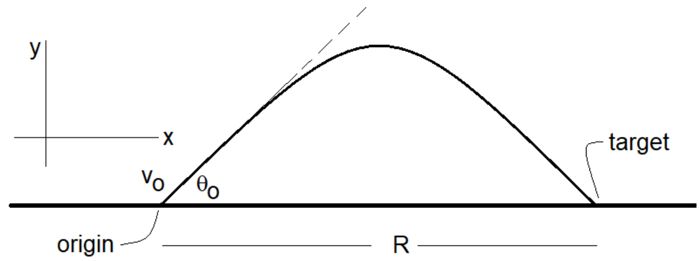
## The Range Equation

[3] Notice that the initial speed is labelled $v_o$ here. A 'nought' subscript denotes a specific value that isn't actually known.

Let's discuss a special case of projectile motion which is of historical interest. In the 17th and 18th century, being a physicist usually meant being an artillery officer. As is usual, we will consider a special case.

DERIVATION 3-2

Consider a flat horizontal plain (which is also a plane) on which are located a battery and a target. Given an initial projection angle $\theta_o$ (*elevation*) and launch speed $v_o$ (*muzzle velocity,* for guns or cannon), how far will the projectile land from the gun (*range*, R)?




Because the plane is horizontal and flat, $y_o = y_f = 0$. We'll launch our projectile from the origin, for convenience.

| | | |
|---|---|---|
| $x_i = 0$ | | $y_i = 0$ |
| $x_f = R$ | ← | $y_f = 0$ |
| $v_{xi} = v_o \cos\theta_o$ | | $v_{yi} = v_o \sin\theta_o$ |
| $v_{xf} = ?$ | | $v_{yf} = ?$ |
| $a_x = 0$ | | $a_y = a_g$ |
| $t = ?$ | | |

Once again, we'll try our 80% Rule, starting on the y-side to find the time with KEq 3:

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 \ .$$

Insert some values and substitutions:

$$0 = 0 + v_o \sin\theta_o \, t + \tfrac{1}{2}a_g \, t^2 \, ,$$

$$0 = \left(v_o \sin\theta_o \ + \tfrac{1}{2}a_g \, t\right) t \, .$$

For the right side here to equal zero, either t = 0 (which is uninteresting; we already know the object was on the ground at the start of the problem), or

$$v_o \sin\theta_o \ + \tfrac{1}{2}a_g \, t = 0,$$

in which case

$$t = \frac{-2 \, v_o \sin\theta_o}{a_g} \quad .$$

This may look a bit strange. Is the time actually negative? Did we hit the target before we launched the projectile? No, we're O.K. Having said that, I hate negative signs, so I'll take the absolute value of the negative acceleration and that will cancel the negative sign in the numerator:

$$t = \frac{2\, v_o \sin\theta_o}{|a_g|} \quad .$$

So, this is the time for the entire trip. How far does the projectile travel horizontally in that time. Back to KEq. 3.

$$x = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$R = 0 + v_o\cos\theta_o t + 0 = v_o\cos\theta_o \frac{2\, v_o \sin\theta_o}{|a_g|} = \frac{v_o^2(2\sin\theta_o\cos\theta_o)}{|a_g|} \quad .$$

Finally, we'll use a trig identity to make this prettier: 2 sinα cosα = sin(2α). This brings us to the final result of

$$R = \frac{v_o^2 \sin(2\theta_o)}{|a_g|} \quad .$$

Remember that this result is valid only when the assumed conditions are met, particularly that the launching and landing spots must be at the same altitude. Otherwise, you will need to treat this as a projectile motion problem to be solved from scratch.

DISCUSSION

In 'real life,' we would also worry about a number of effects that would make the result above invalid, particularly for large ranges. Can you think of at least three?

EXAMPLE 3-4

A ball is thrown at 20 m/s at an angle of 25° above the horizontal over a flat surface. How far from the launch point will the ball land?

This is straight plug-and-chug:

$$R = \frac{v_o^2 \sin(2\theta_o)}{|a_g|} = \frac{20^2 \sin(2 \times 25^o)}{10} = \frac{20^2 \sin(50^o)}{10} = 30.6\text{ m} \quad .$$

EXAMPLE 3-5

Let's go the other way. The launch speed is 50 m/s and we wish to hit a target 160 m away on a flat surface. At what angle should the object be launched?

Re-arranging the Range Equation for theta,

$$\theta_o = \tfrac{1}{2}\arcsin\left(\frac{R|\,a_g\,|}{v_o^2}\right) = \tfrac{1}{2}\arcsin\left(\frac{160 \times 10}{50^2}\right) = \tfrac{1}{2}\arcsin(0.64) = \tfrac{1}{2}(40^o) = 20^o\,.$$

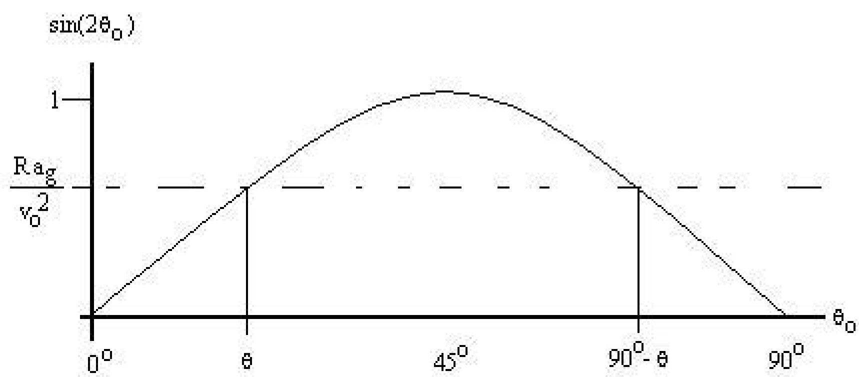
DISCUSSION

Examine the Range Equation again. For a given launch speed $v_o$, what launch angle will result in the largest range?

If we start at $0^o$, the range will be zero; the projectile will just hit the ground right away. As we increase the elevation angle, the range will increase until the sine function maxes out at 1. What launch angle $\theta_o$ does that correspond to? If that angle results in the maximum range, what happens when we go above that angle?

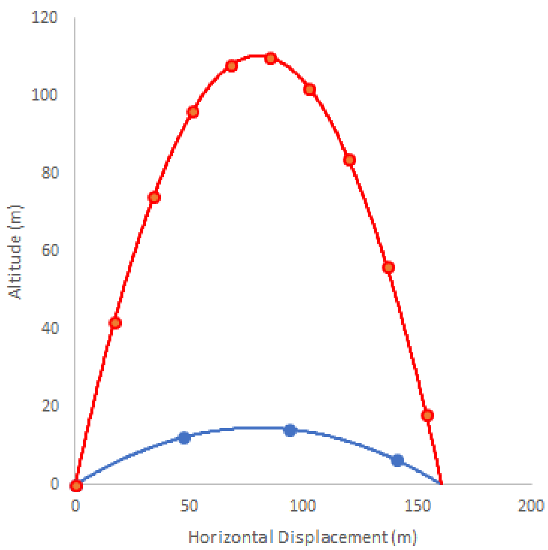
Let's plot the sine of twice the launch angle against the launch angle, $\theta_o$. Solving the example above meant finding an angle such that

$$\sin(2\theta_o) = \frac{R|\,a_g\,|}{v_o^2}\ ,$$




as can be seen on the graph; the solution is at the intersection of the two curves. But, if you follow the dotted line over to the right, you will see that there is a <u>second angle</u> that fulfills this requirement. Since this curve is symmetric, the second larger angle should be the *complement* of the smaller one. So, there are actually two answers to the example above, the $20^o$ we found, OR $70^o$, the complement of $20^o$. Except of course for $45^o$, which is its own complement, there should be two answers to these problems.

So, how is this possible? At a low angle, the projectile is not in the air long, but it has a high x-component of velocity, while at a high angle, the projectile spends a lot of time in the air, but has a correspondingly lower x-component of velocity. These two effects combine to give the same final x displacement as for the low angle case. In the same way, the lower angle launch has a smaller initial y velocity component than the higher angle launch, and so will not reach as high an altitude. The lower angle is useful in tank warfare, where it is important to hit the other guy before he gets off a shot at you, while the second is good if there are obstacles around your target. In the example above, the travel times of the two paths are




$$t = \frac{2\, v_o \sin\theta_o}{|a_g|}$$

$$t_{20^o} = \frac{2(50)\sin(20^o)}{10} = 3.42 \text{ seconds}$$

$$t_{70^o} = \frac{2(50)\sin(70^o)}{10} = 9.40 \text{ seconds} .$$

The figure at left shows the trajectories of this object for each of the angles, 20º and 70º. The dots represent intervals of one second. One can see that if two such objects were launched simultaneously, the one launched at 70º would still be rising when the other arrived at its target.

HOMEWORK 3-2

Derive an expression (that is, start with the kinematic equations) in terms of $v_o$, $\theta_o$, and $a_g$ for the maximum altitude H reached by a projectile. Use this result to calculate the maximum altitudes for the object launched in Example 3-5 for each angle (20º and 70º). Check your results against the graph above.

EXERCISE 3-3

Our target is 350 meters away along a flat surface. Our launcher will throw the projectile with an initial speed of 55 m/s. At what angle (or angles) could we launch in order to hit the target?

HOMEWORK 3-4

You're playing golf on a flat fairway. The green is 150 m away, and you can send the ball away at 60 m/s. At what angle or angles could you hit the ball for a hole-in-one?

HOMEWORK 3-5

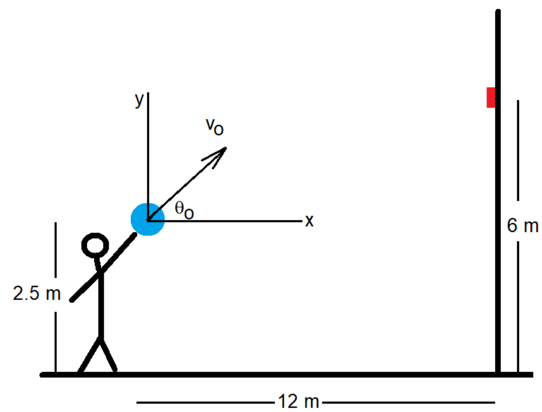
Show that, for a projectile thrown at an initial angle $\theta_o$ above a flat horizontal plane, the maximum altitude H is related to the range by

$$H = \frac{R \tan \theta_o}{4} \;.$$

HOMEWORK 3-6

To score in Sportsball, you must successfully throw the sportsball at a target on the wall from a distance of 12 m. The target is 6 meters above the floor, and you release the ball with a speed of 16 m/s at an altitude of 2.5 m above the floor. At what angle or angles $\theta_o$ from the horizontal should you throw the ball. You may find this relationship useful:

$$\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta} .$$

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## UNIFORM CIRCULAR MOTION

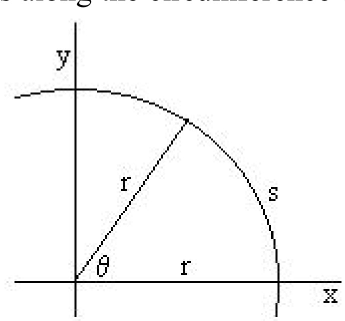
Consider an object moving at constant speed v in a circle of radius r; forget about gravity for now.

DISCUSSION

Does this object have a constant velocity? Which kind of a quantity is velocity? What are the two parts of velocity? Do they both need to be constant for the velocity to be constant? This means that the object is doing what?

Let's find that quantity. We'll do it two ways, one we could almost call' traditional,' and the other a bit less straightforward, but which will leave us with some additional useful relationships.

Before we start, let's define a quantity we will find useful through the rest of this course. Consider our object moving in a circle. Suppose that it has moved a distance s along the circumference of a circle of radius r, where arc s subtends an angle θ. The *arclength* relationship tells us that

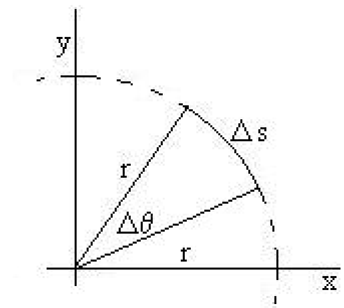



$$s = r\,\theta_{RADIANS} \;.$$

A *radian* is the angle such that the arclength s is equal to the radius r, or about 57.3°. Clearly, if we halve the angle, we also halve the distance along the arc, so that θ and s are proportional by the factor

r. As an extreme example, there are $2\pi$ radians in a circle, since the circumference (the arclength all the way around) is $2\pi r$.

We should next find a way of describing changes in the object's position, or the *angular distance* $\Delta\theta$, so that




$$\Delta s = r\,\Delta\theta \;.$$

If we consider the instantaneous time rate of change of each side of the equation above, we obtain

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \;,$$

Since the radius is a constant. The left side we recognize as the speed $v_T$, We add the 'T' subscript because the velocity is tangent to the circle. On the right side we will define the *angular speed* $\omega$ (omega), the angular distance *per* unit time:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \;.$$

This gives us a choice in describing the motion of the object, in terms of either its speed around the circle or its angular position as seen from the center of the circle:

$$\boxed{v_T = r\,\omega \;.}$$

EXAMPLE 3-6

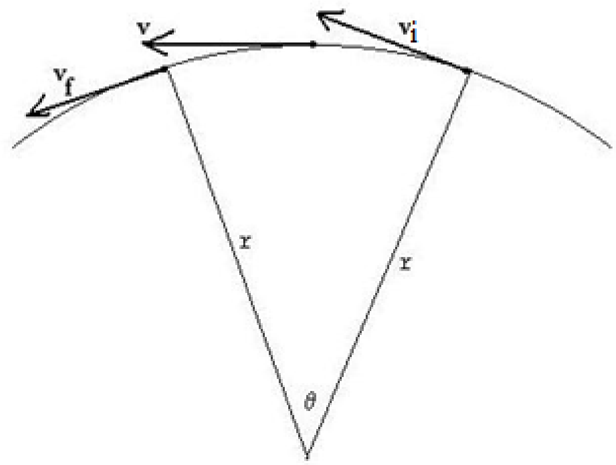
Consider a race car moving around a circular track at 70 m/s. If the radius of the track is 300 meters, what is the car's angular speed as seen from the center of the curve?

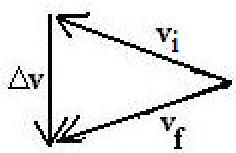
$$v_T = r\,\omega \;\;\rightarrow\;\; \omega = \frac{v_T}{r} = \frac{70}{300} = 0.23\ \text{radians/second}.$$

The following derivations will be valid only for objects moving with constant speed. We'll deal with the other case later in the course.

DERIVATION 3-3

Consider a particular point in the object's path, where the velocity vector $\vec{v}$ is of course tangent to the circle: let's also look just before this point ($\vec{v}_i$) and just after this point ($\vec{v}_f$), so that it takes time $\Delta t$ to get from the former to the latter point. These three vectors have the same magnitude (speed is constant) even though the directions are different. Since the direction of the velocity changed, there was an acceleration. How big and in what direction is the change, $\Delta\vec{v}$? Let's move the velocity vectors around so that they are tail to tail:




The vector $\Delta\vec{v}$ is the vector that needs to be added to $\vec{v}_i$ to get the final result, $\vec{v}_f$. As the figure shows, $\Delta\vec{v}$ points toward the center of the circle (*centripetal*). Since $\vec{a}_{AVERAGE} = \Delta\vec{v}/\Delta t$, the direction of the acceleration is the same as for $\Delta\vec{v}$, and so this is a centripetal acceleration, $\vec{a}_C$. Strictly speaking, we just found the average acceleration over this interval, but if we were to make the time interval $\Delta t$ smaller and smaller, then the average value approaches the instantaneous value at our point of interest, as we've seen before.

Now for the magnitude of $\vec{a}_C$. Since the change in velocity and the acceleration are in the same direction, we can write of their magnitudes that

$$a_C = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} .$$

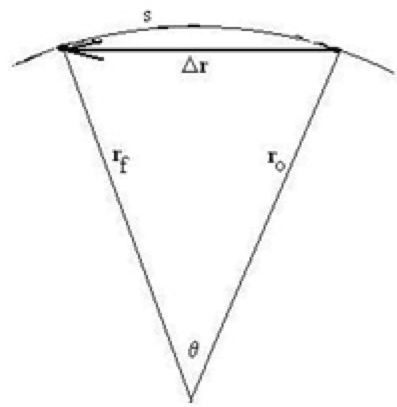
First, we have to realize that the angle between the two velocity vectors $\vec{v}_i$ and $\vec{v}_f$ is the same as the one labelled θ in the original figure. We can argue this because the velocity vectors, being tangent to the circle, are always at right angles to their corresponding location vectors (the r's), so that as r swings through a given angle, then v must swing through the same angle.

If that's true, then we see that we have two *similar triangles*, one in real space, the other in velocity space; each is isosceles (since $v_i = v_f = v$ and $r_f = r_i = r$) and they have the same apex angle). In that case, we can write a relationship involving the lengths of the sides of these triangles:

$$\frac{\Delta r}{r} = \frac{\Delta v}{v} \quad \to \quad \Delta v = \frac{v\,\Delta r}{r}.$$

Then,

$$a_C = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{v\,\Delta r}{r\,\Delta t} = \frac{v\,s}{r\,\Delta t} = \frac{v\,\frac{s}{\Delta t}}{r} = \frac{v(v)}{r} = \frac{v^2}{r},$$




where we argue that, as the time interval is made smaller and our starting and ending positions become closer together on the circle, the displacement magnitude |Δr| approaches in value the distance s traveled by the object along the arc of the circle.

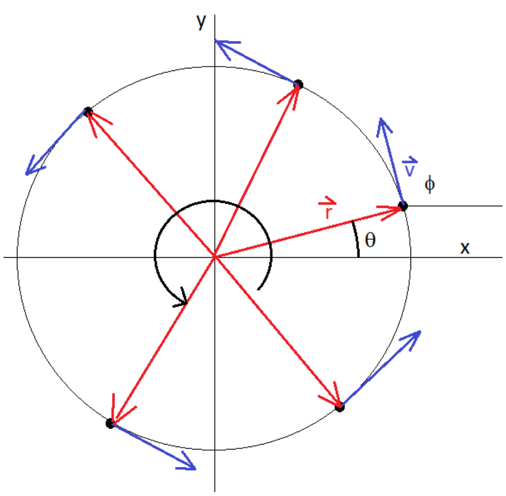
So, in short, an object moving at <u>constant speed</u> about a circular path has an acceleration that points toward the center of the circle (centripetal) and has magnitude $v^2/r$. However, we can rewrite this result by substitution: $v^2/r = (\omega r)^2/r = \omega^2 r$.

$$a_C = \frac{v_T^2}{r} = \omega^2 r \text{ towards the center of the circle}.$$

DERIVATION 3-4*

Here is a shorter, but more conceptual derivation for centripetal acceleration.[4] I like it because, when we're done, we can make use of it to derive some useful mathematical relationships for later in the course.

Consider a point mass travelling around a circle at constant speed v and angular speed ω. Use vector $\vec{r}$ to indicate the position of the object relative to the center of the circle. Remember from above that the direction of $\vec{v}$ is always changing, but it is also always tangent to the circle and perpendicular to $\vec{r}$, and constant in magnitude. The position of the object can be described using the angle theta from above, as measured from the x-axis. The direction of the velocity can be described using the angle ϕ (phi), which is always 90° more than theta.



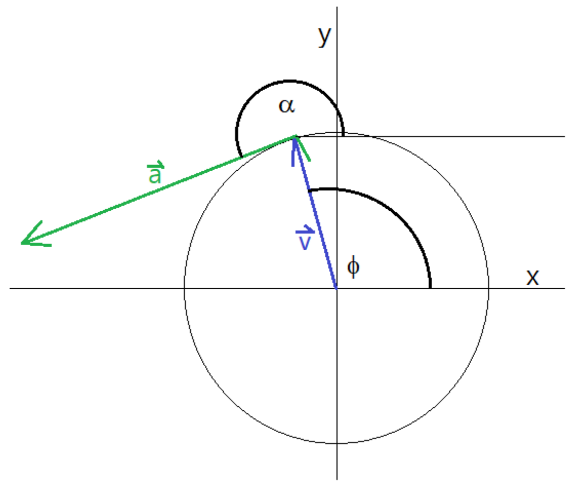

For convenience, let's define a function called the *Instantaneous Time Rate of Change* of a variable, such that, for example,

$$\text{ITRC}(\vec{r}) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \vec{v}\ .$$

[4] This was adapted from Marcel Wellner, *Elements of Physics* (New York, Plenum Press, 1991), 129-131. This book presents many interesting alternate ways of looking at elementary mechanics.

Note that this is nothing new, we're just calling it something else for convenience.

O.K., we're ready to go. What we have in $\vec{r}$ is a vector of constant magnitude whose direction is rotating CCW at constant rate ω. Its ITRC has a constant magnitude that is ω times its own magnitude and a direction rotating CCW at constant rate ω, 90° ahead of its own. Since there's nothing particularly unique about $\vec{r}$ in terms of being a vector, I think we can assert that a similar relationship will hold for any such vector. Aha! The velocity here is exactly just such a vector! Since $\vec{a}$ = ITRC ($\vec{v}$), we know that the acceleration will be a vector of constant magnitude ωv = $\omega^2$r, rotating CCW at constant angular speed ω. The acceleration is 90° ahead of the velocity, which is in turn 90° ahead of the position vector $\vec{r}$; since $\vec{r}$ always points outward from the center of the circle, the acceleration, which is then 180° ahead of the position vector, must point opposite to $\vec{r}$ inward to the center of the circle, *i.e.*, it is centripetal.




MATHEMATICAL DIGRESSION*

If you're O.K. with the last discussion, we can make use of the concept there to develop some useful relationships for later in the semester. Consider a vector $\vec{A}$ with constant magnitude A rotating CCW at a constant angular speed ω. At any moment, its direction angle θ can be written as[5]

$$\theta = \omega t \ .$$

Let $\vec{B}$ be the ITRC of $\vec{A}$. As discussed, the magnitude of $\vec{B}$ will be B = ωA and the direction angle ϕ will be θ + 90°. However, this relationship should be true of the components of $\vec{A}$ and $\vec{B}$ as well:

$$\mathrm{ITRC}(A_x) = B_x \quad \text{and} \quad \mathrm{ITRC}(A_y) = B_y \ .$$

So, if

$$A_x(t) = A\cos(\theta) = A\cos(\omega t) \quad \text{and} \quad A_y(t) = A\sin(\theta) = A\sin(\omega t)\,,$$

[5] For an object starting at the origin and moving linearly with constant velocity, KEq. 3 reduces to x = vt. This is analogous.

Then,

$$\text{ITRC}\big(A\cos(\omega t)\big) = \text{ITRC}\,\big(A_x(t)\big) = B_x(t) = B\cos(\phi) = (\omega A)\cos(\omega t + 90^o)$$
$$= -(\omega A)\sin(\omega t)$$

$$\text{ITRC}\big(A\sin(\omega t)\big) = \text{ITRC}\,\big(A_y(t)\big) = B_y(t) = B\sin(\phi) = (\omega A)\sin(\omega t + 90^o)$$
$$= (\omega A)\cos(\omega t)\ .$$

Or, in summary,

$$\boxed{\text{ITRC}\big(A\cos(\omega t)\big) = -\omega A\sin(\omega t) \quad \text{and} \quad \text{ITRC}\big(A\sin(\omega t)\big) = \omega A\cos(\omega t)\ .}$$

These statements should then be true for <u>any</u> sinusoidally varying function (not just vectors), so long as ω is constant.

EXERCISE 3-4

Verify the steps converting sine to cosine and cosine to sine above. Make use of these trig identities:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\ .$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\ .$$

DISCUSSION*

Describe the jerk, kick, and lurch of an object moving in such a circle. What can you say about their magnitudes and directions?

EXAMPLE 3-7

What is the acceleration of a car that starts from rest and attains a speed of 35 m/s while traveling in a straight line for 100 m? What is the acceleration of a car travelling at a constant 35 m/s while driving in a circle of radius of 100 m? In which case do you think the tires would be more likely to slip?

Let's let the direction of motion be along the x-axis. From Section 2,

$x_i$ = 0 m
$x_f$ = 100 m
$v_{xi}$ = 0 m/s (starts from rest)
$v_{xf}$ = 35 m/s
$a_x$ = ? ←
t = ?

Looks like KEq 4 may work.

$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{35^2 - 0^2}{2(100 - 0)} = 6.1\ \text{m/s}^2\ .$$

For the circular motion,

$$a_C = \frac{v^2}{r} = \frac{35^2}{100} = 12.3\ \text{m/s}^2\ .$$

We might well assume that the situation with the higher acceleration would be the one more likely to have the tires slip.

EXAMPLE 3-8

Suppose you're on a roller coaster with a loop-de-loop of radius 45 m. As you go over the top while upside-down, you notice that your bottom has just barely lost contact with your seat. How quickly is the roller coaster car moving at the top of the loop?

If there is no other agency than gravity acting on you at that point, your acceleration will be 10 m/s$^2$ downward, which at this point is toward the center of the circle. Then,

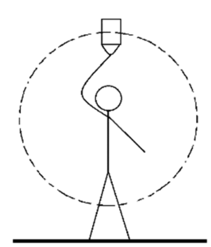
$$a_C = \frac{v^2}{r} \quad \rightarrow \quad v = \sqrt{a_C r} = \sqrt{10(45)} = 21.2\,\frac{\text{m}}{\text{s}}\ .$$

EXERCISE 3-5

Suppose that the moon were a perfect sphere of radius 1740 km. The gravitational field strength $g_{MOON}$ on the surface of the moon is about 1/6 that at the surface of the earth (We know this because we've been there.). How quickly would you need to launch a satellite so that it just skims along the surface of the moon?

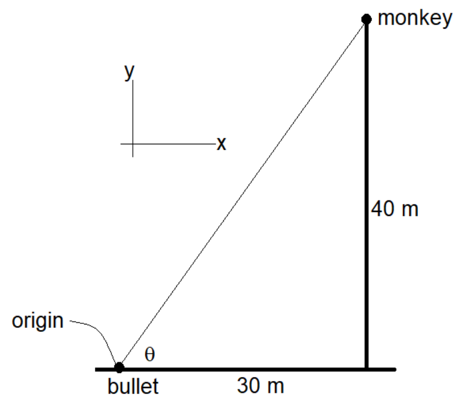
HOMEWORK 3-7

This was a demonstration when I took PHYS I. The professor took a pail of water and swung it in a vertical circle with the intent that the water would stay in the bucket, even when the bucket was inverted. That actually didn't work out well for him. What is the minimum number of revolutions *per* second necessary for Professor Buechner to stay dry?



We shall return for further discussion of centripetal acceleration in a later section.

EXERCISE 3-1 Solution




The idea here is that we need to show that, at some point, the bullet and the monkey are in the same place at the same time. The angle theta will be arctan(40/30) = 53°. We have two objects, and so we need a corresponding number of kinematic equations. The monkey is a bit easier, so let's do that first.

Monkey

| | |
|---|---|
| $x_{Mi} = +30$ m | $y_{Mi} = +40$ m |
| $x_{Mf} = +30$ m | $y_{Mf} = y_{Af} = ?$ |
| $v_{Mxi} = 0$ m/s | $v_{Myi} = 0$ m/s |
| $v_{Mxf} = 0$ m/s | $v_{Myf} = ?$ |
| $a_{Mx} = 0$ m/s$^2$ | $a_{My} = -10$ m/s$^2$ |
| $t = ?$ | |

Arrow

| | |
|---|---|
| $x_{Ai} = 0$ m | $y_{Ai} = 0$ m |
| $x_{Af} = +30$ m | $y_{Af} = y_{Mf} = ?$ |
| $v_{Axi} = v_o \cos(\theta) = 35 \cos(53°) = +21$ m/s | $v_{Ayi} = v_o \sin(\theta) = 35 \sin(53°) = +28$ m/s |
| $v_{Axf} = +21$ m/s | $v_{Ayf} = ?$ |
| $a_{Ax} = 0$ m/s$^2$ | $a_{Ay} = -10$ m/s$^2$ |

We can easily find the time required for the arrow to travel 30 m horizontally by using KEq. 3:

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$30 = 0 + 21t + 0\,t^2 \quad \rightarrow \quad t = \frac{30}{21} = 1.43 \text{ seconds} \ .$$

At this time, both the monkey and the bullet are at x = +30 m. Now at that same time, are they at the same altitude? For the monkey,

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 = 40 + 0\,(1.43) + \tfrac{1}{2}(-10)(1.43)^2 = 29.8 \text{ m} \ .$$

For the bullet,

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 = 0 + 28\,(1.43) + \tfrac{1}{2}(-10)(1.43)^2 = 29.8 \text{ m} \ .$$

And so, yes, the arrow hits the monkey anyway.

EXERCISE 3-2 Solution

If the time is still four seconds, then the building is still 80 m tall.

$$\frac{v_{yf}}{v_{xf}} = \tan\theta = \tan(-53^o) = -1.33 \quad \rightarrow \quad v_{xf} = -0.75\, v_{yf}\ .$$

And, since the final x velocity is the same as the initial, KEq. 1 tells us that

$$v_{xi} = v_{xf} = -0.75\, v_{yf} = -0.75\,(v_{yi} + a_y t) = -0.75\,(0 + (-10)(4)) = 30\ \text{m/s}\,.$$

Lastly, $x_f$ is given by KEq. 3 as

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 30(4) + 0 = 120\ \text{m}\ .$$

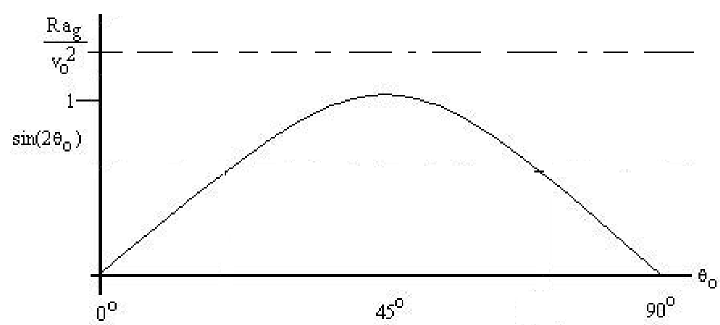
EXERCISE 3-3 Solution

The problem meets the conditions for using the Range Equation, so let's go for it.

$$R = \frac{v_o^2 \sin(2\theta_o)}{|\,a_g\,|}$$

$$\theta_o = \tfrac{1}{2}arcsin\left(\frac{R|\,a_g\,|}{v_o^2}\right) = \tfrac{1}{2}arcsin\left(\frac{350 \times 10}{55^2}\right) = \tfrac{1}{2}arcsin(1.16) = (ERROR)$$

How many times did you retry taking the arcsine? You didn't make a mistake. What angle has a sine of 1.16? Graphically, you're trying to find the intersection of these two curves, and it isn't happening. The physical interpretation of this is that it is impossible to hit the target under these conditions.




EXERCISE 3-4 Solution

For this, $\alpha = \omega t$ and $\beta = 90^o$. Then,

$$\cos(\omega t + 90^o) = \cos\omega t \cos 90^o - \sin\omega t \sin 90^o = \cos\omega t\,(0) - \sin\omega t\,(1) = -\sin\omega t\ .$$

$$\sin(\omega t + 90^o) = \sin\omega t \cos 90^o + \cos\omega t \sin 90^o = \sin\omega t\,(0) + \cos\omega t\,(1) = \cos\omega t\ .$$

EXERCISE 3-5 Solution

The acceleration of an object at the earth's surface is 10 m/$s^2$ towards the earth's center. If the moon's gravity is the only agency acting on the satellite, then we might assume that this satellite's acceleration will be $a_g = 10/6 = 1.7$ m/$s^2$ downward, towards the center of its circular orbit. Don't forget to convert the moon's radius into meters:

$$1740 \text{ km } \times \frac{1000 \text{ m}}{1 \text{ km}} = 1.74 \times 10^6 \text{m} \ .$$

Then,

$$a_C = \frac{v^2}{r} \quad \rightarrow \quad v = \sqrt{a_C r} = \sqrt{1.7(1.74 \times 10^6)} = 1720 \frac{\text{m}}{\text{s}} .$$

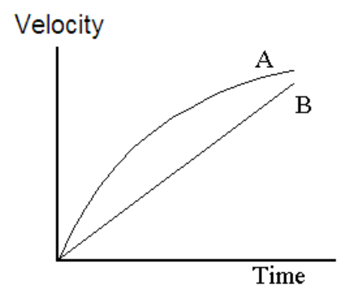
Sample Exam I

MULTIPLE CHOICE (4 pts each)

1) Define up as being positive. Suppose an object is moving downward, but slowing down. Then,

   A) the velocity is negative and the acceleration is positive.
   B) the velocity is negative and the acceleration is negative.
   C) the velocity is positive and the acceleration is positive.
   D) the velocity is positive and the acceleration is negative.
   E) the velocity is negative and the acceleration is zero.

2) Consider the figure, which reports the velocities of two cars in a race on a straight highway as a function of time. Which of the statements below is (or are) true?




   (1) There is a time other than t=0 when both cars have the same displacement from the start line.
   (2) There is a time other than t=0 when both cars have the same velocity.
   (3) There is a time when both cars have the same acceleration.

   A) (1) only
   B) (2) only
   C) (3) only
   D) (1) and (2) only
   E) (2) and (3) only

3) Consider two vectors, $\vec{A}$ $\uparrow$ and $\vec{B}$ $\longrightarrow$. Which of the following choices best represents the general direction of $\vec{B} - \vec{A}$?

   A) $\nearrow$ B) $\nwarrow$ C) $\swarrow$ D) $\searrow$ E) $\longleftarrow$

4) If a cannonball is fired at an angle of 53 degrees above the horizontal and leaves the muzzle with a speed of 400 m/s, what is the magnitude of the acceleration of the ball three seconds into its flight (neglect air resistance and assume that the ball is still in flight)? Pick the closest answer.

   A) 0 $m/s^2$
   B) 10 $m/s^2$
   C) 60 $m/s^2$
   D) 180 $m/s^2$
   E) 240 $m/s^2$

5) We found in class that the range R of an object thrown with initial speed $v_i$ at an elevation $\theta_o$ over a flat plain is given by (if air drag is ignored)

$$R = \frac{v_o^2 \sin(2\theta_o)}{|a_g|} .$$

Find the angle(s) at which one could launch an object with initial speed 400 m/s and have it land 10,000 m downrange.

A) 34° and 56°
B) 19° and 71°
C) 26° and 64°
D) 45° only
E) There are no such angles.

**PROBLEM I** (20 pts)

Starting from the definitions of the average velocity and of the average acceleration,

$$v = \frac{x - x_i}{t} \quad \text{and} \quad a = \frac{v - v_i}{t}$$

and the relation,

$$v_{AVE} = \frac{v_i + v_f}{2} ,$$

*derive* the relation,

$$x = x_i + v_i t + \tfrac{1}{2} a t^2 .$$

Be sure to show all effort for full credit.

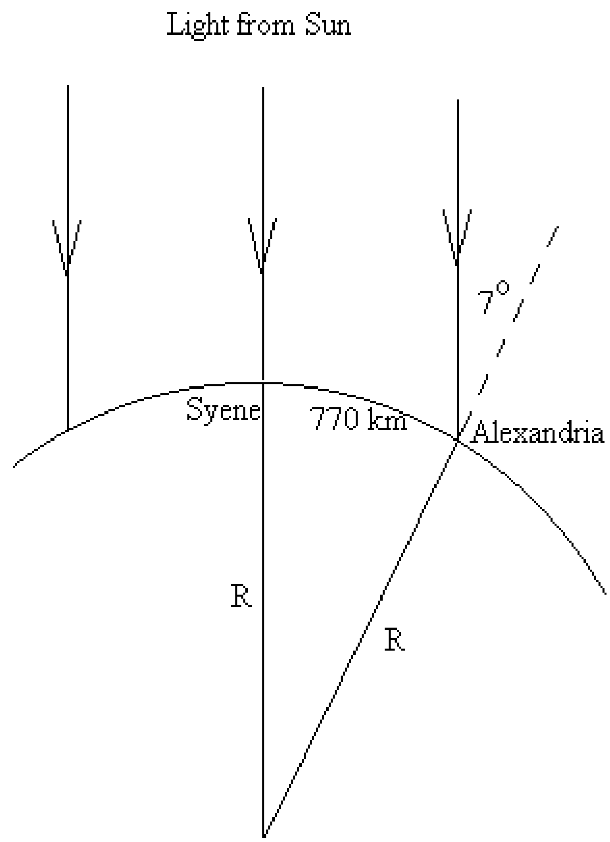
**PROBLEM II** (20 pts)

*Using the component method*, find the magnitude and direction angle of vector $\vec{D}$.

$\vec{A}$ = 6m , $\theta_A$= -75°
$\vec{B}$ = 5m, $\theta_B$ = 45°
$\vec{C}$ = 8m, $\theta_C$ = - 135°
$\vec{D} = \vec{A} + \vec{B} - \vec{C}$

**PROBLEM III** (20 pts)

The size of the earth has been known since antiquity. Eratosthenes assumed that since the sun is very far from the earth, light rays from the sun are essentially parallel (one can show the sun is distant by observing solar and lunar eclipses.). He noticed that when the sun was directly overhead at Syene (in southern Egypt), it was 7° away from overhead in Alexandria, 770 km to the north. From the information given, calculate the radius of the earth. An approximate solution earns most of the points.




**PROBLEM IIII** (20 pts)

A ball is thrown downward with speed 9 m/s from the top of a 40m tall building.

A) How long will it take for the ball to hit the ground? (10 pts)

B) How will be the ball's velocity just before hitting the ground? (10 pts)

## SECTION 4 – RELATIVE MOTION

### Relative Velocities

On occasion, it is useful to consider the motion of an object with respect to an origin/coördinate system which is itself in motion relative to some third reference frame. A simple example is that of the 'people mover' at the airport, a giant conveyor belt that carries weary passengers along the length of the concourse, while also providing those in a rush a little extra speed as they run down the walkway. For example, consider such a walkway (W) which moves with a velocity of +2 m/s with respect to the ground (G). We'll represent this with this notation: $v_{W,G} = +2$ m/s; the first letter indicated which object we're examining, and the second what it is moving 'with respect to.' Now, think of a person (P) walking in the same direction at +1 m/s along the walkway: $v_{P,W} = +1$ m/s. I don't think anyone would argue that the person's velocity relative to the ground is +3 m/s, which would imply that

$$\vec{v}_{P,G} = \vec{v}_{P,W} + \vec{v}_{W,G} \ .$$

Let's test this for some other scenarios. Suppose the person were to walk the wrong way on the walkway at -1 m/s. Then

$$\vec{v}_{P,G} = \vec{v}_{P,W} + \vec{v}_{W,G} = -1 + 2 = +1 \text{ m/s} .$$

The person would still be going in the same direction as before, although more slowly. Still happy?

One of the hardest aspects of relative velocity is to determine which two quantities get added to obtain the third. Let's look more closely at the notation. Two of the velocity terms have the P in the first position and two have the G in the second position. But, one has the W in the first position and the other has the W in the second position. It's the ones with the letters in different positions that are added to obtain the third. If you do it correctly, the outer subscript letters on the right side will match the subscript letters on the left side.

Here's another useful fact: $\vec{v}_{A,B} = -\vec{v}_{B,A}$. As an example, suppose that I'm driving I-83 to York, Penna at 120 kilometers *per* hour (kph) and pass someone parked on the shoulder. That person sees me going northward at 120 kph and himself as stationary, but I see myself as stationary and see him moving southward at 120 kph.

EXAMPLE 4-1

> An airplane with an *airspeed* of 400 kph files eastward with an 85 kph tailwind. What is the *ground speed* of the plane?
>
> Well, here we've introduced a couple of terms that may need explaining. Airspeed is the speed of the plane as measured relative to the air, and of course the ground speed is measured relative to the ground. The wind is measured relative to the ground, and a tailwind moves in the same direction as the plane.

We have three objects to consider: the plane; the ground; and the air. Let eastward be the positive direction. Then,

$v_{P,G} = ?$ ←
$v_{P,A} = +400$ kph
$v_{A,G} = +85$ kph

We note that it is the A that changes position, so we write that

$$\vec{v}_{P,G} = \vec{v}_{P,A} + \vec{v}_{A,G} \ .$$

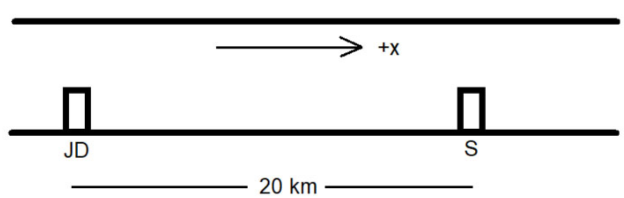
The solution is straightforward:

$$\vec{v}_{P,G} = \vec{v}_{P,A} + \vec{v}_{A,G} = 400 + 85 = 485 \text{ kph}.$$

EXERCISE 4-1

Back on the 'people mover,' Person A walks the correct way at a speed of 1 m/s, while Person B walks at 2 m/s the correct way on the return walkway. What is the relative speed between the brothers? Assume each walkway moves at 2 m/s relative to the ground.

EXAMPLE 4-2




Consider a river flowing in a straight course at 8 kph. Joe has a dock (labeled JD) and want to make a run to the store (S) 20 km downstream for 'supplies.' If the boat can travel 12 kph in still water, how long will it take Joe to make a round trip? We'll assume his order is already waiting for him, so he can immediately turn around.

There are again three objects to worry about: the boat; the water; and the ground. But also there are two parts to the problem: there (I) and back again (II). The velocities are $v_{B,G}$, $v_{B,W}$, and $v_{W,G}$. Since we want the time for each of these trips, we need to work in the displacement. So,

$$\vec{v}_{B,G} = \frac{\Delta\vec{x}_{B,G}}{t} = \vec{v}_{B,W} + \vec{v}_{W,G} \quad \rightarrow \quad t = \frac{\Delta x_{B,G}}{v_{B,W} + v_{W,G}} \ .$$

Let's make downstream the +x direction and use the notation of Sections Two and Three, *i.e.*, let the sign of the value indicate the direction.

On the way downstream:

$v_{B,W} = +12$ kph
$v_{W,G} = +8$ kph
$\Delta x_{B,G} = +20$ km

$$t_I = \frac{\Delta x_{B,G}}{v_{B,W} + v_{W,G}} = \frac{+20}{+12+8} = 1 \text{ hour} .$$

On the way upstream:
$v_{B,W} = -12$ kph
$v_{W,G} = +8$ kph
$\Delta x_{B,G} = -20$ km

$$t_{II} = \frac{\Delta x_{B,G}}{v_{B,W} + v_{W,G}} = \frac{-20}{-12+8} = 5 \text{ hours} .$$

This is then a total of 6 hours.

HOMEWORK 4- 1

An escalator is 20 m long. If a person simply stands on the 'up' side, it takes 30 seconds to ride to the top. If a person walks up the escalator at a speed of 0.6 m/s relative to the escalator, how long will it take him to get to the top? If the same person walks down the 'up' side at the same relative speed as before, how long will it take him to arrive at the bottom?

HOMEWORK 4-2

An airplane is to fly from City A due west to City B to pick up cargo, then return to City A. It will take exactly one hour to load the plane at B, and this entire trip should be done in the shortest time possible. The plane has a maximum airspeed of 300 kph, and encounters an 80 kph westerly wind[1] for the entire trip. If the distance between A and B is 1800 km, how long does the entire trip require?

Well, that was one dimensional motion. Let's move on to two dimensional problems. We saw back in Section One that if

$$\vec{C} = \vec{A} + \vec{B} ,$$

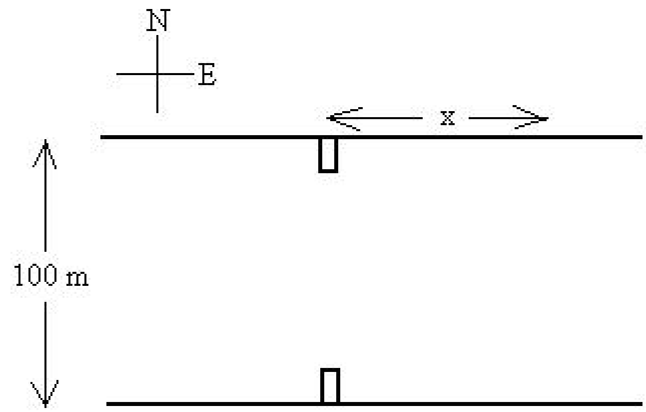
then

$$C_x = A_x + B_x \quad \text{and} \quad C_y = A_y + B_y ,$$

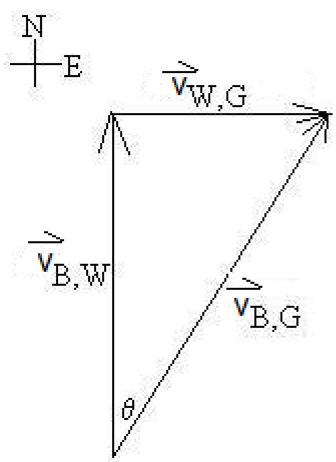
and we can treat a two dimensional problem as two one dimensional problems.

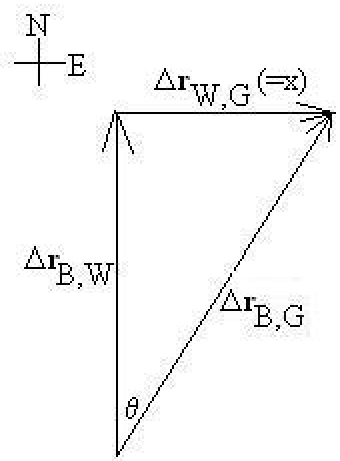
[1] A westerly wind blows from the west, towards the east.

EXAMPLE 4-3




Consider a boatman who wishes to cross a river (100 metres wide) from one dock north to another exactly opposite. His boat will make 10 m/s in calm water. The velocity of the water is 8 m/s eastward. He aims his boat exactly northward and sets off. How far downstream (x) will he actually land, in what compass direction did he actually travel, and how long will it take him to get there?




We have three objects to consider: the boat; the ground; and the water. The velocity equation is then

$$\vec{v}_{B,G} = \vec{v}_{B,W} + \vec{v}_{W,G} \ .$$

In this case, the quickest solution may be to realize that, if the velocities are all constant, a displacement component diagram can be constructed where each term is parallel to the corresponding velocity term. The two triangles so formed are then similar, and so there is a proportionality of the lengths of the sides:

$$\frac{x}{100\ \text{m}} = \frac{8\ \text{m/s}}{10\ \text{m/s}} \quad \rightarrow \quad x = \frac{100\ (8)}{10} = 80\ \text{m} \ .$$

The direction traveled can be found using the tangent of the angle θ:

$$\tan\theta = \frac{8}{10} = 0.8 \quad \rightarrow \quad \theta = 38.7^{o} \ .$$

For the time, we consider that the motion northward (in this case) is independent of the motion eastward; it would take 10 seconds to cover 100 meters at 10 m/s. Or, it would take 10 seconds to cover 80 meters at 8 m/s.

EXERCISE 4-2

Now, suppose that, having learned his lesson, he tries again to cross directly to the other side. In what direction should he aim his boat (relative to north) to arrive exactly at the other dock, and how long will it take him?

HOMEWORK 4-3

A plane needs to leave City A on time and arrive at City B on time exactly seven hours later. City B is 1500 km due east of City A. There is a southerly wind blowing at 120 kph. With what airspeed and in what direction should the pilot head the plane?

HOMEWORK 4-4

A plane needs to leave City A on time and arrive at City B on time exactly eight hours later. City B is 2500 km due east of City A. There is a wind blowing at 120 kph toward 37 degrees west of north. With what airspeed and in what direction should the pilot head the plane?

## Relative Accelerations*

We've just discussed relative velocities, and of course we introduced vector addition as 'relative displacements,' so is there such a thing as *relative acceleration*? Well, you betcha.

EXAMPLE 4-4

A heavy two-meter stick (S) with a vertical orientation is dropped from rest over a tall cliff. At that same instant, an ant (A) starts to accelerate up the stick from the bottom end. From the markings on the meterstick and on his wristwatch, he sees that he covers 0.8 meters in 1.1 seconds. At that time, how far has the ant fallen with respect to the cliff face (C)?

Let upward be positive x. The ant's acceleration relative to the stick ($a_{A,S}$) is found from KEq 3:

$x_{ASi} = 0$ m
$x_{A,Sf} = 0.8$ m
$v_{A,Si} = 0$ m/s
$v_{A,Sf} = ?$
$a_{A,S} = ?$ ←
$t = 1.1$ sec

$$x = x_i + v_i t + \frac{1}{2} a t^2$$

$$a_{A,S} = 2\frac{x - x_i - v_i t}{t^2} = 2\frac{0.8 - 0 - 0(1.1)}{1.1^2} = 1.32 \text{ m/s}^2 \quad .$$

The acceleration of the meter stick[2] is -10 m/s$^2$. Then, the acceleration of the ant with regard to the cliff face is

$$\vec{a}_{A,C} = \vec{a}_{A,S} + \vec{a}_{S,C} \ .$$

$$a_{A,C} = 1.32 + (-10) = -8.68 \ \text{m/s}^2 \ .$$

Then,

$x_{ACi} = 0$ m
$x_{A,Cf} = ?$ ←
$v_{A,Ci} = 0$ m/s
$v_{A,Sf} = ?$
$a_{A,S} = -8.68$ m/s$^2$
$t = 1.1$ sec

$$x = x_i + v_i t + \frac{1}{2} a t^2$$

$$x_{A,C} = 0 + 0(1.1) + \frac{1}{2}(-8.68)(1.1^2) = -5.25 \text{ m} \ .$$

DISCUSSION 4-1

Suppose that a passenger is sitting in a train waiting at the station. You are standing on the platform spying on him. Suddenly, a frog jumps straight upward from the floor of the train car. What shape path will the frog seem to follow as seen by you? As seen by the passenger?

Suppose that a passenger is sitting in a train that is traveling through the station at constant velocity. You watch as the train passes. Suddenly, a frog jumps as before. What shape path will the frog seem to follow as seen by you? As seen by the passenger?

Suppose that a passenger is sitting in a train that is traveling through the station with a constant acceleration $a_T$, specifically the train is speeding up. You watch as the train passes. Suddenly, a frog jumps as before. What shape path will the frog seem to follow as seen by you? As seen by the passenger?

EXERCISE 4-3*

Show that the path of the frog as seen by the accelerating passenger is parabolic and is consistent with the notion of relative accelerations. The most general form of the equation of a parabola is

[2] Well, it's actually just a little bit higher; we'll deal with that in a later section.

$$A^2y^2 + B^2x^2 - 2(f_x(A^2+B^2) + AC)x - 2\left(f_y(A^2+B^2) + BC\right)y - 2(AB)xy + \left((A^2+B^2)\left(f_x^2 + f_y^2\right) + C^2\right) = 0 \quad ,$$

where ($f_x$, $f_y$) are the coördinates of the focus and the line given by

$$Ax + By + C = 0 \quad \rightarrow \quad y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$$

is the *directrix*. For our purposes, we need remember only that the directrix is perpendicular to the axis of symmetry of the parabola.

---

EXERCISE 4-1 Solution

Here we have five objects to keep track of: Person A, Person A's walkway (AW), Person B, Person B's walkway (BW), and of course the ground. Let's let A's direction be positive x. We then know

$\vec{v}_{A,AW} = +1$ m/s
$\vec{v}_{AW,G} = +2$ m/s
$\vec{v}_{B,BW} = -2$ m/s
$\vec{v}_{BW,G} = -2$ m/s

Let's do each person relative to the ground, as we did in the example.

$$\vec{v}_{A,G} = \vec{v}_{A,AW} + \vec{v}_{AW,G} = +1 + (+2) = +3\ \mathrm{m/s}\,.$$

$$\vec{v}_{B,G} = \vec{v}_{B,BW} + \vec{v}_{BW,G} = (-2) + (-2) = -4\ \mathrm{m/s}\,.$$

We want to know $v_{A,B}$. Be careful.

$$\vec{v}_{A,B} = \vec{v}_{A,G} + \vec{v}_{G,B} = \vec{v}_{A,G} - \vec{v}_{B,G} = (+3) - (-4) = +7\mathrm{m/s}\,.$$

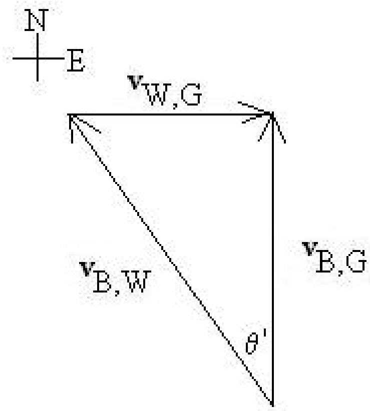
We actually could have done this in one go:

$$\vec{v}_{A,B} = \vec{v}_{A,AW} + \vec{v}_{AW,G} + \vec{v}_{G,BW} + \vec{v}_{BW,B} = \vec{v}_{A,AW} + \vec{v}_{AW,G} - \vec{v}_{BW,G} - \vec{v}_{B,BW}$$
$$= +1 + 2 - (-2) - (-2) = 7\mathrm{m/s}\,.$$

EXERCISE 4-1 Solution

The common error here is simply to flip the triangle over. But what should be done is to deform the triangle by sliding $\mathbf{v}_{W,G}$ over until the sum, $\mathbf{v}_{B,G}$, is pointing due north:

Before, the two short sides of the right triangle were $\mathbf{v}_{B,W}$ and $\mathbf{v}_{W,G}$, but now those vectors are the <u>hypotenuse</u> and a short side, respectively. So,




$\theta' = \arcsin(8/10)$ and $|\mathbf{v}_{B,G}| = 10\cos\theta'$.

Finish the calculations yourself.

EXERCISE 4-3 Solution

At the moment of launch, the frog is moving forward (say, the +x direction) at speed $v_T$ because of the motion of the train, and upward at speed $v_U$. Once the frog leaves the floor, the only agency acting on it is gravity, and we've already worked out the trajectory for your point of view in Section Three:

$$y = 0 + v_o\sin\theta_o\frac{x}{v_o\cos\theta_o} + \frac{1}{2}a_g\left(\frac{x}{v_o\cos\theta_o}\right)^2 = \frac{v_U}{v_T}x + \frac{a_g}{2v_T^2}x^2 .$$

However, since the passenger is accelerating with respect to the frog, he sees (or thinks he sees) the frog accelerating backwards, since $\vec{a}_{B,M} = -\vec{a}_{M,B}$. To find the trajectory as seen by the passenger, we need to start from scratch.

| | |
|---|---|
| $x_i = 0$ (why not?) | $y_i = 0$ (same here) |
| $x_f = ?$ | $y_f = ?\ \leftarrow$ |
| $v_{xi} = 0$ (as seen by the man) | $v_{yi} = v_U$ |
| $v_{xf} = ?$ | $v_{yf} = ?$ |
| $a_x = -a_T$ | $a_y = a_g$ |
| $t = ?$ | |

Once again, we'll start with the x-side and find the time:

$$x = x_i + v_{xi}t + \tfrac{1}{2}a_xt^2$$

$$x = 0 + 0 + \tfrac{1}{2}(-a_T)t^2 \quad \rightarrow \quad t = \sqrt{\frac{2x}{-a_T}} .$$

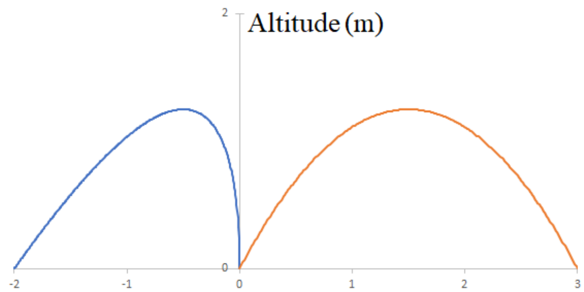
This result is O.K. because x will become negative as the frog 'falls behind' the passenger. Now to the y-side and to substitute in for the time:

$$y = y_i + v_{yi}t + \tfrac{1}{2}a_yt^2$$

$$y = 0 + v_U\sqrt{\frac{2x}{-a_T}} + \frac{a_g}{-a_T}\,x .$$

$$y + \frac{a_g}{a_T}\,x = v_U\sqrt{\frac{2x}{-a_T}}\ .$$

Here is a sample curve for some representative values of $a_T = 4$ m/s$^2$ , $v_{iT} = 3$ m/s, and $v_U = 5$ m/s for the trajectory of the frog as seen from the passenger's point of view and from your point of view standing on the platform.




Square both sides to get

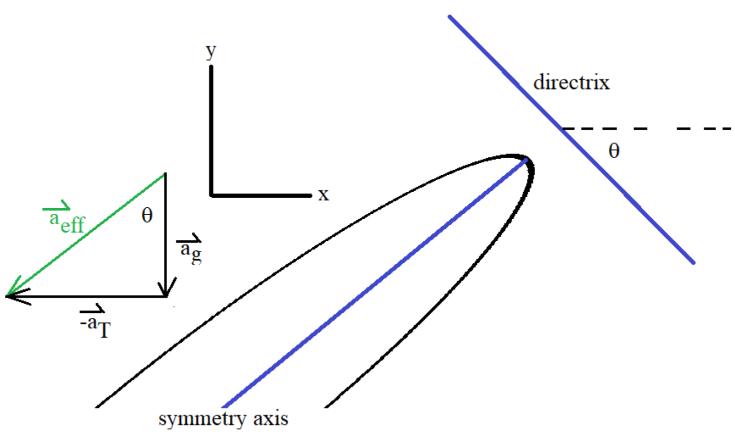
$$y^2 + 2\,\frac{a_g}{a_T}\,xy + \left(\frac{a_g}{a_T}\right)^2 x^2 + \frac{2v_U^2}{a_T}\,x = 0\ .$$

First, is this a parabola? This does match the general form of the equation for a parabola as given.[3] The difference is that its axis of symmetry is not vertical but tilted. Continuing,

$$A^2y^2 + B^2x^2 - 2\left(f_x(A^2+B^2) + AC\right)x - 2\left(f_y(A^2+B^2) + BC\right)y - 2(AB)xy$$
$$+ \left((A^2+B^2)\left(f_x^2+f_y^2\right) + C^2\right) = 0\ .$$

By comparing the coëfficients of each power of x and y here with those in our trajectory solution, we can determine the values of A, B, C, $f_x$, and $f_y$. Luckily, we need only A and B today:

$$A = 1\ ,\ \ B = -\,\frac{a_g}{a_T}\ .$$




The directrix is then a line with a slope of

$$\frac{-A}{B} = -\frac{-a_T}{a_g} = \frac{a_T}{a_g},$$

and that indicates a parabola with its symmetry line tilted from the y-direction by an angle θ with tangent ($-a_T/a_g$).

How might we interpret this? The passenger would observe the direction of acceleration of the

---

[3] Strictly speaking, we showed that the square of the trajectory function is a parabola; since that equation puts the same restrictions on the values of x and y as the original (there are no sign changes), that too is parabolic.

flying frog to be at an angle θ from the horizontal, but he also sees that the free-fall acceleration magnitude would be given by

$$a_{eff} = \sqrt{a_T^2 + a_g^2} \ .$$

For the specific example above, the gravitational acceleration would seem to be directed at an angle of 22° 'behind' the vertical and be 10.8 m/s$^2$.

## SECTION 5 – THE FIRST PICTURE

Students often get irritated with me when I point out that, up to this moment, we've really not done any physics at all. We did discuss the acceleration of objects due to gravity near the earth's surface, but the rest was really just definitions. Kinematics is the descriptive study of the motions of objects, but we'd like to know <u>why</u> objects move the way they do. To understand this, at least to the degree allowed in our one semester course, we'll be looking at three 'pictures,' or ways of looking at problems. To many people, these three pictures seem separate and unconnected, but in fact, they are all really the same, just twisted around a bit. Each is particularly well suited to addressing certain types of problems; the other two can of course be used as well, but not as efficiently.

In the introduction, I mentioned that the course is structured much like a traditional two-column proof Geometry course. In this section, we will introduce the two axioms on which the pictures are built. In Physics, we call these axioms *laws*. What exactly is a law? Like the axions of Geometry, laws are ideas that we observe never to be false and which we then assume are true. The purpose of experimentation is to try to show that these laws are false; the more unsuccessful we are at that, the more confidence we have that they are true. Of course, sometimes, we fool ourselves into thinking that something is true, then find out that it was actually a special case, or that our experiments just weren't accurate enough.[1] The notions many students have when they start a course in classical mechanics are often referred to as *Aristotelian*; part of the purpose of these classes is to disabuse students of these Aristotelian notions.

### Dynamics

*Dynamics* is the study of <u>why</u> objects move the way they do, particularly with regard to *forces*. The root of the word is Greek for force.[2]

DISCUSSION 5-1

> How would you define a force? What definition were you given in grade school? Is that good enough? Is force a scalar or a vector? Is there a difference between pushing something to the left and pushing it to the right?

DISCUSSION 5-2

> Let's start by making some observations. We'll place a book on the table. Watch the book carefully. What is the book doing? We often use the expression 'at rest' to describe this situation. What will it be doing an hour from now? What about next semester? What about when I finally retire? If we want to book to be not at rest, what must happen?

---

[1] An excellent example of this is that some of the ancients believed that the earth revolves around the sun. They reasoned that, if this were true, a certain effect would be observed. The effect was in fact not observed (it was too small to be detected with the techniques available at the time) until two millennia later.

[2] Indeed, a *dyne* is a unit of force, one we won't be using.

Suppose I next toss the book onto the table. What phrase might we use to describe the book? Once the book hits the table, what happens, and why? What stopped the book?

What is necessary to change the motion of an object? If we say that the motion is constant, which quantity from Section 2 are we really referring to? Which quantity from Section 2 measures the change in the motion?

CHEESY EXPERIMENT 5-1

Let's do a quick experiment to see how this works. We've decided that a force is necessary to change an object's motion or velocity, that is, it causes acceleration. I have a cart on wheels (to minimize the effect of friction, whatever that is) and a force-o-meter marked off in some weird units. I have some confidence that the force-o-meter works, in that as I pull harder, the numbers increase on the dial. As I pull on the cart, you can hear from the sound of the wheels that it's accelerating. If I pull with twice as much force, the acceleration is higher. Briefly, as F↑, a↑.

DISCUSSION 5-3

We should try to be a bit more explicit in the relationship between acceleration and force. If no force means no acceleration, and more force means more acceleration, what is the simplest relationship between them that you can think of? What about the directions of the force and acceleration?

If we assume that, perhaps, the universe behaves as simply as possible, we might conjecture that the acceleration is proportional to the force. Then, we make a hypothesis that $\vec{a} \sim \vec{F}$. We may be wrong, of course. It may turn out that the acceleration is proportional to $F^2$ or $F^3$. Maybe the force and acceleration are actually not in the same direction.[3] At some point, we'll do this experiment much more carefully and find out.

What else can we change? We don't want to base our notions on just one experiment; the results may have been coïncidental. Best to vary as many parameters as possible.

CHEESY EXPERIEMENT 5-2

Let's repeat by keeping the force constant and doubling the mass of the cart. What do you notice? Is the acceleration larger or smaller when the mass is increased?

DISCUSSION 5-4

What relationship would you say exists between the acceleration and the mass of an object?

We might write that as m↑, a↓. Note something interesting. In chemistry, you're told that the mass is a measure of how much material there is in an object. Here, we see that the mass of an

[3] Just wait for Physics III!

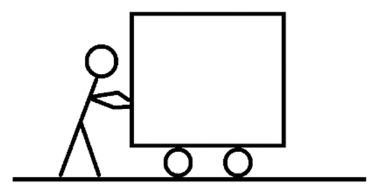
object is a measure of how hard it is to accelerate that object. The simplest relationship between the two would be that the acceleration and the mass are *inversely proportional*: a ~ 1/m. Of course, we may be wrong, but we at least have a hypothesis.

We've performed some simple experiments and conjectured that the acceleration is proportional to the applied force if the mass is held constant, and inversely proportional to the mass of the force is constant. Let's synthesize these ideas into one:

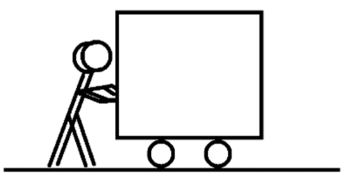
$$\vec{a} \sim \frac{\vec{F}}{m} \;.$$

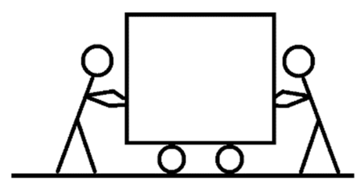
If we choose the correct units, we can make the proportionality an equality. Let the force necessary to accelerate one kilogram at one meter/second$^2$ be called one *newton*.

DISCUSSION 5-5



What if there is more than one force acting on an object? Consider poor Joe, who has to push a crate across the warehouse floor by applying a force F to the right. The next day, Joe gets his twin brother Jeb to help. How much force is applied to the crate by the boys? What would you expect the acceleration of the crate to be today, compared to that of yesterday? On the next day, Jeb isn't available, so Joe asks his other twin brother Jake to help out. Jake, however, seems to never quite 'get it.' How much force is applied to the crate in this situation? What would you expect the crate's acceleration to be on this day? So, what must we do when there is more than one force?





Putting all of these ideas together gives us the s*econd law of motion*:

$$\vec{a} = \frac{\sum_n \vec{F}_n}{m} \;.$$

Note that when we sum the forces,[4] we include only the forces that are actually acting on the object of interest. Other forces influence the motions of <u>other</u> objects. I'll show you a way of keeping track of which forces act of which objects.

Lastly, although the form of the second law given above is conceptually the better, we are going to re-arrange it so that it is more convenient to use for problem solving:

[4] You may not be familiar with this notation. $\sum_n \vec{F}_n$ means simply to add up all the forces with n as a counting number: $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$ . Remember to add the forces as vectors.

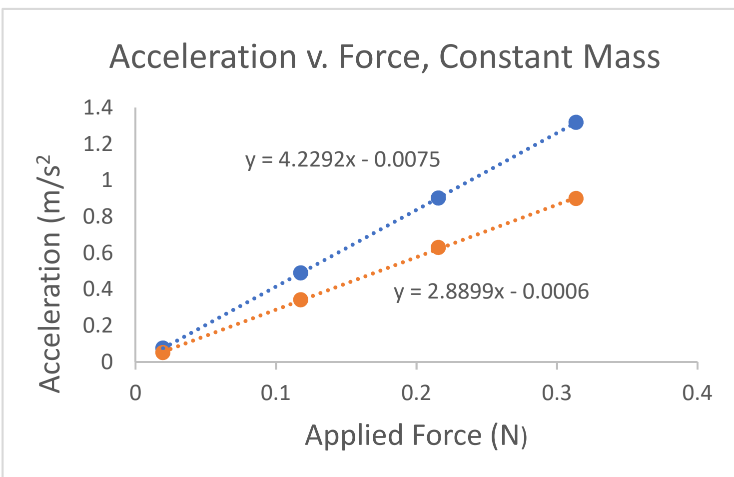
$$\sum_n \vec{F}_n = m\vec{a}\ .$$

The reason I like this is that all the effort of solving a problem is done on the left side, and the right side is always mass times acceleration. Let me emphasize that $m\vec{a}$ is not a force. Forces are the cause, and acceleration is the result.

You may ask, 'what's the *first law of motion*?' Well, it's a special case of the second. If there are no forces acting on an object, its acceleration is zero and the velocity will be constant. If the object is at rest, it will remain at rest, and if it's moving, it will continue that *rectilinear motion*.

EXPERIMENT 5-3

We placed a mass, which we call a glider, on a horizontal air track. The track acts much like an air hockey table. There are small holes through which air is forced to lift the glider off the track surface to minimize friction (whatever that is). A force was applied to the mass and the resulting accelerations were measured. Here are the results, plotted in two ways. In the first graph, each line represents runs with a constant mass; the fact that the lines pass through the origin (well, the intercepts are very small compared to the values plotted) indicates that the respective accelerations and applied forces are proportional. What's more, matching the equation for a line to our hypothesized relationship leads us to predict that the slope should be the inverse of the mass:




$$y = (\text{slope})x + \text{intercept}$$

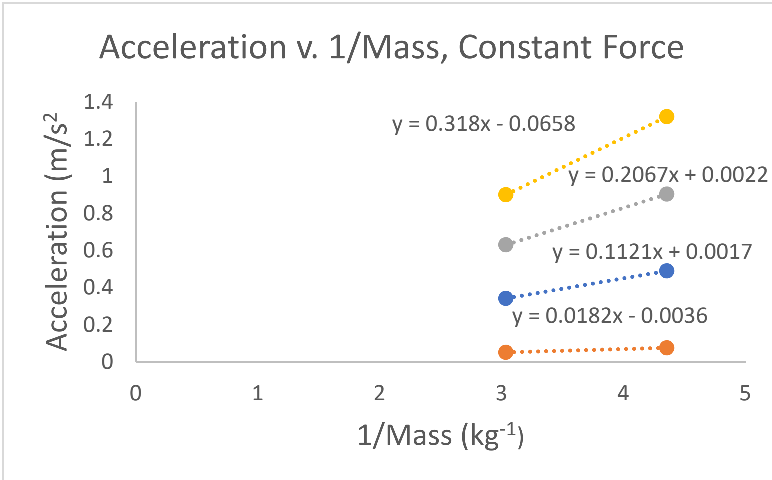
$$a = \left(\frac{1}{m}\right)F + 0\ .$$

| Actual Mass | Mass from graph | *Per cent* difference |
|---|---|---|
| 0.2295 kg | 0.2365 kg | + 3.1 % |
| 0.3293 kg | 0.3460 kg | +5.1 % |
| | | |

To gain a bit more confidence, let's plot these data differently, acceleration *v.* inverse mass. In this case, we're trying to fit the data to this relationship:

$$a = (F)\frac{1}{m} + 0\ .$$

Once again, we should see lines passing through the origin with the slopes equal to the respective applied forces.




We can see that the intercepts are quite small, compared to the smallest acceleration values, so the proportionality condition seems to be met. Let's look at the forces.

| Actual force applied | Force from graph | *Per cent* difference |
|---|---|---|
| 0.3137 N | 0.3180 N | + 1.4 % |
| 0.2156 N | 0.2067 N | - 4.1 % |
| 0.1176 N | 0.1121 N | - 4.7% |
| 0.0196 N | 0.0182 N | - 7.1 % |

Without doing an uncertainty analysis, we can't really determine if our hypothesis is justified, but I think that, perhaps, we may have some confidence that it is correct, subject to future, more careful validation.

HOMEWORK 5-1

A net force of 45 N is applied to a mass of 16 kg. What will be the mass's acceleration? How much force should be applied to a 27 kg mass to give it the same acceleration?

The *third law of motion* seems to be the one students have the most trouble with, although it really is the easiest to understand: If object A exerts a force on object B, then B exerts a force on A that is of the same type, equal in magnitude, and opposite in direction. Mathematically, we write that

$$\vec{F}_{B,A} = -\vec{F}_{A,B} \ .$$

Think about this scenario: A speeding car A rear-ends a parked car B at a red light; the parked car B is accelerated forward because of the force exerted by A, while A slows down due to the force exerted backwards on it by B. Two forces that fulfill this description are referred to as a *third law pair*. To be a third law pair, the forces must fit the description given above, *e.g.*, A pushes B and B pushes on A.

CHEESEY EXPERIMENT 5-4

Let's do a quick check of this concept with two *force-o-meters*.

Third Law.MOV

We see that the forces each exerts on the other are equal in magnitude. Later in Section 7, we'll show some experimental results that will help grow confidence in the third law.

DISCUSSION 5-6

Our book is still sitting on the table. There is a gravitational force exerted on the book by the earth (this is called the book's *weight*) and a *force of contact* acting upward on the book from the table. If the acceleration of the book is zero (it's not moving), then what can we say about the two forces just mentioned? Do these two forces form a third law pair? If not, what <u>are</u> the other halves of each pair? Did you give them the A on B, B on A test?

If there's doubt on what constitutes a third law pair, just change the subjective and objective parts of the sentence around. It's not too hard to believe that if the table pushes up on the book, then the book pushes down on the table. Harder perhaps to believe that if the earth pulls down on the book, the book pulls up equally on the earth. Indeed, if I were to drop a book, the earth would accelerate upward to meet the book!

DISCUSSION 5-7

Why don't we notice the earth moving upward toward the book?

Forces that form third law pairs are often called action-reaction forces. I don't like this terminology, because it gives the impression that one force occurs first, then the other. Third law pair forces occur simultaneously.

HOMEWORK 5-2

A positively charged proton (mass = 1 dalton)[5] repels a positively charged alpha particle (mass = 4 daltons) with a force of 0.5 pico-newtons. What force does the alpha particle exert on the proton?

## TYPES OF FORCES

In this course, you will encounter several types of forces. We'll start with three of them, then add in the others when we're ready.

### Weight

[5] More or less. A dalton is 1/12 the mass of a carbon 12 atom. A pico-newton is one quadrillionth of a newton.

We mentioned above that there is a force on objects that are near the surface of the earth that is associated with gravity, which we shall call the weight, $\vec{W}$. When an object is in free fall, the only force acting on it is $\vec{W}$, which we know from lab causes the object to experience an acceleration of $a_g$ downward. Using the second law, we can write that

$$\vec{W} = m\,\vec{a}_g \ .$$

Since all objects will fall with this same acceleration, the force necessary to give an object a certain acceleration is proportional to the mass of the object by some factor which we'll call $\vec{g}$, the *gravitational field strength*:

$$\vec{W} = \vec{g}m \ .$$

Note that $\vec{g}$ must be a vector quantity (pointing downward). What is the value of g? The fact that

$$\vec{W} = \vec{g}m = m\,\vec{a}_g \quad \rightarrow \quad \vec{g} = \vec{a}_g \ .$$

However, $\vec{g}$ and $\vec{a}_g$ are two different quantities; they have the same value, the same dimension, and the same direction, but different units. Since g is the gravitational force *per* unit mass the earth exerts on an object near its surface, $\vec{g}$ is 9.8 newtons/kg, downward.[6]

DISCUSSION 5-8

> Drop a ball. As it falls, is there an acceleration? Is it specifically $a_g$? Is there a gravitational field? Now, make the ball smaller and drop it. Is there any acceleration? Is there a gravitational field? How strong is it? Keep making the ball smaller and smaller until there is no ball left at all. Is there an acceleration? Is there a gravitational field? How strong is it?
>
> Next, place the ball on a table. Is there an acceleration? Is there a gravitational field? How strong is it?

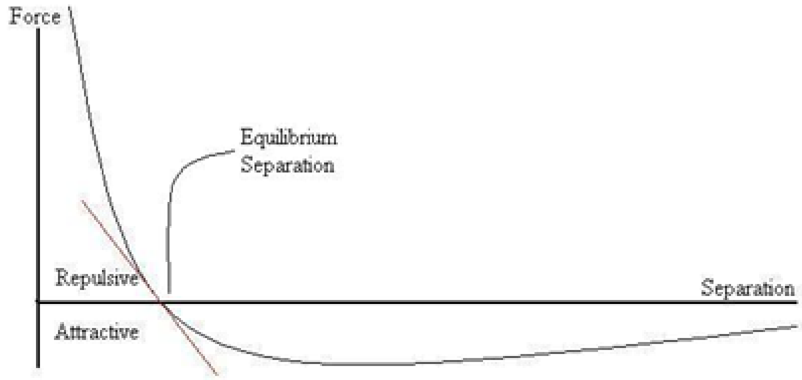
Now, as we did for $a_g$, we will round off the value of g to 10 N/kg for the purposes of homework and exams. In lab, we will be more careful.

## (Normal) Force of Contact

Another type of force is the *contact force*, which is due to the fact that two objects are actually touching one another. We will be considering two different contact forces in this course, but we'll start with the one that is perpendicular (normal) to the surface of contact. The nature of this type of force can be thought of as being due to the electronic bonds between atoms or molecules in each

[6] This may seem like a big deal over nothing, but there is an analogous situation in Physics 2 that generally gives students a hard time. Better to start thinking this way now.

of the materials. You may have learned in chemistry that the forces between these particles looks a bit like this as a function of the separation:




The forces of repulsion and attraction cancel at the equilibrium separation (F = 0). Around this point, it can be shown that the system acts much like balls connected by springs. So, as the two macroscopic objects come into contact, the 'springs' are compressed and produce forces which act to push the objects back apart. Or the objects could be glued together and then pulled apart, so that the 'springs' stretch and try to pull the objects back together. In either case, the forces are due to contact between the objects and are directed perpendicularly to the interface between the objects.

As an example, suppose you arrive home and set your bookbag on your couch. At first, the bag will move downward into contact with the couch. As the bag pushes into the cushion, it compresses the springs there, which, as we'll see later, start to push back upward. In the end, the bag comes to rest with the springs under it compressed. If we think of the atoms in an object as balls connected by springs (instead of electric bonds) we can imaging the same thing happening in microcosm.

## Tension

Often, we speak of the *tension* in a string or rope. We'll define the tension to be the force the string exerts on the object it's attached to. In this course, we usually assume that the strings are massless and inextensible (they don't stretch).

Let's make an argument that the tension at each end of such a string is the same as at the other end (except of course opposite in direction). Consider a rope used in a tug of war game. The team on the right pulls to the right with force $F_R$, and by the third law of motion, the string exerts the same magnitude force (the tension $T_R$ at that end, by our definition) on the team. Likewise, the team on the left exerts a force $F_L$ on the rope, and the other half of that third law pair is the tension $T_L$ on the left end of the rope. If the rope is massless, any difference in applied net force would cause an infinite acceleration. Hence, $F_R = F_L$ and $T_R = T_L$.

Strings and ropes are often looped over wheels. This does nothing more than change the direction of the tensions at the ends, so long as the wheel is itself frictionless and massless. Situations where the wheel is not frictionless or massless will be treated later in the course.

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## Applications of The Laws of Motion

First, several comments on notation. In kinematics, we used the sign of a vector's value to indicate the direction of the vector. If an object moved in the +x directions at 3 m/s, we said that $v_x = +3$. If it moved in the -x direction, we said $v_x = -3$. We're going to stick with that notation for kinematic quantities here, but we will do something different with forces. We will always write the magnitude of the force, but insert the correct sign in front of the force to indicate the direction. In particular, we will always use positive 10 N/kg as the value for g regardless of whether up or down is positive. Second, although we were very careful in past sections to measure the angle for finding components CCW from the x axis, we shall abandon that approach at this point. Now, we will just make use of the most convenient angle and take care of the signs as described above.

DISCUSSION 5-9

> Consider a man standing on a spring scale. What does the scale actually measure? Suppose the man holds the scale up against the wall and pushes on it horizontally. Does the scale measure the man's weight?

This is one of the first things we need to be able to do, decide what forces act on an object. The man's weight does not act on the scale, the man's weight is the force of (gravitational) attraction between the earth and the man, and that force acts on the man. The man and the scale are in contact with one another, and it is the normal force of contact that the scale measures. For example, if I were to place the scale on the wall and lean against it, the scale would not be measuring my weight. To keep track of the forces acting on an object, we can use a *free body diagram*, which is just an accounting tool to isolate each object for analysis. Draw each of the forces with its tail at the center of the body under consideration (here, the man). Here we see the weight (force of gravity of the earth acting on the man) and the normal force (force of contact of the scale acting on the man). What about the force of the man acting on the scale? Well, that's a force on the scale, not on the man; that force would go on the free body diagram of the scale..

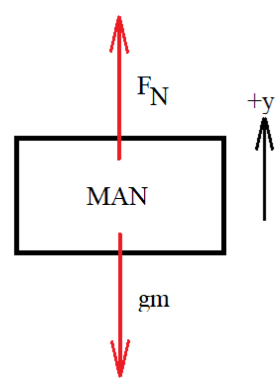
EXAMPLE 5-1

If the man is not moving, his acceleration is zero, and so we can write, using the second law and making upward positive, that

$$\sum_n \vec{F}_n = m\vec{a}$$

$$+F_N - gm = 0$$

$$F_N = gm \ .$$




Since, by the third law, the normal force on the man from the scale is the same magnitude as that of the normal force of the scale on the man, the reading on the scale is numerically equal to the weight of the man, but it is not the weight of the man.

Now, let's put the man and the scale in an elevator that is accelerating upward. The diagram is similar to the one above. Writing the second law results in:

$$\sum_n \vec{F}_n = m\vec{a}$$

$$+F_N - gm = ma$$

$$+F_N = ma + gm$$

So, we see that if the elevator is accelerating upward, the scale reading will be higher than the man's weight, while it will be lower if the elevator's acceleration is downward.

DISCUSSION 5-10

While you may not be gnurdy enough to ride in an elevator with a scale, you probably have noticed this effect. There are sensors in your body that can tell you when one layer of you is compressed against another layer of you. How do you 'feel' when your elevator starts to move upward? Do you feel as if you are heavier? If you were standing on a scale, what would it read? What about when your elevator starts to descend?

HOMEWORK 5-3

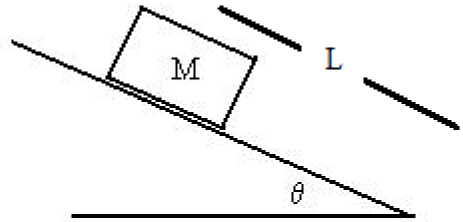
An 90 kg man stands in an elevator. What force does the floor of the elevator exert on the man if
a) the elevator is stationary?
b) the elevator accelerates upward at 1.2 $m/s^2$?
c) the elevator rises with constant velocity 3 m/s?
d) while rising, the elevator decelerates at 0.5 $m/s^2$?
e) the elevator descends with constant velocity of 2.5 m/s?

HOMEWORK 5-4

A softball (mass = 0.19 kg) is thrown directly upward so that it leaves the pitcher's hand at 4.5 m/s. The pitcher's hand moved through 1.5 meters as he threw the ball. What force did the pitcher exert on the ball?

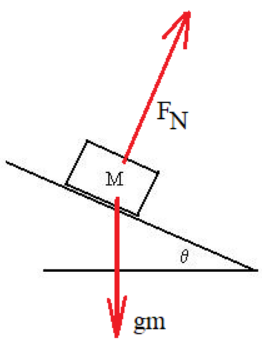
EXAMPLE 5-2

Here is a problem we shall use as the model for presenting solutions. We will be revisiting it from time to time.



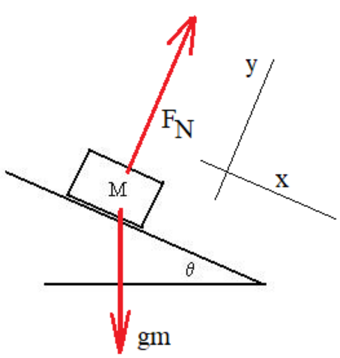

Consider a block of mass M on a frictionless plane inclined at an angle θ from the horizonal. If the block is released from rest from a point a distance L up the incline, how quickly will it be moving when it reaches the bottom? Let theta be $37^o$, M be 5 kg, and L be 2 meters.

Next, we can analyze the forces acting on the mass. We can do an inventory, much as we did for kinematics. Is there a weight? Sure, and remember that there is only one weight *per* object. Is the mass touching something? It's touching only the incline, so there is one normal force of contact. Are there any strings or rope or the like? No. The free body diagram for the mass is then:




Notice that the normal force of contact is perpendicular to the surface. A common misconception is that the normal force points upward, and is probably due to the fact that students start by trying problems like the elevator situation above.

Next, we'll pick a coördinate system. There is a need for some experience here, but here is a hint: we certainly expect the block to accelerate along the plane, and not to either jump off the plane or burrow into it. You may remember Rule One from Section 3: choose a system such that the acceleration is along one of the axes. It will be much easier to solve this problem if we orient the axes parallel and perpendicular to the plane. It's not impossible to solve the problem otherwise, but it's a lot tougher mathematically. If the problem is such that the acceleration is zero, then this aspect is not so important and other considerations can be examined.

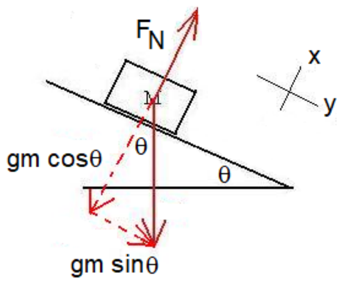



Now, we can write the second law as

$$\vec{F}_N + \vec{g}m = \vec{a} \ ,$$

which is not very useful, since we then have one equation with two unknowns. However, we decided back in Section 1 that if two vectors are equal, then their components must be independently equal. So we'll break this equation into two separate equations, one for x and one for y.

First, the components. It would probably be a good idea to review. We've decided to make the coördinate system as shown in the figure, based on the object's presumed acceleration direction. $F_N$ is already completely in the +y direction, but gm is mixed. We must replace gm with two other vectors that, when added together, equal the original vector, with one parallel to the x-axis and the other parallel to the y-axis. Note that we are <u>not</u> using the angle as measured from the +x axis to find the components, but instead are simply making use of the trig identities with the angle we are given in the triangle. We will assign the proper sign to the directions of these forces when we write the second law equations. Since the x-component of the weight points in the positive x-direction, we'll place a plus sign in front of it. Similarly, the normal force is in the +y-directions, but the y-component of the weight in the negative y-direction.




$$x: \quad + gm\sin(\theta) = ma_x$$

$$y: \quad + F_N - gm\cos(\theta) = ma_y = 0$$

In this case, the y equation is not useful, but the x equation tells us that

$$a_x = g\sin(\theta) \ .$$

Keep in mind that in our method, g is always positive; we took care of the direction by placing the correct sign in front of the terms in the second law equations.

DISCUSSION 5-11

When we finish with a major section of a solution, we should ask if the result makes some sense. If theta were zero, what would the acceleration be? If theta were ninety degrees, what would the acceleration be? Do those values make sense?

EXAMPLE 5-2 Continued

So, for the values given in the problem, the acceleration will be

$$a_x = g\sin(\theta) = 10\sin(37^o) = 10(0.6) = 6\ \mathrm{m}/s^2\ .$$

Now that we have the acceleration (and it's constant!), we will choose a kinematic equation to find the final speed. Let's place the origin at the starting location.

$x_i$ = 0 (starts from the origin)
$x_f$ = 2m
$v_{xi}$ = 0 (starts from rest)
$v_{xf}$ = ? ←
$a_x$ = +6 m/s2
t = ?

KEq 4 seems like a good choice.

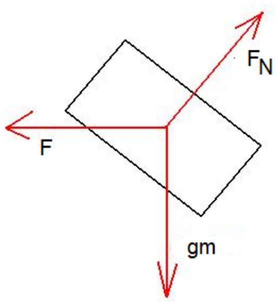
$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$$

$$v_{xf} = \sqrt{v_{xi}^2 + 2a_x(x_f - x_i)} = \sqrt{0^2 + 2(6)(2-0)} = +\,4.9\ \mathrm{m/s}\,.$$

We take the positive root because the box is sliding in the +x direction.

In the previous example, we alluded to 'other considerations' in choosing a coördinate system. As stated, it's generally best to align the axes so that the acceleration is along one of them. If the acceleration is zero, this is not so important, and a judicious choice of axes might reduce the amount of math we need to do. This is Rule Number Two. Consider this problem:

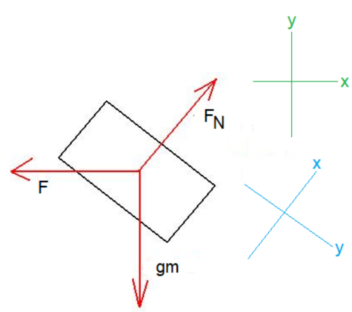
EXAMPLE 5-3

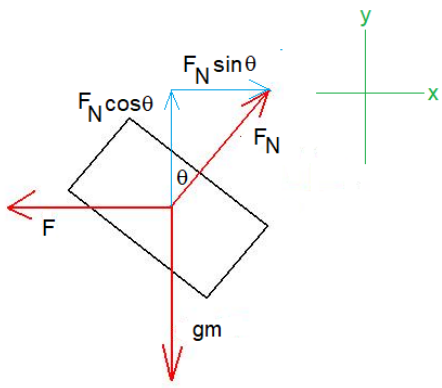



Apply a force F horizontally to the block of the previous problem so that the block remains stationary. Find F and the normal force, $F_N$.

Let's start with a free body diagram. It bears repeating that the normal force is perpendicular to the surface between the box and the incline. Also, since the object is motionless, there is no acceleration and Rule Number One is moot.

Now, if you were to set up the tilted, blue coördinate system as in the previous example, you might notice that you would have to decompose two vectors. However, if you chose the green system, only one vector would require decomposition. That may make the algebra easier in that there would be fewer terms to manipulate (four *vs* five). Let's go with the green system and find the components of $F_N$. A small amount of geometry shows that if the ramp is inclined at angle theta, then the normal force is inclined theta from the vertical.






Now we write the second law for x and y:

$$x: \quad + F_N \sin(\theta) - F = ma_x = 0$$

$$y: \quad + F_N \cos(\theta) - gm = ma_y = 0 \ .$$

At this point, it is worth making a general comment. Note that the acceleration terms are zero. This is because this particular object is stationary. This is not always the case. A common error is to write a = 0 for every problem.

Here's a neat mathematical trick. We're going to divide these two equations. First, some re-arrangement.

$$F_N \sin(\theta) = F$$

$$F_N \cos(\theta) = gm$$

Then, we divide the left sides and set that equal to the quotient of the right side:

$$\frac{F_N \sin(\theta)}{F_N \cos(\theta)} = \frac{F}{gm}$$

$$\tan(\theta) = \frac{F}{gm}$$

$$F = gm\tan(\theta) = 10(5)\tan(37^o) = 37.5\text{ N} \ .$$

Returning to either of the original equations results in

$$F_N \cos(\theta) = gm \quad \rightarrow \quad F_N = \frac{gm}{\cos(\theta)} = \frac{10(5)}{\cos(37^o)} = 62.2\text{ N}$$

MATHEMATICAL JUSTIFICATION

Suppose that A = B and C = D ≠ 0. It should be O.K. to say that

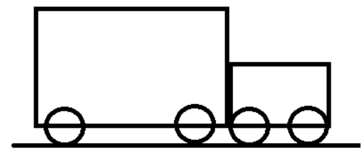
$$\frac{A}{C} = \frac{B}{C} \ ,$$

but since C = D,

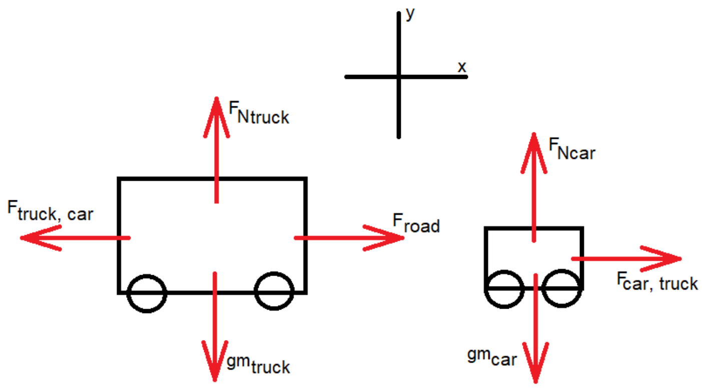
$$\frac{A}{C} = \frac{B}{D} \ .$$

Our next step is to investigate solving problems with more than one body. Remember that only the forces that actually act on an object can affect its motion. Other forces affect the motions of other objects.

EXAMPLE 5-4



Two of my uncles once started one of their cars by pushing it with the other car (It works fairly well with manual transmission cars. Don't try it with automatics.). Suppose Bud's truck has a mass of $m_T = 2500$ kg and Russell's car has a mass of $m_C = 1800$ kg. Bud's truck has a force from the road of 8000 N pushing it forward.[7] If the truck is in contact with the car, what will be the acceleration of the vehicles and what force will each exert on the other?




We have two objects. We should isolate them from each other and from all other objects (such as the earth) so that we can analyze the forces acting on each. Of course, each car has a weight, and a normal force acts on each upward from the ground. For the truck, there is a force forward from the ground and a force backward from the car, while the car experiences a force forward from the truck. Since the presumed acceleration is to the right, we'll use Rule One and choose the coördinate system shown. The third law of motion says that if the truck exerts a force on the car, then the car exerts a force of the same magnitude in the opposite direction, so $F_{TRUCK,\,CAR} = F_{CAR,\,TRUCK}$. We must write a set of second law equations for <u>each</u> object.

Truck

x: $+F_{ROAD} - F_{TRUCK,CAR} = m_{TRUCK}a_x$

y: $+F_{N\ TRUCK} - gm_{TRUCK} = m_{TRUCK}a_y = 0$

Car

x: $+F_{CAR,TRUCK} = m_{CAR}a_x$

y: $+F_{N\ CAR} - gm_{CAR} = m_{CAR}a_y = 0$

The y equations are of no use, so we'll add the x equations to eliminate the forces acting between the vehicles.

$$F_{ROAD} - F_{TRUCK,CAR} = m_{TRUCK}a_x$$

[7] We'll discuss the nature of this force later in this section. If you like, for now you could say the truck's tires are pushing the truck forward.

$$F_{CAR,TRUCK} = m_{CAR}a_x$$

$$F_{ROAD} = (m_{TRUCK} + m_{CAR})a_x$$

$$a_x = \frac{F_{ROAD}}{m_{TRUCK} + m_{CAR}} = \frac{8000}{2500 + 1800} = 1.86\ \mathrm{m/s^2}\ .$$

Now, we go back to find the contact force. We can use either equation, but why not use the simpler one?

$$F_{CAR,TRUCK} = m_{CAR}a_x = 1800(1.86) = 3348\ \mathrm{N}\ .$$

MATHEMATICAL JUSTIFICATION

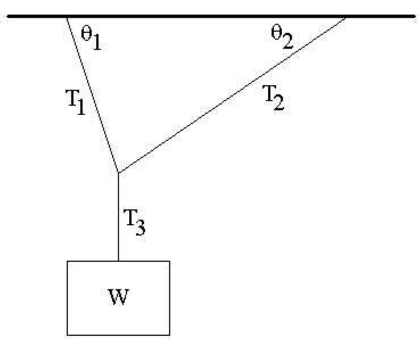
Suppose that A = B and C = D. It should be O.K. to say that

$$A + C = B + C$$

but since C = D,

$$A + C = B + D\ .$$

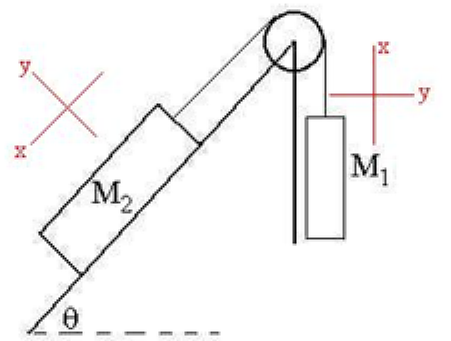
HOMEWORK 5-5



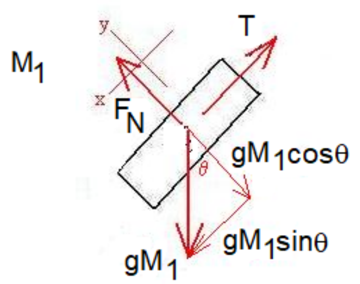

An object with a weight of 250 N is hung from the ceiling as shown. Find the tension in each of the wires if $\theta_1 = 53^o$ and $\theta_2 = 30^o$. Hints: There are two objects to investigate, the mass itself, and the knot where the wires meet.

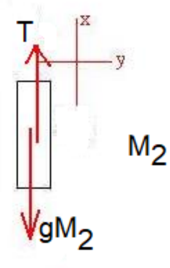
EXAMPLE 5-5

This example introduces a couple more new notions. Consider two blocks as shown with the inclined surface being without friction (whatever that is). Since there are two bodies, we will have to have two free body diagrams and two sets of second law equations. There looks to be a complication in choosing a coördinate system, though; no matter how the x and y axes are oriented, at least one acceleration will need to have two components, and there will have to be several equations relating the accelerations of each block to each other. We can avoid this by using a *fractured coördinate system*. For example, if mass one slides up the incline by one meter , mass two must descend exactly one meter. Remember that our strings are inextensible. By using the system shown below, we can minimize the tedium of relating all the necessary quantities and use simply Δx, v, and a to describe the motions of the masses along their respective x axes, while asserting that there is no motion in the respective y. We will also assume that the wheel has no effect other than to change the direction of the string; it is massless and frictionless. I will use the term *magic* to mean this.[8] The combination of a magic string with a magic wheel means that the tension is the same at both ends of the string. The problem is, find the acceleration of the masses and the tension in the string. When a problem is worded like this, the expectation is that the tension will not appear in the answer for the acceleration, and the acceleration will not appear in the answer for the tension; the answers should be in terms of only those quantities given in the problem, plus some obvious ones such as g.




Let's do free body diagrams. Note that the angle marked θ in this diagram is the same as the original angle of inclination.




So, write a second law equation for each mass:

$$M_1 \ \text{x:} \ \ +gM_1\sin(\theta) - T = M_1 a_x$$

$$M_1 \ \text{y:} \ +F_N - gM_1\cos(\theta) = M_1 a_y = 0 \ ,$$

$$M_2 \ \text{x:} \ -gM_2 + T = M_2 a_x$$

$$M_2 \ \text{y:} \ \ \text{No Forces} \ .$$

This last 'equation' is written to assume me that you have checked the y-direction for $M_2$ and there is just simply nothing going on there.

Once again, we're going to add the x equations:

$$gM_1\sin(\theta) - T = M_1 a_x$$

$$-gM_2 + T = M_2 a_x$$

$$gM_1\sin(\theta) - gM_2 = (M_1 + M_2)a_x \ .$$

This is an efficient way to eliminate the tension terms. The acceleration is then

[8] In Section 8, we will deal with non-magic wheels.

$$a_x = \frac{M_1 \sin(\theta) - M_2}{M_1 + M_2} g \ .$$

Let's stop and think about whether this makes some sense. The acceleration should be inversely proportional to the total mass (check). If $M_1$ is much larger then $M_2$, the blocks should slide to the left, and vice versa. Is there some condition whereby the blocks could just balance? Seems O.K.

Now to find the tension. Substitute the acceleration result back into the simpler of the two x equations:

$$-gM_2 + T = M_2\left(\frac{M_1 \sin(\theta) - M_2}{M_1 + M_2} g\right) ,$$

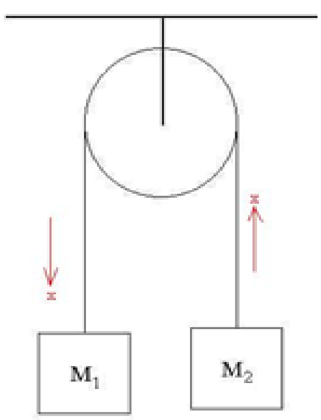
$$T = M_2\left(\frac{M_1 \sin(\theta) - M_2}{M_1 + M_2} g\right) + gM_2 \ .$$

Technically, this expression meets the requirement I put on the solutions. If this were an exam question, for example, I would accept it. It is possible, though, to make it prettier.

EXERCISE 5-1

Simplify the expression above for the tension. Get in some practice with algebra.

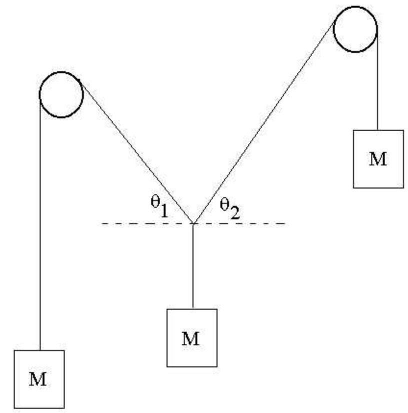
EXERCISE 5-2




Consider the *Atwood's Machine*, comprising two masses connected by a massless string over a magic wheel. Find the acceleration of the masses and the tension in the string. Let $M_1$ be 5 kg and $M_2$ be 7 kg. Give yourself no more than one minute to obtain the correct answer.

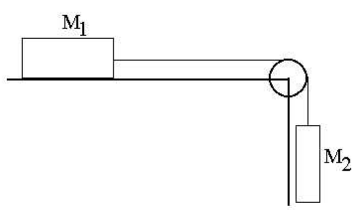
Note the difference between 'answer' and 'solution.' The solution would take much longer than a minute to produce.

HOMEWORK 5-6




Three identical masses are hung from strings as shown in the figure. The strings and wheels are magic. Find the two angles $\theta_1$ and $\theta_2$. HINT: again, consider the knot as a separate object.

HOMEWORK 5-7




A box ($M_1$) sits on a frictionless table and is connected by a magic string over a magic wheel to another mass ($M_2$), as shown. Find the acceleration of the masses and the tension in the string.

Let's now consider some objects in uniform circular motion. To review, such an object, moving in a circle at constant speed, has an acceleration towards the center of the circle (centripetal acceleration) with its magnitude given by

$$\mathbf{a_C = \omega^2 r = \frac{v^2}{r}} \ .$$

As we have seen earlier in this section, accelerations are caused by forces. There must, therefor, be a force or at least a force component towards the center of the circle. We call this of course a *centripetal force*. This sometimes causes confusion; better to call this 'a force that acts centripetally.' The reason I say this is that, often, students will correctly draw in the force that acts centripetally, but then add in an additional centripetal force as if it is separate from the actual forces. The general rule for this is, if you can't identify what is exerting the force, it's probably not actually there.

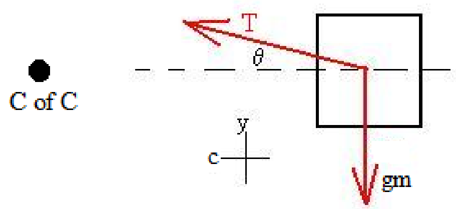
Some forces of course are directed away from the center of the circle; we refer to these as *centrifugal forces*. This is also something to be careful about.

To be consistent with Rule One, we will make one axis in the direction of the acceleration, that is, we will place the c-axis as positive toward the center of the circle. Centripetal forces will have a plus sign inserted in front of their magnitudes, and centrifugal forces a negative sign.

DISCUSSION 5-12

Suppose that I swing a small mass m on a string around in a horizontal circle of radius r. What forces are acting on the mass? Perform the checklist. Is there weight? Is the mass touching anything? Is there a string? Draw a free body diagram for the mass. What if I slow the speed of the object down, does that change your ideas about the directions?

EXAMPLE 5-6




Consider the problem in the discussion above. Find the relationship among theta, the speed, and the radius of the circle. Draw the free body diagram, indicate the center of the circle (C of C) and choose the coordinate system so that the c-direction is in the direction of the acceleration toward the center. We'll then make y be vertical. Then, we'll need to decompose the tension into c and y components. The second law equations are then

$$\text{c:}\quad T\cos\theta = ma_C = m\frac{v^2}{r}$$

$$\text{y:}\quad T\sin\theta - gm = ma_y = 0\ .$$

Can you see why the string must be at an angle above the horizontal? If it were not, there would be a net downward force and the mass would accelerate downwards. Let's re-arrange and divide the y-equation by the c-equation:

$$T\sin\theta = gm$$

$$T\cos\theta = m\frac{v^2}{r}$$

$$\tan\theta = \frac{gr}{v^2}\ .$$

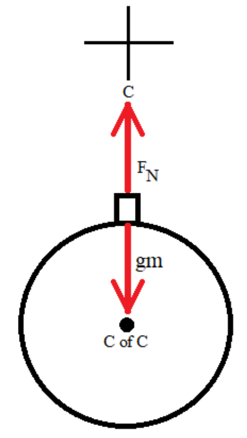
With this result, we see that the angle of the string will decrease as the speed increases. How quickly would the mass need to move to make the string horizontal?

EXAMPLE 5-7

Suppose that you're riding a Ferris wheel of radius 20 m. The operator decides to have fun with the rubes and speeds up the wheel. At how many revolutions per minute would the wheel need to spin in order for you to feel apparently weightless when at the top of the wheel?

First, let's recall what apparently weightless means. It does not mean there is no weight; we can't just shut that off. Like in the discussion of the elevator above, it means that the normal force of contact between you and your seat is zero and you are in free-fall. Second, since revolutions per minutes is angular speed, we'll use that form for the centripetal acceleration. The weight is pointing toward the center of the circle but the normal force is away from the center, so

$$+gm - F_N = ma_C = m\omega^2 r \ .$$

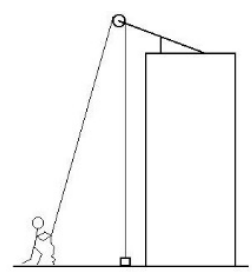



Setting $F_N$ to zero and solving for omega,

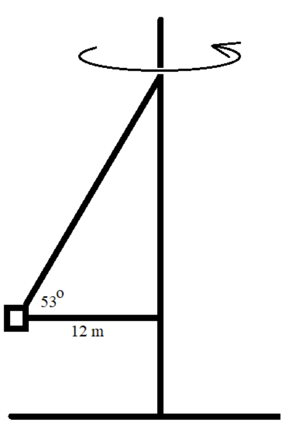
$$\omega = \sqrt{\frac{g}{r}} = \sqrt{\frac{10}{20}} = 0.71\frac{\text{rad}}{\text{s}} \times \left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) \times \left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 6.75\ \text{rev/min}.$$

HOMEWORK 5-8

Suppose you want to lift a 12 kg mass from the ground to an altitude of 15 meters as quickly as possible. The problem is that the rope will break if its tension exceeds 160 N. What's the shortest amount of time in which the mass can be lifted without the rope breaking?



HOMEWORK 5-9
A ride at the firemen's field days (carnivals, down here) comprises a seat connected to a central column by a horizontal strut and a strut connected at a 53° angle, as shown. The lower strut is 12 meters long. If the seat plus passenger has a mass of 120 kg and the ride rotates at 2.4 revolutions per second, what is the tension in each strut?




## Pseudo-forces*

As the term suggests, these are forces that don't actually exist. Suppose you're sitting in your car at a red light. When the light turns green, you accelerate forward. You may feel as if there is a force pushing you back into your seat. If you have some trinket hanging from your rearview mirror, you may think that there is something pulling it backwards. These forces don't actually exist. What's actually happening is that the car accelerates forward and exerts a force forward on you to move you forward along with itself. You sink into the seat, the sensors in your body feel your layers being squished together, and your body interprets this as a force pushing backwards. VIDEO.

Similarly, when a car rounds a corner to the right, as an example, you may feel as if you are being pushed outward against the door. In reality, the door is curving to the right while your body has a propensity to move in a straight line. The door is actually pushing you to the right, helping you to move in the circle along with the car.

As you may guess, pseudo-forces are imagined by observers who are in an accelerating frame of reference. Generally, if you can't identify the source of a force, it's probably actually a pseudo-force.

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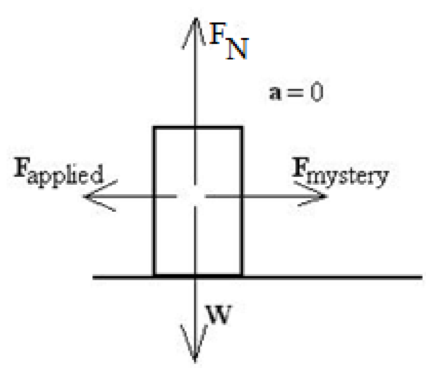
HOMEWORK 5-10

> An ornament (m = 0.3 kg) hangs from the rearview mirror of a car. When the car accelerates forward along a horizontal road, the die appears to swing backward so that the string supporting it makes an angle of 6º with the vertical. What is the acceleration of the car? HINT: We're not concerned about *how* the die swung back, only that it *has* swung back.

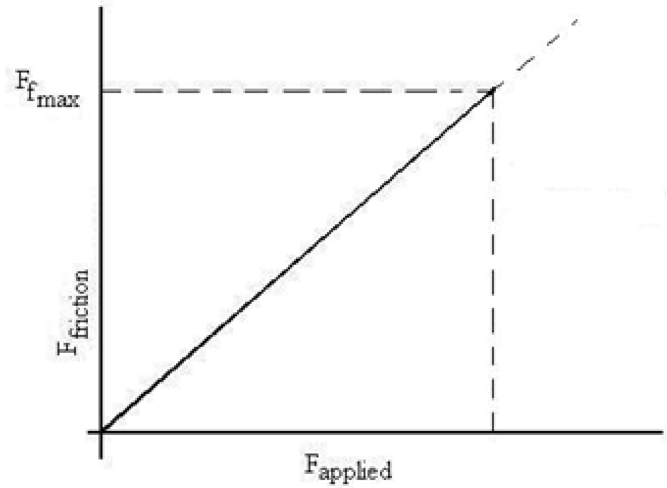
DISCUSSION 5-13

> Can you think of a way this effect could be used as, for example, a safety device in an automobile?

## Friction



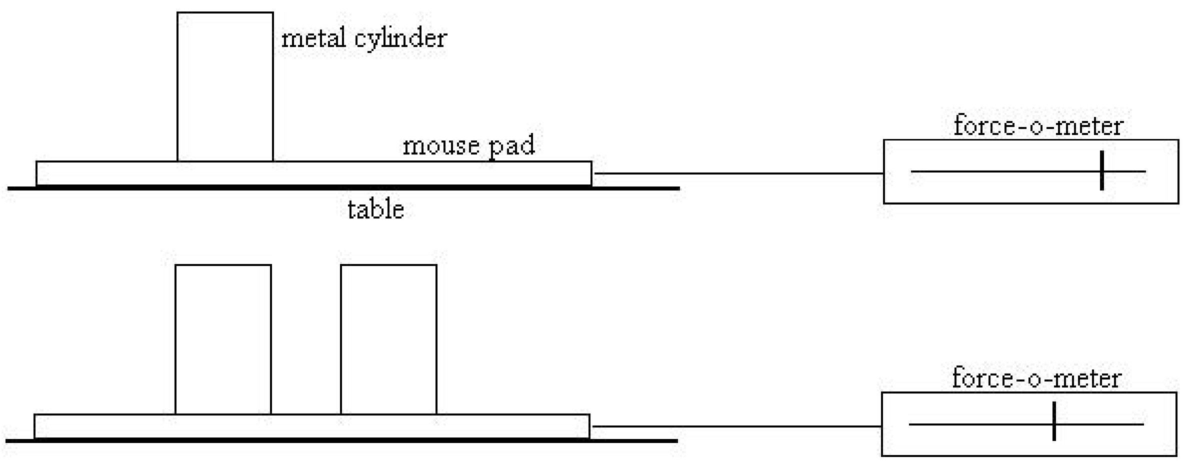

Now, we consider the fourth force, another contact force, that we've been dancing around since Section 2. *Friction* occurs at the interface between two surfaces and is directed along the surface (not perpendicular to it, as for normal contact forces), opposite to the direction in which the surfaces are sliding or want to slide. There are two types of friction that we will consider: *kineti*c and *static*. It is a common misconception that an object must be stationary to experience static friction or moving to experience kinetic friction. What is important is whether the surfaces in question are sliding against one another or not. Let's start by considering an object at rest on the desk; clearly the sum of the forces acting on this object (weight and normal force from the desk) is zero, since there is no acceleration. If we apply a small force horizontally to the object, we may be mildly surprised that it does not accelerate; if the second law is to remain correct, there must be yet another mystery force acting oppositely to our applied force that causes




the total horizontal force to be zero (second law, $F_{appl} - F_{mystery} = ma = 0$). What's more, the magnitude of that force changes as we change our applied force; it's always just big enough to cancel our force. That is, if we apply 2 newtons, it applies 2 newtons, if we apply 5 newtons, it applies 5 newtons. We'll call this force *friction*. A graph of this situation might look like this, a line of slope one passing through the origin:

Furthermore, we see that, if we continue to increase our applied force, there comes a point at which this frictional force reaches a maximum value; we know this because we can apply enough force to make the object move, and that requires a net non-zero force. How big is this maximum frictional force and what quantities determine its value?

CHEESEY EXPERIMENT 5-5




In this experiment, we placed a one kg metal cylinder atop a mouse pad 'sled' and slowly increased the applied force as measured by our force-o-meter. If we apply a non-zero force and the mass doesn’t move, then from the second law, we know how much frictional force is applied. The point of interest here, of course, is the value of the applied force for which the sled just begins to move, which is then also the maximum frictional force. In this case, that force was 2 N. What if the mass is doubled to 2 kg? What do you think the maximum force will be? It was in fact 4 N.

DISCUSSION 5-14

On what parameters of this experiment do you think the maximum friction force depends? Suppose I place my hand on the mousepad and repeat the experiment to obtain 1.6 N. I repeat the experiment but this time obtain 26 N. Did the mass of my hand change? What did change? Is this notion consistent with what we saw in the first part of the experiment? When I doubled the mass, what else doubled?

It appears that the magnitude of the maximum possible frictional force is proportional to the magnitude of the normal force pushing the two surfaces together. We usually use the Greek letter μ (mu) as the constant of proportionality and call it the *coëfficient of friction*:

$$F_{f\,MAX} = \mu F_N \ .$$

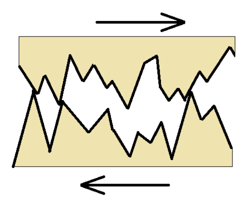
EXPERIMENT 5-5 CONTINUED

Let's flip the mousepad over and repeat. The results are listed in the table.

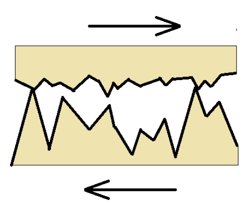
| Normal Force | Maximum Frictional Force (pad right side up) | Maximum Frictional Force (pad upside down) |
|---|---|---|
| 0 N | 0 N | 0 N |
| 9.8 N | 2N | 3 N |
| 19.6 N | 4 N | 6 N |
| coëfficient of friction | 0.20 | 0.31 |

You may notice that the coefficient is different for the two parts of the experiment. What is different between the two situations?

The value of the coëfficient of friction is always for a pair of surfaces. We can't say that the coefficient for the tabletop is 0.4, it must be the value for the tabletop and the bottom of the mousepad. Since the surface of the mousepad is rougher than the top, that coëfficient will be higher.



What is the nature of friction? In this case, it is once again the springiness of the bonds between atoms. Surfaces are never perfectly smooth, and comprise 'valleys' and 'hills.' As the surfaces try to slide against one another, the points of lateral contact will exert forces to prohibit the sliding. If one of the surfaces is smoother, there will be fewer points of contact, and it will require less force to initiate the slide. You can try this yourself by using your fingers.



DISCUSSION 5-15

Does friction disappear once the surfaces start to slide? Suppose that you are moving a couch across a floor. Do you need to continue to apply a force in order to keep the couch moving once it's started? Is it harder to get the couch to start to move, or to keep it moving?

So, it appears that there are two types of friction. One when the surfaces are not sliding, as discussed above, and one when the surfaces are already sliding. To distinguish these, let's call the first (above) *static friction* and the new one *kinetic friction*. Let's repeat the experiment above for sliding surfaces.

EXPERIMENT 5-5 CONTINUED

This time, we'll get the mass moving, then apply enough force to keep it moving at a constant velocity. Then the acceleration will be zero and once again, the applied force will be the same size as the frictional force.

| Normal Force | Frictional Force (pad right side up) | Frictional Force (pad upside down) |
|---|---|---|
| 0 N | 0 N | 0 N |
| 9.8 N | 2.5 N | 3 N |
| 19.6 N | 5.0 N | 10 N |
| coëfficient of friction | 0.26 | 0.51 |

We see that the frictional force is proportional to the normal force. However, the proportionality constants are different from their respective values in the static case. We'll need to distinguish them as $\mu_S$ and $\mu_K$:

$$F_{fK} = \mu_K F_N \ .$$

We will assume that, unlike static friction, kinetic friction always has this value. Remember that in the first half of the experiment, we found the maximum value of the static friction. At this point, it's worth doing a brief review.

- There is a force of contact called friction which acts along the interface of two objects (as opposed to perpendicular to the interface, as for the normal force of contact).

- If the surfaces are not sliding against one another, we call the friction static. This static friction force is only as big as it needs to be to prevent the surfaces from sliding against one another, but only up to a maximum value that depends on the natures of the two surfaces and on how hard they are being pushed together:

$$F_{fS} \leq \mu_S F_N \ .$$

  Because of the inequality in the relationship for static, we concentrate on situations where the surfaces are 'about to slide,' or there is some similar condition so that we know that we are at the critical point when the equality holds true. You should justify this when to do homework or exam problems.

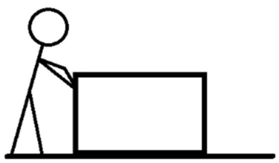
- If the surfaces are already sliding, we have kinetic friction, in which case

$$F_{fK} = \mu_K F_N \ .$$

DISCUSSION 5-16

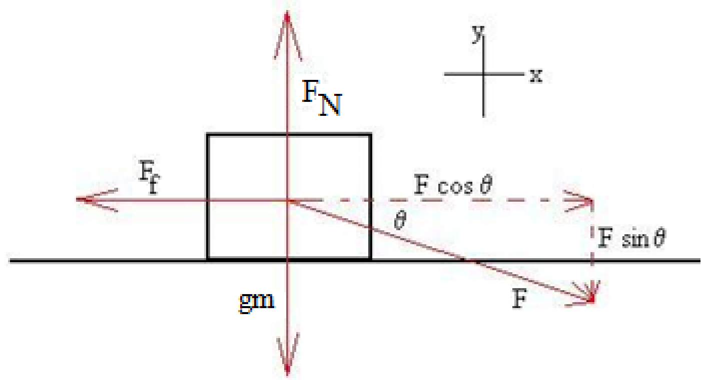
What are the units for the coefficient of friction?

EXAMPLE 5-8



You're pushing a 800 N box at constant velocity across a floor by applying a 200 N force F applied at an angle of 25° below the horizontal. What is the coëfficient of kinetic friction between the box and the floor?

Because of the wording of the problem (F keeps the box moving), we assume that the block is moving toward the right. Correspondingly, the frictional force is to the left. The velocity is constant, so all components of the acceleration are zero. We therefor make use of Rule Two in choosing a coördinate system. The second law equations are then




$$+F\cos\theta - F_{FK} = ma_x = 0$$

$$-F\sin\theta + F_N - gm = ma_y = 0\ .$$

We also need an equation for the friction. Since this is kinetic friction, there is no question that we use the equal sign:

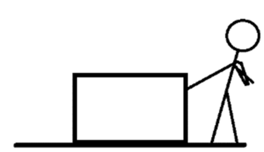
$$F_{fK} = \mu_K F_N\ .$$

The solution is fairly straightforward substitution:

$$\mu_K = \frac{F_{fK}}{F_N} = \frac{F\cos\theta}{gm + F\sin\theta} = \frac{200\cos(25^0)}{800 + 200\sin(25^o)} = 0.20$$

DISCUSSION 5-17

Your boss sees you moving the crate in this manner and says, "You need to work smarter, not harder. Get a rope and pull the crate at 25° above the horizontal." Will this require less force? What physical reasons (in words) would back this up?



EXERCISE 5-3

Go ahead and calculate how much force would be necessary at 25° above the horizontal to keep the crate moving at a constant velocity. Was your boss correct?

DISCUSSION 5-18

What question should you ask yourself next? Usually, this question is answered using calculus. Can you think of a way to solve this without calculus?

EXAMPLE 5-9

Can we find what angle would result in the lowest required force to keep the crate moving at a constant velocity? The force necessary to keep the crate moving as a function of the applied angle is given by

$$F = \frac{\mu_K gm}{\cos\theta + \mu_K \sin\theta} .$$

We would like to minimize the force, which means maximizing the denominator in the expression. This is a problem that screams for calculus, which we are forbidden to use. So, we'll have to be a bit cleverer. Let the coëfficient of friction be represented by the cotangent of an angle phi.

$$\cos\theta + \mu_K \sin\theta = \cos\theta + \cot(\varphi)\sin\theta = \cos\theta + \frac{\cos(\varphi)}{\sin(\varphi)}\sin\theta$$
$$= \frac{\sin(\varphi)\cos\theta + \cos(\varphi)\sin\theta}{\sin(\varphi)} = \frac{\sin(\varphi + \theta)}{\sin(\varphi)}$$

Now, since $\sin(\varphi)$ is a constant (it depends on $\mu_K$), we need only worry about the numerator here, which maxes out at one when the sum of the angles is 90°. So, the angle that would require the least force to keep the crate moving would be

$$\theta_o = 90^o - \varphi = 90^o - \operatorname{arccot}(\mu_K) = 90^o - \operatorname{arccot}(0.20) = 11.3^o$$

Static friction problems are generally harder than kinetic problems. First of course, we may not be at the critical point of the surfaces being just about to slide, and often we don't even know which way they will try to slide. Let's look at a fairly standard problem and see what kinds of questions can be asked and how we might deal with them.

HOMEWORK 5-11

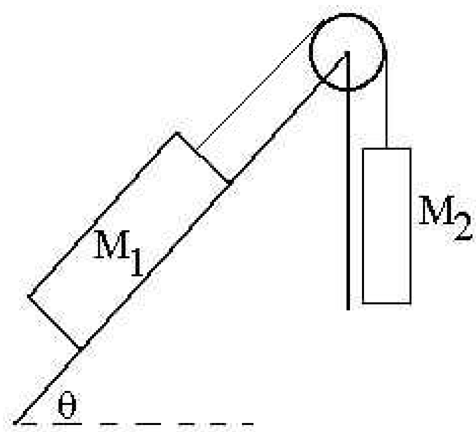
A mover finds that a 120 kg dresser requires a 70 N horizontal force to set it in motion across the floor, but only 55 N to keep it moving with constant velocity. Find the static and kinetic coëfficients of friction between the bottom of the dresser and the floor.

HOMEWORK 5-12

What minimum force is necessary to drag a crate of mass 60 kg across a floor at constant velocity with a rope inclined at $37^o$ above the horizontal? The coëfficient of kinetic friction is 0.7.

EXAMPLE 5-10

Consider once again two blocks connected by a light string over a magic wheel, with one block on a rough incline.

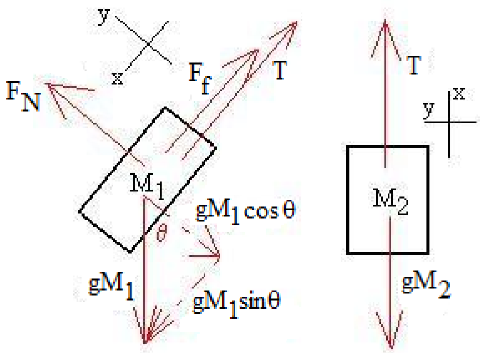



What kinds of questions could be asked? One might wonder

- what is the largest value of $M_1$ (or the smallest value of $M_2$) for which $M_1$ will not slide down the plane?
- what is the smallest value of $M_1$ (or the largest value of $M_2$) for which $M_1$ will not slide up the plane?
- what is the smallest value of $\mu_S$ for which $M_1$ will not slide down (or up) the plane?
- what is the smallest (or largest) value for $\theta$ for which $M_1$ will not slide up (or down) the plane?
- what is the direction and magnitude of the acceleration, assuming that the blocks are in motion?
- about many things not listed here.

Many of these types of problems require us to assume or guess which way the blocks would slide if there were no friction, since we need to be able to assign a direction to the frictional force that would keep that from happening. Sometimes, the direction is obvious, other times we will need to solve the problem first with $\mu_S = 0$. Let's take a run at several of these.

Write the second law, assuming that the blocks are moving, or about to move, with $M_1$ sliding down the incline. Keep in mind, we may well be wrong about this. Even worse, the blocks may be moving one way but accelerating the other!




For $M_2$, we have

$$+T - gM_2 = M_2 a_x$$

No Forces

For $M_1$,

$$-T - F_f + gM_1 \sin\theta = M_1 a_x$$

$$+F_N - gM_1 \cos\theta = M_1 a_y = 0$$

$$F_f = \mu F_N \ .$$

Note that I haven't specified which type of friction, although if it's static, the system must be just about to move. Nor have I specified zero acceleration. I have assumed that the frictional force is directed up the incline. Let's do some math.

$$F_f = \mu F_N = \mu(gM_1 \cos\theta)$$

Substitute this into the $M_1$ x equation.

$$-T - \mu\, gM_1 \cos\theta + gM_1 \sin\theta = M_1 a_x$$

Then, as usual, add the $M_2$ x equation to eliminate the tension.

$$-T - \mu\, gM_1 \cos\theta + gM_1 \sin\theta = M_1 a_x$$

$$+T - gM_2 = M_2 a_x$$

$$-gM_2 - \mu\, gM_1 \cos\theta + gM_1 \sin\theta = (M_1 + M_2) a_x$$

$$g\big(-M_2 + M_1(\sin\theta - \mu \cos\theta)\big) = (M_1 + M_2) a_x$$

Let's answer some of the questions that were listed above.

What is the largest value of $M_1$ for which $M_1$ will not slide down the plane? Since the masses are not yet moving, $a_x = 0$, the friction is static, and the equation simplifies to

$$M_2 = M_1(\sin\theta - \mu \cos\theta)$$

$$M_{1MAX} = \frac{M_2}{\sin\theta - \mu \cos\theta}$$

What is the smallest value of $M_1$ for which $M_1$ will not slide up the plane? We don't have to redo the entire problem. The only thing that changes from the first question is the direction of the frictional force, and we can fix that with a mathematical trick by replacing μ with -μ. So, we immediately know that

$$M_{1MIN} = \frac{M_2}{\sin\theta + \mu\ \cos\theta}\ .$$

What is the smallest value of $\mu_S$ for which $M_1$ will not slide down (or up) the plane? Once again, the acceleration is zero and the friction is static, so

$$M_2 = M_1(\sin\theta - \mu\ \cos\theta)$$

$$\mu_{S\ MIN} = \frac{\sin\theta - \dfrac{M_2}{M_1}}{\cos\theta}\ .$$

What is the smallest value of $\mu_S$ for which $M_1$ will not slide up the plane? We'll use our math trick again, since the only difference in the equations will be the direction of the frictional force.

$$\mu_{S\ MIN} = -\left(\frac{\sin\theta - \dfrac{M_2}{M_1}}{\cos\theta}\right) = \frac{\dfrac{M_2}{M_1} - \sin\theta}{\cos\theta}\ .$$

I'm sure you're getting the idea here. Now, what if the masses start from rest and do slide? What would be the acceleration? That's going to depend on which way they slide. So, first we need to find the direction of acceleration without friction, then put it back in to find the actual acceleration:

$$a_{x\ \mathrm{NO\ FRICTION}} = \frac{-M_2 + M_1\sin\theta}{(M_1 + M_2)}\, g$$

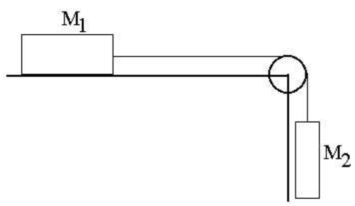
$$a_x = \frac{\left(-M_2 + M_1(\sin\theta - \mu_K\ \cos\theta)\right)}{(M_1 + M_2)}\, g$$

If the sign of this 'no friction' acceleration is positive, we're already O.K. and we return to the solution as given. If the 'no friction' acceleration is negative, then we need to reverse the sign of the coefficient of friction, then compute:

$$a_x = \frac{\left(-M_2 + M_1(\sin\theta + \mu_K\ \cos\theta)\right)}{(M_1 + M_2)}\, g$$

Of course, if you're told which direction they are moving in, you just pick the correct sign for $\mu_K$.

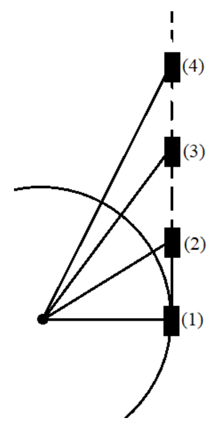
HOMEWORK 5-13




A 15 kg mass ($M_1$) is connected by a magic string over a magic wheel to a 5 kg mass ($M_2$) as shown in the figure. When the masses are released, $M_2$ falls from rest a distance of 2.2 meters in 3 seconds. What is the coëfficient of kinetic friction between $M_1$ and the table surface?

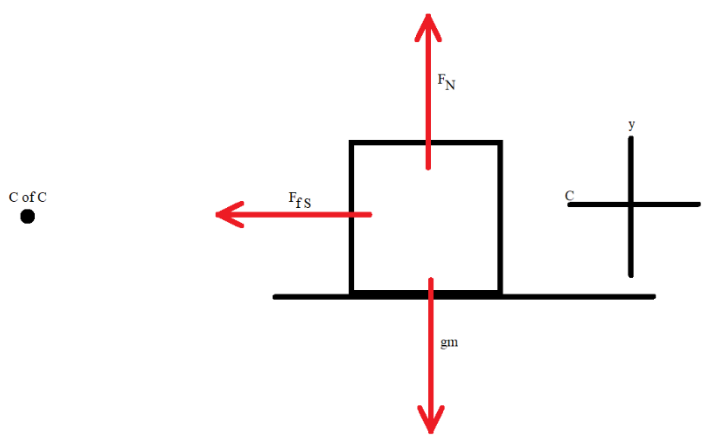
EXAMPLE 5-11

Suppose you're bored early on a Sunday morning and you decide to drive your 1800 kg car in circles in the parking lot at the local mall. How quickly can you drive the car in a circle of radius 50 m?




So, what forces act on the car? Obviously, there is weight. There is a normal force from the pavement upward, and there is friction. If you've tried this on an icy surface, you know the car will simply travel in a straight path. In what direction is the friction? Well, the car is 'trying' to move in a straight path, which means it's trying to slide away from the center of the circle. Since the friction opposes that attempt, it must point toward the center of the circle. You can even feel this force with your hands VIDEO. Which type of friction is this, static or kinetic? Be careful! Let's say the coëfficient of friction is 0.85.

Let's do a free body diagram.




Rule One tells us to place the c-axis toward the center of the circle. The second law equations are

$$\text{c:}\ +F_{fS} = ma_C = m\frac{v^2}{r}$$

$$\text{y:}\ +F_N - gm = ma_y = 0$$

$$F_{FS} = \mu_S F_N \quad \text{crit. sit.}$$

We can use the equality in the static friction equation because we're looking for the maximum speed, *i.e.*, the tires are just about to slip. Re-arranging and substituting,

$$v^2 = \frac{r\,F_{fS}}{m} = \frac{r\,\mu_S F_N}{m} = \frac{r\,\mu_S gm}{m} = r\,\mu_S g = 50(0.85)10 = 425$$

$$v_{MAX} = \sqrt{425} = 20.6\ \mathrm{m/s}\,.$$

Notice that the mas of your car didn't matter.

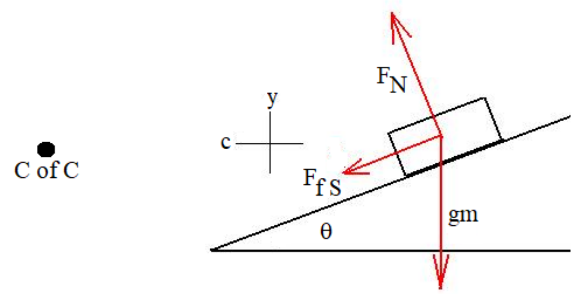
DISCUSSION 5-19

What does MDOT do to help you get around sharp curves on the highway? Think especially about cloverleafs. How does this help?

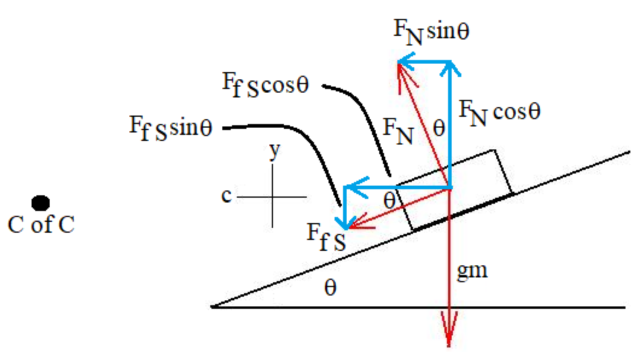
EXAMPLE 5-12

Turn 1 at the Talladega Superspeedway is inclined at a 33° angle from the horizontal and has a radius of 330 m (this depends of course on which lane you are in). Brand new tires have a coëfficient of static friction with the track surface of 1.3. What is the theoretical maximum speed a 1,636 kg car can travel and still negotiate this turn?

Of course, we begin with a free body diagram. As in the previous example, this car would tend to move toward the outside of the circle, and so the static frictional force would act to oppose this sliding and so push it toward the center. Following Rule One, the c-axis is toward the center of the circle and then the y-axis is vertical. We will need to decompose both the frictional force and the normal force.




The second law equations become:




$$c:\; + F_{fS}\cos\theta + F_N\sin\theta = ma_C = m\frac{v^2}{r}$$

$$y:\; + F_N\cos\theta - F_{FS}\sin\theta - gm = ma_y = 0$$

$$F_{fS} = \mu_S F_N \quad \text{crit. sit.}$$

Let's re-arrange, substitute, and divide.

$$+\mu_S F_N\cos\theta + F_N\sin\theta = m\frac{v^2}{r}$$

$$+F_N\cos\theta - \mu_S F_N\sin\theta = gm$$

$$\frac{\mu_S F_N\cos\theta + F_N\sin\theta}{+F_N\cos\theta - \mu_S F_N\sin\theta} = \frac{m\frac{v^2}{r}}{gm}$$

$$\frac{\mu_S\cos\theta + \sin\theta}{\cos\theta - \mu_S\sin\theta} = \frac{v^2}{gr}$$

At this point, of course, we could solve for any of the variables contained in this relationship. Proceeding to find the maximum speed results in

$$v_{MAX} = \sqrt{gr\frac{\mu_S\cos\theta + \sin\theta}{\cos\theta - \mu_S\sin\theta}} = \sqrt{10(330)\frac{1.3\cos33^o + \sin33^o}{\cos33^o - 1.3\sin33^o}} = 203\,\mathrm{m/s}\,.$$

The record is about 96 m/s.

Suppose in the previous example, we had wanted to find the slowest possible speed so as not to slide to the center of the track. Since the only thing different about the problem is the direction of the frictional force, we can take this result and flip the sign of the coefficient of friction:

$$v_{MIN} = \sqrt{gr\frac{-\mu_S\cos\theta + \sin\theta}{\cos\theta + \mu_S\sin\theta}}\ .$$

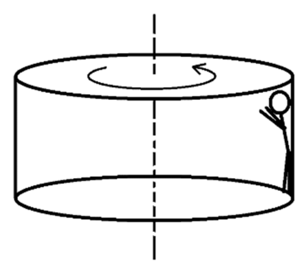
With these particular numbers, I expect that we'll be taking the square root of a negative number, which means the car could be parked on the incline and not slip, but with a coëfficient corresponding to an icy surface, there may well be a real minimum speed.

HOMEWORK 5-14

A dime (m = 2 grams) sits at the edge of the platter of a record player (They've made a comeback, so I know you know what that is). What is the minimum coëfficient of static friction that will keep the coin on the platter as it spins. A standard LP is 30 cm in diameter and rotates at $33^1/_3$ revolutions *per* minute.

HOMEWORK 5-15

Again at the field days, a patron enters a circular room of radius 5 m. The room starts to spin and speeds up to 7 radians/second, at which time the floor drops away. What must be the minimum coëfficient of static friction so that the passenger does not slide down to a certain death?



## Outside the Safe Zone*

Let's take a crack at a problem that does have drag. Consider a small metal ball of mass m dropped from a great height. In the absence of air, it will accelerate downward uniformly at some value around 9.8 m/s$^2$. The drag force is commonly assumed to be proportional to the speed of an object, and of course depends on the size, shape, and orientation of the object as well as the properties of the fluid though which it is travelling. The direction is opposite to the motion of the object through the fluid. Let's write NII for the vertical direction, with down positive:

$$+\mathrm{gm} - \mathrm{bv} = \mathrm{ma}.$$

Qualitatively, we can see that, just after release when the speed is extremely small, the acceleration will be equal to $a_g$. However, as the ball falls and picks up speed, the acceleration will become less and less (still downward though of course). Eventually, the ball acquires speed v = gm/b, at which point the acceleration becomes zero, and the velocity accordingly becomes constant. This ultimate speed is known as the *terminal velocity* of the ball. Of course, this value will vary from object to object and even on the orientation of the object, if it is not spherical.

Let's be a bit more analytical about this problem and solve for the velocity as a function of time. In doing so, we'll find the solution to a problem that re-occurs often in this course.

Re-arranging the NII equation above,

$$\mathrm{v} = \mathrm{v_I} + \mathrm{v_{II}} = \frac{\mathrm{gm}}{\mathrm{b}} - \frac{\mathrm{m}}{\mathrm{b}}\mathrm{a}\,.$$

Clearly, the expression that will ultimately represent the velocity has a constant part ($v_I$) and a changing part ($v_{II}$), such that $v = v_I + v_{II}$. By observation, we see that $v_I = gm/b$. Next, the ball's acceleration is the instantaneous time rate of change of the velocity, but because $v_I$ is constant, it is also the ITRC of $v_{II}$ alone. Mathematically,

$$\begin{aligned}
\mathrm{a} = \mathrm{ITRC(v)} &= \lim_{\Delta t\to 0}\frac{\Delta \mathrm{v}}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{\Delta(\mathrm{v_I}+\mathrm{v_{II}})}{\Delta t} = \lim_{\Delta t\to 0}\left(\frac{\Delta \mathrm{v_I}}{\Delta t}+\frac{\Delta \mathrm{v_{II}}}{\Delta t}\right) = \lim_{\Delta t\to 0}\frac{\Delta \mathrm{v_I}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta \mathrm{v_{II}}}{\Delta t} \\
&= 0 + \lim_{\Delta t\to 0}\frac{\Delta \mathrm{v_{II}}}{\Delta t}\;\;.
\end{aligned}$$

So, now we have that

$$\lim_{\Delta t\to 0}\frac{\Delta \mathrm{v_{II}}}{\Delta t} = \left(-\frac{\mathrm{b}}{\mathrm{m}}\right)\mathrm{v_{II}}\;\;.$$

DISCUSSION 5-20

> We're trying to find a function whose ITRC is proportional to itself. You may remember studying such a function in high school, although then, the proportionality constant was positive. I'll give you a hint: right now, in 2020, the proportionality constant in the U.S. is horrifically low at +0.0005, while in Switzerland, the constant is a very Swiss -0.0021.

The solution to this equation is

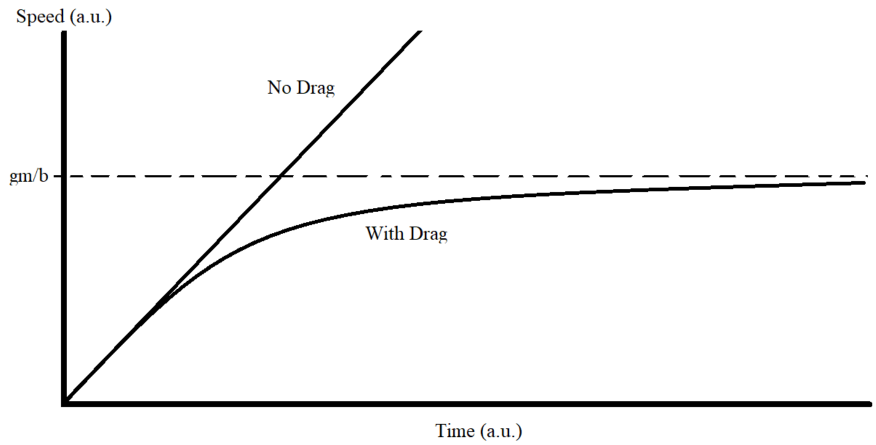
$$v_{II}(t) = v_o e^{-\frac{bt}{m}} ,$$

with $v_o$ the initial value of $v_{II}$ (but not of v!) and e is a well-known number in mathematics equal approximately to 2.7183. Solve for $v_o$, knowing that $v(t=0) = 0$:

$$v = \frac{gm}{b} + v_o e^{-\frac{bt}{m}}$$
$$0 = \frac{gm}{b} + v_o e^{-\frac{b0}{m}}$$
$$v_o = -\frac{gm}{b} .$$

So, finally,

$$v(t) = \frac{gm}{b}\left(1 - e^{-\frac{bt}{m}}\right) .$$




MATHEMATICAL JUSTIFICATION*

We have a function F(t) with the ITRC of F proportional to F (constant C). Call the initial value of F at t = 0 $F_o$. What we'll do is start at $F_o$, at t = 0 and estimate F at a time $\Delta t$ later by extending a line tangent to the curve; we'll call that value $F_1$. For this line, $F_o$ is the y intercept and $CF_o$ is the slope. $F_1$ is <u>not</u> expected to be the actual value of the function at that time.

$$F_1 = F(\Delta t) = F_o + CF_o\,\Delta t = F_o(1 + C\Delta t) .$$

Let's repeat this process to estimate the function's value at t = 2$\Delta$t. We'll use the slope calculated from our estimate of $F_1$:

$$F_2 = F(2\,\Delta t) = F_1 + CF_1\,\Delta t = F_1(1 + C\Delta t) = F_o(1 + C\Delta t)^2 .$$

We can use an inductive argument that the $n^{th}$ estimate will be

$$F_n = F(n\,\Delta t) = F_o(1 + C\Delta t)^n .$$

We have every reason to think that each subsequent iteration of this process takes us further and further from the correct values, so we'll need to do this in extremely small steps, *i.e.*, take the limit as $\Delta t \to 0$. To do that, we're going to make a few substitutions. First, the actual time t is the product of the number of steps we've taken and the size of each step: t = n $\Delta$t. We'll also define q to be 1/(C $\Delta$t). Then, for motivations that should be obvious in a moment or two, we can write that n = Ct/C$\Delta$t = qCt. Note that, as $\Delta t \to 0$, $q \to \infty$.

$$F(n\,\Delta t) \to F(t) = \lim_{\Delta t \to 0} F_o(1 + C\Delta t)^n$$

$$F(t) = \lim_{q\to\infty} F_o \left(1 + \frac{1}{q}\right)^{q\,Ct} = F_o \left(\lim_{q\to\infty}\left(1 + \frac{1}{q}\right)^{q}\right)^{Ct}$$

The value of the limit is very well know in mathematics, but if you don't recognize it, you can obtain an approximate value by letting q be a fairly high integer and obtain 2.718281… .This number appears so often that it has its own symbol, e. The result then is that

$$F(t) = F_o e^{Ct} \; .$$

You might recognize this as the formula for continuously compounded interest in a savings account. In 2020, many European central banks have negative interest rates. In the U. S., negative rates at the Federal Reserve are forbidden by the Constitution.

---

EXERCISE 5-1 Solution

$$T = M_2\left(\frac{M_1 \sin(\theta) - M_2}{M_1 + M_2} g\right) + gM_2$$

$$T = \left(\frac{M_1 M_2 \sin(\theta) - M_2^2 + M_2(M_1 + M_2)}{M_1 + M_2}\right) g$$

$$T = \left(\frac{M_1 M_2 \sin(\theta) - M_2^2 + M_2 M_1 + M_2^2}{M_1 + M_2}\right) g$$

$$T = \left(\frac{M_1 M_2 \sin(\theta) + M_2 M_1}{M_1 + M_2}\right) g$$

$$T = \left(\frac{M_1 M_2}{M_1 + M_2}\right)(\sin(\theta) + 1) g$$

EXERCISE 5-2

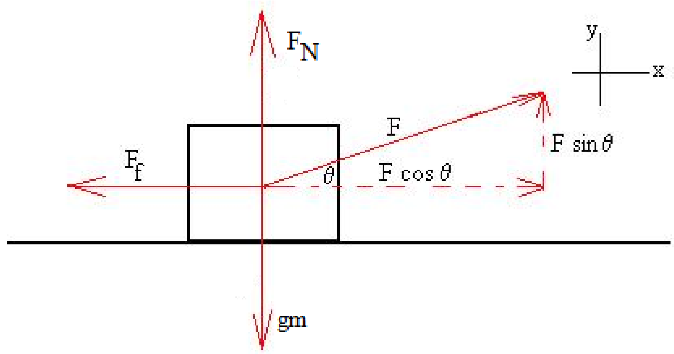
Well, you just did this problem. In this case, though, the angle of the 'incline' is 90°.

$$a_x = \frac{M_1 \sin(\theta) - M_2}{M_1 + M_2} g = \frac{5\sin(90^o) - 7}{5 + 7} 10 = -1.67\ \mathrm{m/s^2}.$$

$$T = \left(\frac{M_1 M_2}{M_1 + M_2}\right)(\sin(\theta) + 1)g = \left(\frac{5(7)}{5 + 7}\right)(\sin(90^o) + 1)10 = 58.33\ \mathrm{N}$$

EXERCISE 5-3 Solution

The second law equations are almost the same; only the vertical component of the applied force changes direction (it will presumably also change magnitude).




$$+F\cos\theta - F_{FK} = ma_x = 0$$

$$+F\sin\theta + F_N - gm = ma_y = 0 \ .$$

$$F_{fK} = \mu_K F_N \ .$$

This solution is a bit more tedious.

$$F\cos\theta = F_{FK} = \mu_K F_N = \mu_K(gm - F\sin\theta) = \mu_K gm - \mu_K F\sin\theta$$

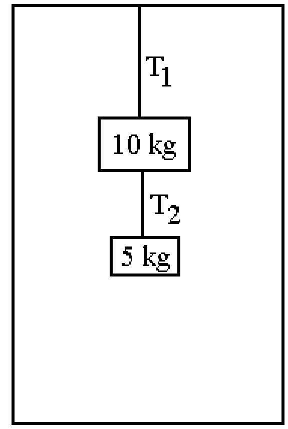
$$F\cos\theta + \mu_K F\sin\theta = \mu_K gm$$

$$F\,(\cos\theta + \mu_K \sin\theta) = \mu_K gm$$

$$F = \frac{\mu_K gm}{\cos\theta + \mu_K\sin\theta} = \frac{(0.2)800}{\cos(25^o) + \mu_K\sin(25^o)} = 161.5\text{ N}$$

Sample Exam II

**MULTIPLE CHOICE** (4 pts each)

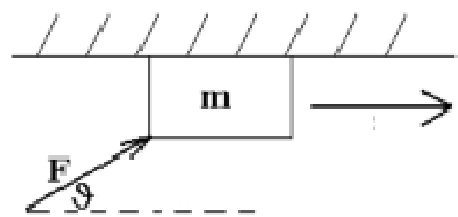



1) Consider the two masses ($m_1$ = 10 kg, $m_2$ = 5 kg) hanging as shown in an elevator. What is the tension in each rope as the elevator accelerates downward at 3 m/s$^2$? Let g = 10 N/kg.

   A) $T_1$ = 105 N; $T_2$ = 35 N
   B) $T_1$ = 195 N; $T_2$ = 65 N
   C) $T_1$ = 100 N; $T_2$ = 50 N
   D) $T_1$ = 150 N; $T_2$ = 50 N
   E) None of the answers above is correct.

2) Choose the answer which bests completes the sentence:
   If an object is at rest, then

   A) no forces act on the object.
   B) any forces which form Third Law pairs cancel each other out.
   C) the mass of the object must be very large.
   D) the sum of all forces acting on the object must be zero.
   E) the weight and the normal force must be equal in magnitude and opposite in direction.




3) Consider a block of mass m which is just about to slide along the ceiling, as shown. The coëfficient of static friction between block and ceiling is $\mu_S$. Which of the following sets of equations follow from Newton's second law?

| | | |
|---|---|---|
| A) $F \sin\theta + mg - F_N = 0$ | $F \cos\theta + F_f = 0$ | $F_f = \mu_S F_N$ |
| B) $F \cos\theta - mg - F_N = 0$ | $F \sin\theta + F_f = 0$ | $F_f = \mu_S F_N$ |
| C) $F \cos\theta - mg + F_N = 0$ | $F \sin\theta - F_f = 0$ | $F_f = \mu_S F_N$ |
| D) $F \sin\theta - mg - F_N = 0$ | $F \cos\theta - F_f = 0$ | $F_f = \mu_S F_N$ |
| E) $F \sin\theta - mg + F_N = 0$ | $- F \cos\theta - F_f = 0$ | $F_f = \mu_S F_N$ |

4) Suppose you would like to launch a satellite so that it orbits the earth in a circle just above the surface (ignore inconvenient considerations such as air resistance and irregular topography). What would the speed of the satellite need to be? The radius of the earth is $6.4\times10^{+6}$ m.

   A) $6.4\times10^{+7}$ m/s
   B) 8000 m/s
   C) 2500 m/s
   D) 800 m/s

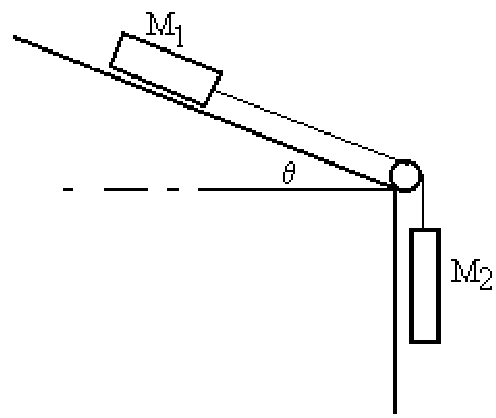
E) 10 m/s

N.B.: If you're rusty on scientific notation, remember that $6.4\times10^{+6} = 6{,}400{,}000$.

5) The most expensive production automobile ever is the *Veyron* by Volkswagen (cost = \$6 million each). The Veyron can slow from 110 m/s to a stop in 10 seconds on a flat road, *i.e.*, it can decelerate at 11.1 m/s$^2$. Assuming that this is due entirely to the brakes (it's not), what minimum coëfficient of static friction between road and tires will allow this? Pick the closest value.

   A) 0
   B) 0.25
   C) 0.9
   D) 1.1
   E) 2.6

**PROBLEM I** (20 pts)




Consider the two masses ($M_1$ = 2 kg, $M_2$ = 5 kg) as shown, one of which is on a smooth surface inclined at an angle of $\theta = 30^\circ$ from the horizontal.

A) What is the acceleration of the masses? (15 pts)

B) Find the tension in the string. (5 pts)

**PROBLEM II** (20 pts)

Billy decides to boat 2000 m downstream and back. His mom tells him he must be back within an hour (= 3600 seconds). The river flows at 1 m/s relative to the ground and Billy's boat moves at 3.0 m/s relative to the water.

A) How much time will it take Billy to travel the 2000 m downstream?

B) How much of his 60 minutes remain for the return trip upstream?

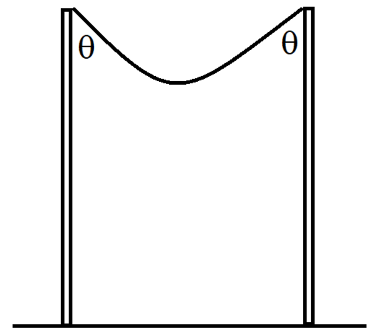
C) With what velocity relative to the water would Billy's boat need to move to return on time? Can he make it back in time?

**PROBLEM III** (20 pts)

Commander Buzz Kutter is floating in space 50 meters from the open airlock of his spaceship, the

Lazy Star. The only thing in his possession not immediately needed for survival is a hammer. Outline a procedure that might allow Kutter to return to his ship. Explain fully.

**PROBLEM IIII** (20 pts)




A uniform string of mass M hangs between the tops of two poles of equal height. At each end, the string makes an angle θ with the pole, as shown. Find the tension in the string at its center.

HINT: Here's a problem where the tension in a string is NOT the same along its length. It is not massless.

## SECTION 6 – THE SECOND PICTURE

We've looked at the motions of objects using forces and accelerations, and if we were lucky enough to have constant accelerations, the kinematic equations. Now, we'll introduce a second picture which we may, or may not, find more convenient to use on certain classes of problems. Please note that this new picture is really nothing more than Newton's second law with a few definitions thrown in; there is a tendency for students to thjnk of this material as disconnected from previous discussions, but it really just a re-arrangement of stuff you already know.

DISCUSSION 6-1

Consider a small toy car sitting on a table at a spot marked 'X'; we'll assume the wheels make its contact with the table frictionless. Observe the car closely. Now, observe the car as it travels through point X. Is it fair to say that the car possesses some quality or property in the latter case which it lacks in the former? How did the car acquire that property?

CHEESY EXPERIMENT 6-1 VIDEO

After the experiment, we concluded/agreed on the following:

- We agreed that there is some quality the object possesses when it's moving through X that it lacks when it's stationary. For want of a better word, let's call that quality *energy* (E).
- Energy is transferred into the object by applying a force. However, the force must act through a displacement. Applying a force to a non-moving object transfers no energy.
- Transferring energy into (or out of) an object is a process; let us call the <u>transfer</u> of energy the *work* (W) done on the object. Work is <u>not</u> a form of energy, it is the transfer of energy. Let's define the work on an object to be positive when energy enters the object and negative when it is removed (why not?)
- The bigger the force, the more energy is transferred: as F↑, W↑. We might even speculate that W is proportional to F. That would certainly be the simplest relationship consistent with our observations. We could be wrong, of course; perhaps $W \sim F^2$ or $F^3$. We'll make the simplest assumption and see if there is a contradiction somewhere in our subsequent experiments.
- The greater the displacement over which the force acted, the more work is done: that is, as $\Delta x$ ↑,W↑ .We might speculate that W is proportional to $\Delta x$.

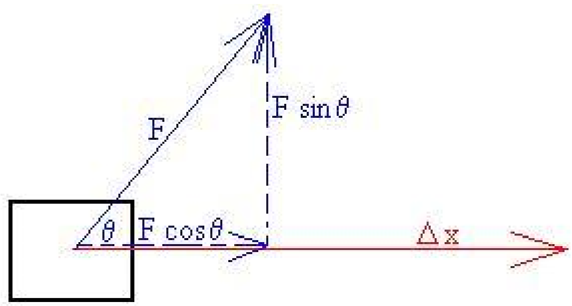
What's more, there is an effect due to the relative orientation of the force with the displacement. We saw that:

- If $\vec{F}$ and $\Delta\vec{x}$ are in the same direction, energy is transferred <u>into</u> the object and we say that positive work was done.
- If $\vec{F}$ and $\Delta\vec{x}$ are in the opposite directions, energy is transferred <u>out of</u> the object and we say that negative work was done.
- If $\vec{F}$ and $\Delta\vec{x}$ are perpendicular, <u>no</u> energy is transferred into the object and we say that no work was done.

DISCUSSION 6-2

> How can we express these last notions in a more mathematical way? Can you think of a function that will give us a positive value when two vectors are parallel, a negative value when they are anti-parallel, and zero when they are perpendicular?

Let's consider an object moving along, say, the x-axis while a force is applied at some angle away from the x-axis, as shown in the figure. When we talked about vectors and components, we said that the components of a vector can replace the original vector. Let's do so for this force. The component parallel to the displacement is F cos θ and the component perpendicular is F sin θ. The former should contribute to the work, while the latter does not. Sounds like just what we need. Indeed, if the angel were greater than 90°, the cosine would provide the negative sign required when the force and displacement are in generally opposite directions.




Let's synthesize these notions into a single mathematical expression, with the assumption that the universe works as simply as possible:

$$W = F\,\Delta x \cos\theta_{F,\Delta x} = \Delta E\,.$$

The unit for work is newtons times meters; we will define one *joule* (J) as the work done by one newton of force acting on an object while it displaces one meter in the same direction. This procedure will increase the energy of the object by one joule.[1]

Now, since the result for the work doesn't depend on the actual directions of the force or the displacement, but only on their relative directions, we might guess that the work is a scaler quantity. We'll confirm this in a page or so. As such, the work can be written as[2]

$$W = \vec{F} \cdot \Delta\vec{x}\ .$$

EXAMPLE 6-1

> Consider a box pulled 4 meters along the flat ground by a rope with tension 58 newtons which is at an angle of 54° above the horizontal. How much work does this force do?
>
> The diagram for this is close to the one above. The work would be

[1] I like to use a bank account as an analogy. Work is like the deposits and withdrawals, while the amount of energy is like the balance. If there is a deposit of $19, the balance increases by $19.

[2] Revisit Section One to review the dot product of two vectors.

$$W = F\,\Delta x \cos\theta_{F,\Delta x} = 58\,(4)\cos(54^{o}) = \boxed{+136.4\,J}\ .$$

HOMEWORK 6-1

A cowboy grabs a rope trailed by a runaway horse and applies a force of 1100 N as he is dragged 37 meters. How much work does the cowboy do on the horse? How much work does the horse do on the cowboy? How do the answers to this question depend on whether the horse stops, slows, or keeps running?

DISCUSSION 6-3

What if several forces act on the object simultaneously? Can you extend the analogy with the back account?

At this point, we're in a strange position. We think we know a bit about transferring energy, but we don't yet know what energy is. Let's see what happens if we apply a number of forces to an object and find the total work performed on it., which in turn should be the change in the object's energy, ΔE.

DERIVATION 6-1

To start off, let's assume a one-dimensional problem with constant forces.

$$\begin{aligned} W_{TOTAL} &= \sum_n W_n = \sum_n \vec{F}_n \cdot \Delta\vec{x} = \left(\sum_n \vec{F}_n\right) \cdot \Delta\vec{x} = (m\,\vec{a}) \cdot \Delta\vec{x} = m(\vec{a} \cdot \Delta\vec{x}\,) \\ &= m\left(\frac{v_f^2 - v_i^2}{2}\right) = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = \Delta\left(\tfrac{1}{2}mv^2\right) = \Delta E\ , \end{aligned}$$

at which point we might jump to the conclusion that

$$E = \tfrac{1}{2}mv^2\ .$$

This is a little dangerous; just because two quantities have the same change in value doesn't mean that they have the same value. For example, there could be some constant term included in the energy that cancels out when calculating the change.[3] However, we have previously decided to go with the simplest explanations, until a contradiction is found. Historically, this was the definition of energy, but as we proceed through this section, we will introduce notions of other types of energy. Seeing as our object possesses this energy due to its motion, let's define this specifically to be the *kinetic energy*, K:

$$K = \tfrac{1}{2}mv^2\ .$$

[3] Perhaps E = ½ mv$^2$ + mc$^2$. The second term will always disappear when ΔE is calculated.

Note that, because the kinetic energy depends on the speed of the object, it is a scalar, not a vector.

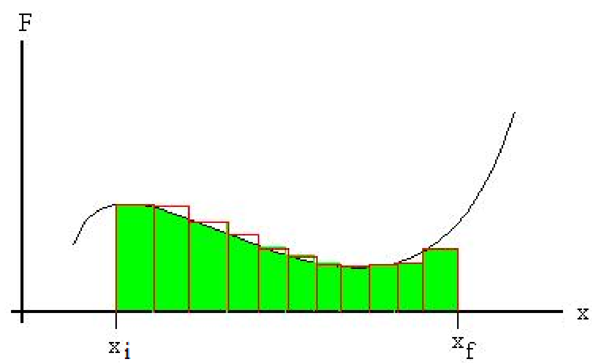
HOMEWORK 6-2

At what speed would a 50 kg person have to run to have the same kinetic energy as a 1500kg auto traveling at 100 km/h?

DISCUSSION 6-4

Suppose a jogger of mass M is trotting along at speed $v_o$, and therefor has kinetic energy $K_o$. What kinetic energy (in terms of $K_o$) would he have if he doubled his pace? If his daughter with half his mass then paces him, how much kinetic energy would she have?

Now let's make thigs a little harder. First, what if the force applied were not constant (or, a *variable force*)? Clearly, more work would be done in some displacement intervals than in others.




DERIVATION 6-2

We need to break the overall displacement down into very many, very small displacements $\Delta x_n$, over which we can consider the force to be relatively constant at value $F_{\Delta x\, n}$; we then find the work done over that interval to be (approximately)

$$W_n = F_n \Delta x_n = \frac{1}{2} m\, v_n^2 - \frac{1}{2} m\, v_{n-1}^2$$

and the net work is then

$$W = \sum_n F_n \Delta x_n = \frac{1}{2} m\, v_1^2 - \frac{1}{2} m\, v_i^2 + \frac{1}{2} m\, v_2^2 - \frac{1}{2} m\, v_1^2 + \cdots + \frac{1}{2} m\, v_f^2 - \frac{1}{2} m\, v_{n-1}^2$$
$$= \frac{1}{2} m\, v_f^2 - \frac{1}{2} m\, v_i^2 = \Delta K \,,$$

as before.

DERIVATION 6-3

And finally, what if the object moved in three dimensions? The force could be written in components, $F_x$, $F_y$, and $F_z$. $F_x$ would make no work contribution due to movement in the y or z directions, $F_y$ would make no contribution due to movement in the x or z directions, and $F_z$ would make no contribution due to movements in the x or y directions, Therefore, making use of Derivation 6-2, the work done by this force would be

$$W = \sum_{n} F_x\,\Delta x + F_y\,\Delta y + F_z\,\Delta z$$

$$= \tfrac{1}{2}\,m\,v_{xf}^2 - \tfrac{1}{2}\,m\,v_{xi}^2 + \tfrac{1}{2}\,m\,v_{yf}^2 - \tfrac{1}{2}\,m\,v_{yi}^2 + \tfrac{1}{2}\,m\,v_{zf}^2 - \tfrac{1}{2}\,m\,v_{zi}^2$$

$$= \tfrac{1}{2}m(\,v_{xf}^2 + v_{yf}^2 + v_{zf}^2 - (\,v_{xi}^2 + v_{yi}^2 + v_{zi}^2)) = \tfrac{1}{2}m\left(v_f^2 - v_i^2\right) = \Delta\left(\tfrac{1}{2}\,m\,v^2\right)$$

$$= \Delta K\ ,$$

as before.

A combination of these last two arguments lets us assert that the result is valid even for variable forces in three dimensions, although in practice that may be quite difficult to calculate.

So, most generally speaking, we have that

$$W_{TOTAL} = \Delta K.$$

This last relationship is called the *work-energy theorem*. Note that it is nothing more than Newton's Second Law, combined with one of the kinematic equations, plus a definition. It is the second 'picture' of the three we shall use to solve problems, the first being forces and accelerations.

| | | |
|---|---|---|
| Net force | causes | change in velocity |
| Net work | causes | change in kinetic energy |
| ? | causes | change in ? |

You may well ask, why bother? Can’t we just solve everything with Newton’s Second? We will find that this picture will be on occasion more convenient to use than forces and accelerations, especially in cases where we don't need to know the time a trip takes, or when the acceleration is not constant.

EXAMPLE 6-2

Throw a ball upward with an initial speed of 12 m/s. How high does it rise (H)?

$$W_{TOTAL} = \tfrac{1}{2}\,m\,v_f^2 - \tfrac{1}{2}\,m\,v_i^2\ .$$

When we used Newton’s second law, we put in all of our effort on the left side finding the forces, but the right side was always $m\vec{a}$. Here, we put all the effort in again on the left side finding the works, and the right side is always ΔK.

The only force acting on the ball is its weight, gm, downward. The displacement is H upward, so our angle between the force and the displacement is 180°. The work done is therefor

$$W_g = (gm)(H)\cos(180^o) = -gmH$$

The ball stops at its highest altitude, so $v_f = 0$. So,

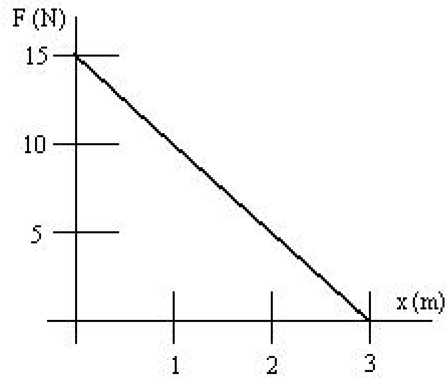
$$-gmH = \tfrac{1}{2}m\,v_f^2 - \tfrac{1}{2}m\,v_i^2 \ .$$

$$H = \frac{v_i^2 - v_f^2}{2g} = \frac{12^2 - 0^2}{2(10)} = 7.2\text{ m} \ .$$

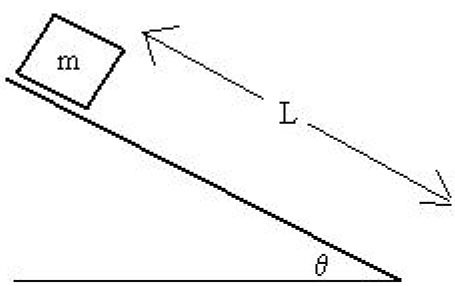
HOMEWORK 6-3

A 3 kg object initially at rest is acted on by a non-constant force which causes it to move 3 m. The force varies with position as shown in the figure.

a) How much work is done on the object by this force?
b) What is the final speed of the object as it arrives at x = 3 m? Assume that the given force is the only force acting on the mass.

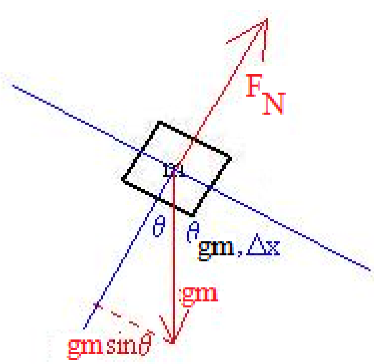



EXAMPLE 6-3




Here's a problem we've seen before to compare the solution methods of Picture One and Picture Two. Consider a block of mass m = 5 kg at the top of a frictionless ramp L = 2 meters long that is inclined at θ = 37° to the horizontal. If the mass starts from rest at the top, how quickly will it be moving when it reaches the bottom? The answer better be 4.9 m/s.

Draw a free-body diagram; the weight and a normal force are the only forces. One thing we <u>don't</u> need to do is choose a coördinate system. Everything is relative to the direction of the displacement. We'll use the WE theorem,




$$W_{TOTAL} = \tfrac{1}{2}m\,v_f^2 - \tfrac{1}{2}m\,v_i^2 \ .$$

Let's look at the works:

$W_N$ = 0, since the force is perpendicular to the displacement;

$W_g = (mg)\,(L)\,(\cos\theta_{mg,\Delta x})$.

What angle should we use for $W_g$? It's not 37°! We want the angle between the force and the displacement, 53°.

$$W_N + W_g = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2$$

$$0 + W_g = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2$$

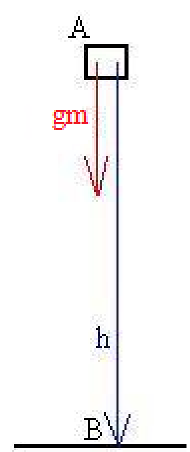
$$v_f = \pm\sqrt{v_i^2 + \frac{2W_g}{m}} = \sqrt{v_i^2 + 2gL\cos(\theta)} = \sqrt{0^2 + 2(10)2\cos(53^o)} = 4.9\ \mathrm{m/s}\,.$$

DISCUSSION 6-5

Suppose that I drop an object from a given height, such as a pen onto the table. The force of gravity (the object's weight) does work and the kinetic energy of the object increases. Now, suppose instead that I slowly lower the object to the table from the same initial altitude. Compare the work done by gravity in the second case to the work done in the first case. Do you understand the difference between the work done by a force and the total work done by all forces on an object?

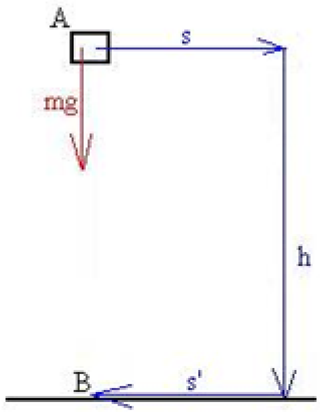
## Conservative and non-Conservative Forces

Let's divide the realm of forces in to two categories: *conservative forces* and *non-conservative forces*. This may seem rather facile, in that I could divide forces in to red and non-red categories, and each force would have to fit into one of them. However, this is a distinction which we will find useful. What we find is that for some forces, the work they do on some object moving from any particular point A to any particular point B is independent of the path taken between A and B. We call this type of force a conservative force. There are a number of alternate ways to define what a conservative force is, but they are all equivalent to each other. Any force for which the work can depend on the path is a non-conservative force.




Let's take the weight of an object as a concrete example. Suppose that I lower a mass m from a height h above the table to the top of the table. I'm only interested at this point in what the weight does, not what any other force, such as from my hand, does. The force is gm downward, and the displacement is h downward, and those two vectors are parallel, so we have that

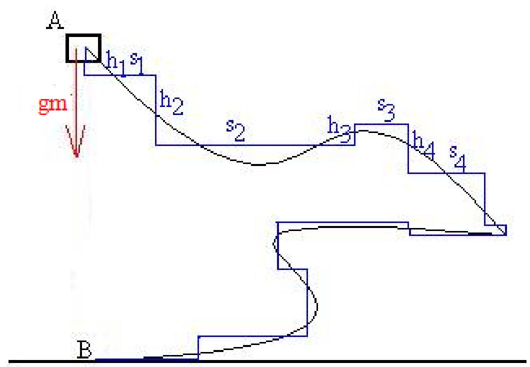
$W_g = gm\,h\cos(0^o) = gmh.$




Now, let's take the object on a little tour of the region. Move it horizontally a displacement s, then down h, then horizontally again s, back to point B. The work done will be

$$W_g = gm\,s\cos(90^o) + gm\,h\cos(0^o) + gm\,s\cos(90^o) = gmh$$

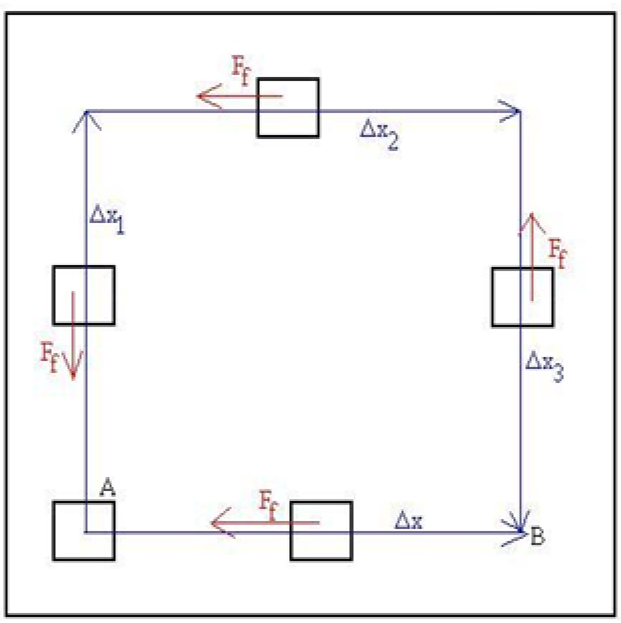
once again.




Let's pick a random path. You might be able to see that we can always approx-imate any path to an arbitrary degree of accuracy with these stepped horizontal and vertical move-ments. From previous discussion, we know that any horizontal movements will correspond to no work being done by gravity. The vertical displacements are each of magnitude $h_n$, some parallel to the weight and some anti-parallel, such that the work done by the weight during each vertical motion is

$$W_g = \sum_n gm\,h_n\cos(\theta_n) = gm\sum_n h_n\cos(\theta_n)\ ,$$

where cos $(\theta_n)$ = +1 if the displacement is downward (parallel to the force) and -1 if the displacement is upward (anti-parallel to the force). We realize that the last summation is simply h, so that the work done by the weight is gmh, as before, and work done by the weight throughout the whole trip is indeed independent of the path taken.




Next, let's consider an example of a non-conservative force: friction. Consider an object being slid across a table top along two paths (let all Δx's be the same magnitude). Remember that we are not concerned with the work done by any other force, such as that of the hand that pushes the block. The frictional force will be (not proven here):

$$F_{fK} = \mu_K\,(gm)$$

So that the work done by friction from Point A to Point B along the direct path is

$$W_{f\,\text{Direct Path}} = \mu_K\,(gm)\,\Delta x\ .$$

If instead, the object is pushed along the other three sides of a square, the same amount of work will be done by friction along <u>each</u> of the sides, so that

$$W_{f\,\text{Long Path}} = 3\,\mu_K\,(gm)\,\Delta x \neq W_{f\,\text{Direct Path}}\ .$$

So, we see that friction is <u>not</u> a conservative force. Just as clearly, neither is the force that pushed the object around on the table.

---

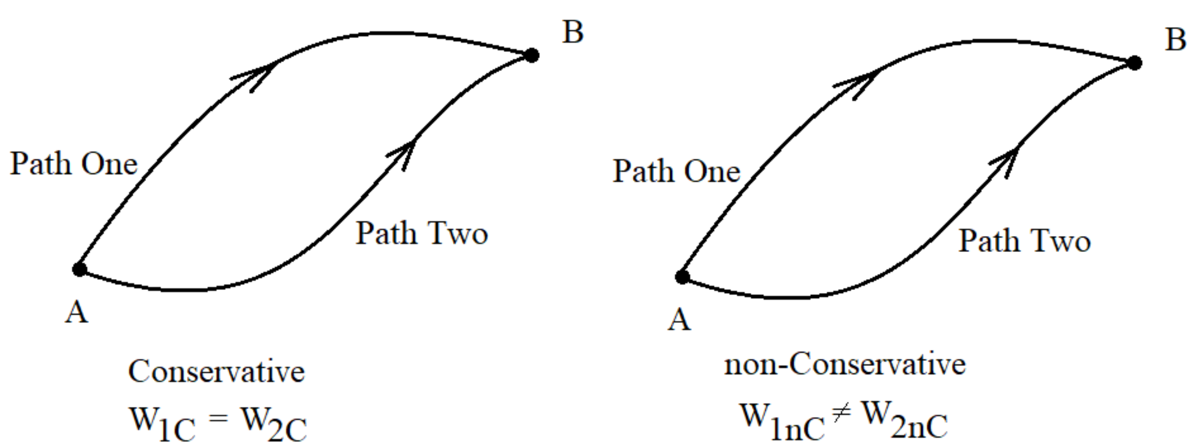
## Potential Energy

A few pages ago, we defined energy as ½ $mv^2$, which is how it was originally defined. It was a few years later that the 'kinetic' was added. We make this distinction because we will introduce a second type of energy, although to my mind, it is only a bookkeeping trick to keep track of some work terms. I admit, though, that the concept of *potential energy* (U) can be extremely useful.

Let's consider the dropped pen again. We can say that during its fall, the pen is acted on only by the force of gravity, which does positive work, and thereby causes an increase in the pen's kinetic energy (work-energy theorem). We can develop an alternate notion, by saying that energy is somehow stored in the pen by virtue of its altitude above the table, and that this potential energy is then converted to kinetic energy as the pen falls. What we find is that any conservative force can have a potential energy function associated with it. For example, if a conservative force does positive work on an object so that the kinetic energy increases, we could alternatively say that the potential energy of the object is decreasing while the kinetic energy is increasing, and *vice versa*. So, for a given conservative force ($F_C$), we require that

$$W_C = -\Delta U\ .$$

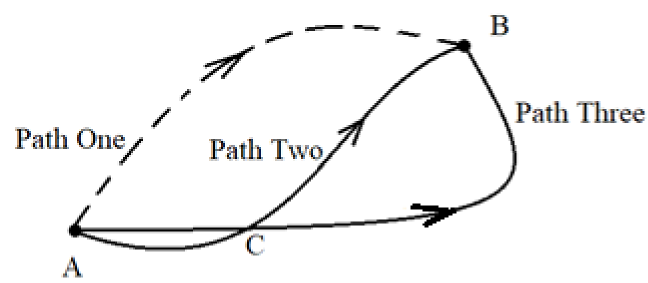
We can do this only for conservative forces. Here's why.




By definition for a conservative force, the work done by the force along any Path One from A to B is the same as along any other Path Two. Since the two paths have only points A and B in common, there must be some numbers associated with the object being at each of these points that provide sufficient information to determine the work. We call these values the potential energy of the object at A ($U_A$) and the potential energy of the object at B ($U_B$).[4] If we tried to do that with the non-conservative force, starting at point A, we would have to conclude that there are two different values associated with point B, or indeed, potentially an infinite number of such values, one for each possible path and amount of work done.

DISCUSSION 6-6

> One might point out that this argument regarding conservative forces is valid only when Path One and Path Two do not cross (they would have more than just two points in common). Of course, we can come up with any number of paths that do cross. Can you provide an argument that takes care of that omission?




Here we go. Let's start with the work-energy theorem, and divide the works on the left into two categories, depending on whether the associated forces are conservative (C) or non-conservative (NC):

$$W_{TOTAL} = \Delta K$$

$$W_C + W_{NC} = \Delta K \ .$$

We'll define the change in potential energy as $\Delta U = -W_{CONS}$, so that

$$-\Delta U + W_{NC} = \Delta K \ .$$

Since work and energy are <u>not</u> the same thing, and because I hate minus signs,

$$W_{NC} = \Delta K + \Delta U \ .$$

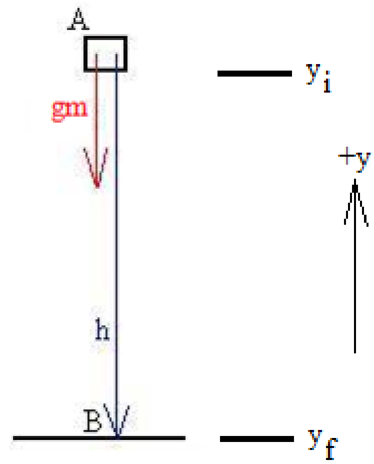
In the same way that I find $\Sigma F = ma$ to be more convenient than the conceptually better $a = \Sigma F/m$, I find this form of the work-energy theorem to be more convenient than the conceptually satisfying version of a few pages back.

Can we figure out what the gravitational potential energy function is? Not really. We can only figure out an expression for its <u>change</u>.

[4] If you read this paragraph carefully, you should have noted that these values are not directly associated with the points A and B themselves, but with our object being located at A and at B. There <u>is</u> a quantity associated with the points themselves, regardless of whether there is an object there or not, but it is typically not covered in PHYS 1. Look for an analogous quantity though in PHYS 2!

DERIVATION 6-4

Consider the specific example of the pen discussed earlier that was lowered from a height H to the table below. We calculated that the work done on the pen by the weight was gmh. Now, to make this work consistently, it's necessary to give up a little freedom of choice; we will require upward to be positive y.[5] Since h is a positive number (the magnitude of the displacement) and since $y_i > y_f$, we can instead write that




$$W_g = gmh = gm(y_i - y_f) = -(gmy_f - gmy_i)$$
$$= -\Delta(gmy) \ .$$

If we keep in mind that we defined U such that

$$W_g = -\Delta U \ ,$$

we might just jump to the conclusion that

$$U(y) = gmy \ .$$

Now, of course, we still have the same problem we had with kinetic energy, that there may be some constant term we're missing that will cancel out when we find ΔU: $U(y) = gmy + U_o$. This time, though, we're going to take advantage of that. Where we pick our origin *(i.e.,* where $y = 0$) is entirely up to us, and so that is where we choose the potential energy to be zero. So, we'll make these choices to be as convenient for us as possible. Generally (80% Rule!) you will want to place $y = 0$ at the lowest level of a problem.[6]

But, let's consider. Suppose I raise a 2 kg object from a tabletop 1 m above the floor to 2 m above the floor. If the zero of potential energy is zero at floor level, I increased U from (10)(2)(1) = 20 joules to (10)(2)(2) = 40 joules. If the zero had been at table level, it went from 0 joules to (10(2)(1) = 20 joules. And if the 3 meter high ceiling had been U = 0, it went from (10)(2)(-2) = -40 joules to (10)(2)(-1) = -20 joules. In each case the change was the same (+20 joules) even if the actual potential energy values were very different.

Some admonitions before we start examples. First, remember that you should not put the potential energy term on both sides of the relationship; it's either a work term on the left, or it's a potential energy change on the right. As I said, this is a bookkeeping trick. Second, remember that there may well be more than one conservative force operating on the object, which would require us to

[5] You may remember doing this when we required radial forces to be counted as positive when toward the center of the circle and negative when away.

[6] Of course, later, we'll see some exceptions*, i.e.* the other 20%!

have more than one $\Delta U$ term. For this course, there are only three conservative forces; all others should be considered to be non-conservative.

---

## Conservation of Mechanical Energy

Let's call the sum of an object's kinetic and potential energies its *mechanical energy*. Let's consider a special case in the absence of non-conservative forces, or at least a situation where no non-conservative forces do work:

$$W_{NC} = \Delta K + \Delta U$$

becomes

$$0 = \Delta K + \Delta U$$

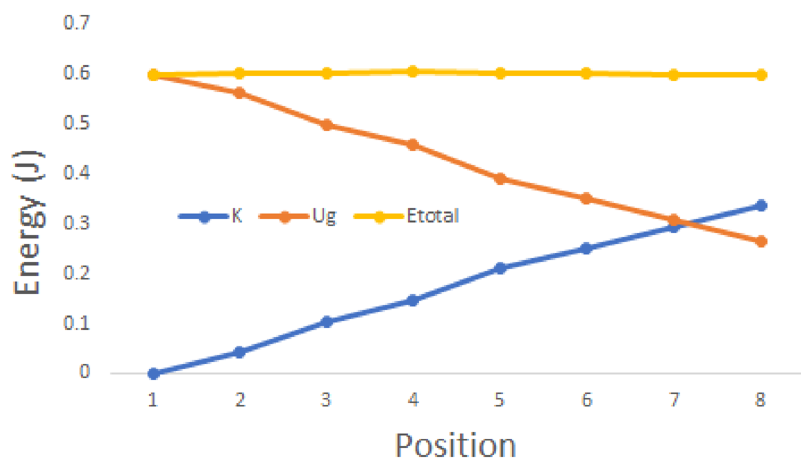
$$0 = K_f - K_i + U_f - U_i$$

$$K_i + U_i = K_f + U_f \ .$$

This is interesting. It says that, in the absence of non-conservative forces (or at least of such forces which do any work), the total mechanical energy is *conserved*, or remains constant. There is, in physics, a great number of quantities that are conserved in the absence of outside agencies. In the present example, the energy may change from from kinetic to potential or *vice versa*, but it is neither created nor destroyed.

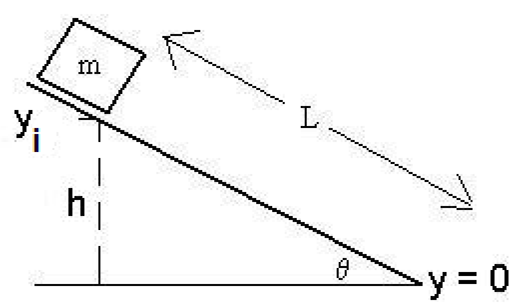
This concept of the conservation of mechanical energy is not the same as *conservation of total energy*, which you may have heard of in your other classes. This is a much more restricted form of that concept. For example, let's look once again at the dropped pen. Just after release, the pen has zero kinetic energy and mgh of potential energy (we'll let U = 0 at the tabletop). Just <u>before</u> hitting the table, U = 0 and K is not zero, and in fact equals numerically mgh. Now in a more general way, we can talk about the conservation of total energy, but only if we broaden the definition of energy. You may remember from your other classes that the molecules in solids can be modeled by balls connected by springs, and that the balls are constantly vibrating, possessing kinetic and potential energy. This kinetic energy is different (in a fashion) from the *translational* kinetic energy discussed above, in that for translational kinetic energy, every particle shared the same velocity vector, but for *vibrational* kinetic energy, the motions are more random. When the pen hit the table, shock waves went out from the impact through both the table and the pen, increasing the vibration of the molecules in each object. This increased *thermal energy* is observed macroscopically as an increase in the *temperatures*[7] of both the table and the pen. Other energy is carried away as *sound,* which eventually warms other objects it hits, such as your eardrum.

EXPERIMENT 6-2




Everything we've done in this section up to this point was based on the conjecture that the work is found by multiplying the displacement of an object by the parallel component of the force acting on it. While the rest of the section has a fairly firm basis, if the original conjecture is incorrect, all that followed may be just as incorrect. So we need some evidence to support the conjecture, and that is usually accomplished by performing an experiment. Here are the results of an experiment measuring the potential and kinetic energies of an object as it slides down a frictionless incline. Note that, as the potential energy U decreases, the kinetic energy K correspondingly increases, but that the total energy (U + K) remains constant as predicted (to within experimental error). In this experiment, the maximum deviation from the average is 0.8%.

EXAMPLE 6-3




You.ve seen this one before. Consider a block of mass m = 5 kg at the top of a frictionless ramp L = 2 meters long, which is inclined at $\theta = 37°$ to the horizontal. If the mass starts from rest at the top, how quickly will it be moving when it reaches the bottom?

As usual, draw a sketch and a free-body diagram. There are two forces acting on the mass: the weight and the normal force. Let's start with the more recent and more useful version of the work-energy theorem:

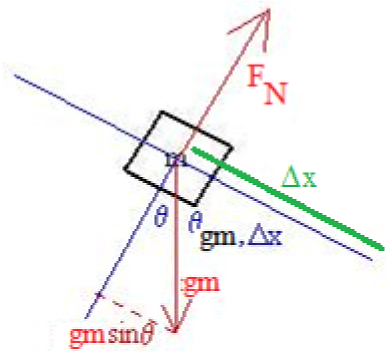
---

[7] While you may have a general idea of what temperature is, we'll define it carefully later in the course.

$$W_{NC} = \Delta K + \Delta U$$

$$W_{NC} = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2 + gmy_f - gmy_i$$

Next, let's consider the works done:




$W_N = 0$ – the force is perpendicular to the path and so the cosine term is always zero.
$W_g$ – this is a conservative force and will be dealt with on the right side of the equation.

There is one piece of information we will need: the initial altitude of the object. Since the $\sin\theta = y_i/L$, $y_i$ is $L \sin(37^o) = 1.2$ m.

So then

$$0 = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2 + gmy_f - gmy_i \ ,$$

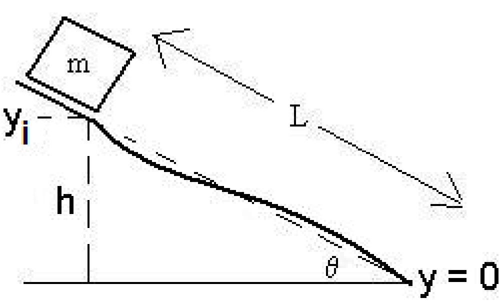
which is always nice.

Following a brilliant suggestion I read somewhere, I'll put y = 0 at the bottom of the ramp. I also realize that the object starts from rest at the top, and so I'll simplify here with justification.[8]

$$0 = \tfrac{1}{2} m v_f^2 - \underbrace{\tfrac{1}{2} m v_i^2}_{\text{starts from rest}} + \underbrace{gmy_f}_{y_f = 0} - gmy_i$$

$$\tfrac{1}{2} m v_f^2 = gmy_i$$

$$v_f = \sqrt{2\, gy_i} = \sqrt{2(10)(1.2)} = 4.9 \mathrm{m/s}\,.$$



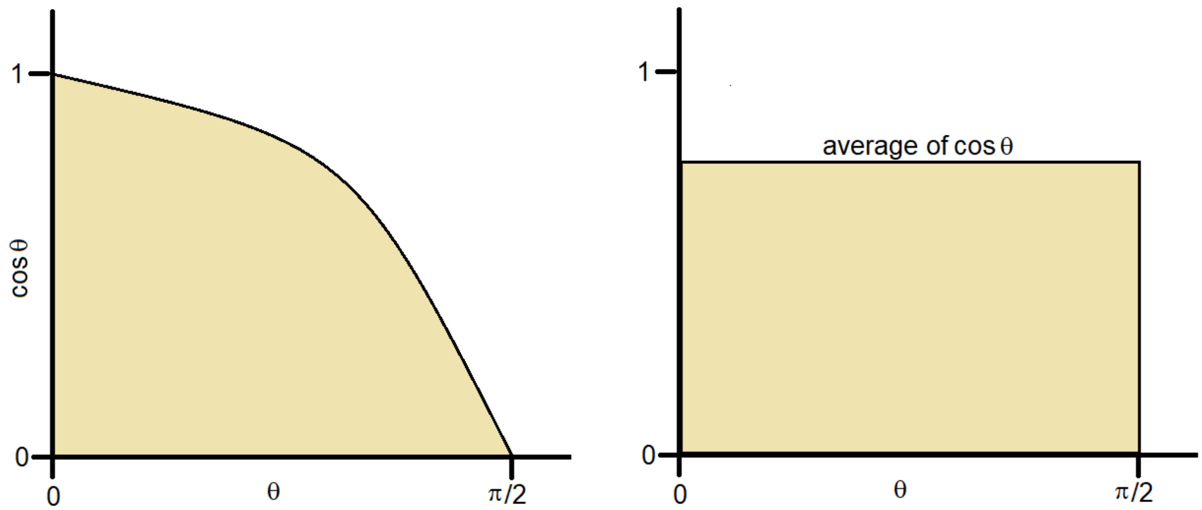

At this time, it is legitimate to ask, "Gee, Dr Baum, we've learned how to do this problem three ways, none of which seems any easier than the others. What's the point?" And here it comes. Suppose that instead of a straight frictionless surface, the incline had instead possessed a 'wavy' surface, as shown in the figure. Here, the mass does not slide uniformly down a straight surface. Let's think about doing this with Newton's second law. Let x be the direction down the incline. The weight will have a constant component along the dotted line shown in the figure, but the normal force will have a varying component in that direction, sometimes down the incline, sometimes up the incline, and sometimes zero, depending on the exact shape of the surface. And that's an oversimplification. If we try to use the original form of the work-energy theorem (following the wavy line), it's

[8] In this section, I'll indicate quantities that are zero in red, and justifying why in the line directly underneath.

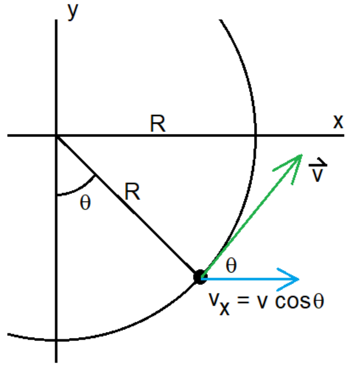
certainly true that the normal force does no work, but the angle that the weight makes with each small interval of displacement as the object slides down will vary. Either way, we would have to know a lot of very specific data about the shape of the curves ramp and do a horrendous calculation. However, because the weight is a conservative force, the work done on the object does not depend on the exact path taken; all we need to know are the potential energies at the start and end of the trip. Here is a perhaps clearer example to illustrate the usefulness of potential energy.

MATHEMATICAL DIGRESSION*

Before we cover the next example, we need to find the average value of the cosine function in the first quadrant, between 0 and $\pi/2$ radians (90º). To do that, we're going to find the area




under the cosine curve. The height of the rectangle containing the same area over the same domain will be the average. As an analogy, think of the left curve as the distribution of grades on an exam (well, let's hope not!). The area under the curve represents the total number of points earned by all students. If we redistribute the points so that every student gets the same grade, then that grade will be the average.




To find the average, consider an object moving at constant speed v around a circle of radius R. To match the parameters of our problem, we'll look at the motion when the object is in quadrant four with theta measured as shown. For convenience, we'll make both v and R, and therefor ω, equal one.[9] This makes the x component of the velocity be

$$v_x = v\cos\theta = \cos(\theta) \ ,$$

and since θ = ωt, we can write that, numerically,

$$\theta = t \ .$$


That means that the graph of the x component of the velocity in quadrant four matches exactly the graph on the left above, and their areas will be the same. What is the area under a velocity vs time graph? It's the displacement, and in this scenario, the object moved from x = 0 to x = R = 1. The area under the original cosine curve is then 1.[10] To find the average value in this interval, consider the area of the rectangle:

$$area = base \times height$$
$$1 = \frac{\pi}{2} h \quad \rightarrow \quad h = \frac{2}{\pi} \ .$$

The average of the cosine function between 0º and 90º is 2/π.

EXAMPLE 6-4

Consider a small ball (mass m) attached to the end of a (magic) string of length L = 1.5 m). The ball is held up at 90º to the vertical and released. How quickly is it moving when it reaches the bottom of its swing?

---

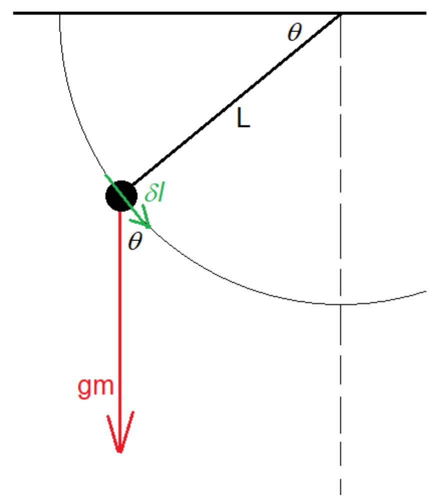
[9] If it bothers you that the validity of the coming result may be limited to this special case, remember that we can simply use different units so that R is 1. That is, if R = 3.2 meters, we can simply define one *noof* to be 3.2 meters and the radius becomes one noof while the initial velocity becomes 1 noof/second. The real requirement here is that ω = 1 rad/sec. Now, if omega were not 1 rad/s, there would simply be a scaling factor that would in the end disappear. If, for example, omega were say 6.7, then the initial velocity would be 6.7 times greater, but the time to erach the x-axis would be 6.7 times shorter; the shape under the curve becomes taller and narrower by the same factor so that the area remains the same.

[10] Modified from "Why does the area under one hump of a sine curve exactly equal 2?" Girl's Angle Bulletin, July 31 2013 https://girlsangle.wordpress.com/2013/07/31/why-does-the-area-under-one-hump-of-a-sine-curve-exactly-equal-2/.

As a demonstration of the usefulness of the concept of potential energy, we're going to do this problem twice. First the long way; keep in mind that we could also use calculus, but that wouldn't make this solution much shorter.

First Solution*

We'll use the original form of the work-energy theorem. There are two forces acting on the mass, the ball's weight and the tension in the string.




$W_T$ = 0 (the tension is always perpendicular to the path).
Wg is tough. In general, the work is F Δx cos $\theta_{F,\Delta x}$. That works if the force is constant and the displacement is along a straight line. But here, the angle between the weight and the direction of motion is continually changing. We must break the path down into very many very small lengths $\delta l$, find the work for each displacement, then add them all up.

$$W_g = \sum_n \mathrm{gm}\,\delta l_n \cos(\theta_n) = \mathrm{gm}\sum_n \delta l_n \cos(\theta_n)$$

We can express the average value of the cosine as

$$\cos(\theta)_{AVE} = \frac{\sum_n \delta l_n \cos(\theta_n)}{\sum_n \delta l_n} \; .$$

We've previously shown that the average value of the cosine between 0° and 90° is 2/π, and the sum of the $\delta l_n$ terms is one-fourth of the circle's circumference, or, πL/2. Then,

$$W_g = \mathrm{gm}\left(\sum_n \delta l_n\right)(\cos(\theta)_{AVE}) = \mathrm{gm}\left(\frac{\pi L}{2}\right)\left(\frac{2}{\pi}\right) = \mathrm{gmL} \; .$$

Then,

$$W_T + W_g = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$
$$W_T = 0 \qquad \text{starts from rest}$$

$$\mathrm{gmL} = \tfrac{1}{2}mv_f^2$$

$$v_f = \sqrt{2gL} = \sqrt{2(10)1.5} = 5.48\text{ m/s} \; .$$

Second Solution

Next, let's use potential energy.

$W_T = 0$ (the tension is always perpendicular to the path
$W_g$ – conservative force, treat as potential energy terms

Set U = 0 at the bottom of the problem

$$0 = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 + gmy_f - gmy_i$$
starts from rest    $y_f$ set to zero

$$\frac{1}{2} m v_f^2 = gmL$$

$$v_f = \sqrt{2gL} = \sqrt{2(10)1.5} = 5.48 \text{ m/s} \; .$$

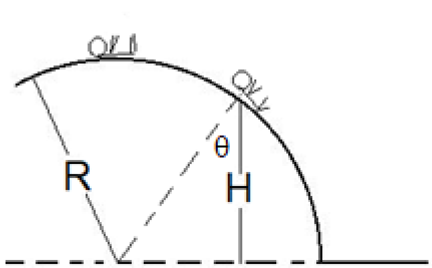
Much shorter.

Generally, you will find that using potential energy is never harder than finding the work directly, and usually much easier.

HOMEWORK 6-4

A pitcher hurls a 0.35 kg sportsball around a vertical circular path of radius 0.6 m, applying a tangential force of 30 N, before releasing it at the bottom of the circle (underhand pitch). If the speed of the ball at the top of the circle was 12 m/s, what will be the speed just after it's released?

EXERCISE 6-1



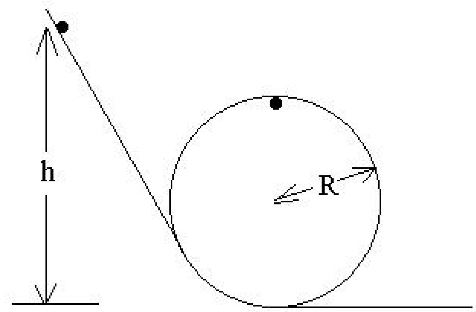

Another classic. Consider a child perched at the top of an igloo, which we will consider to be a hemisphere of radius R covered in slippery snow. He starts with an almost zero speed from the top and travels down the side. At what vertical distance H from the ground will he become airborne?

HOMEWORK 6-5

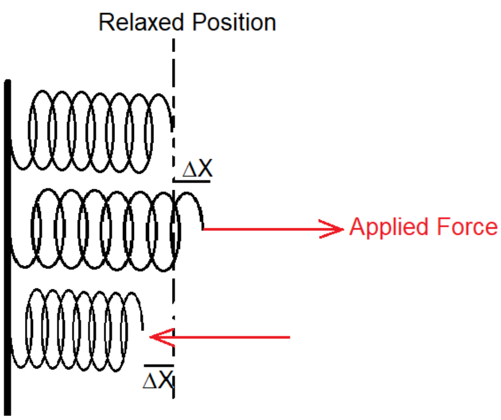
Tarzan swings on a 25 m long vine that was initially inclined at an angle of 25° from the (downward) vertical. What is his speed at the bottom of his swing if he pushed off his branch with an initial speed of 3 m/s?

HOMEWORK 6-6

A point mass block slides without friction on the loop-de-loop track of radius R as shown. From what height h must it be released from rest in order to make it around the loop without leaving the track?




## Springs




Let's next consider our second conservative force. If I take a spring and simply toss it onto the table, you may notice that it always assumes the same length, regardless of whether I compress it or stretch it before I toss it. Let's refer to this as the spring's relaxed condition and the length its *relaxed length*. In order to stretch or compress the spring, I must apply some force. In this course, at least for now, we shall assume that all springs obey *Hooke's relationship:* the force necessary to stretch (or compress) a spring from its relaxed state is proportional to the amount of stretching (or compression). In more mathematical terms:

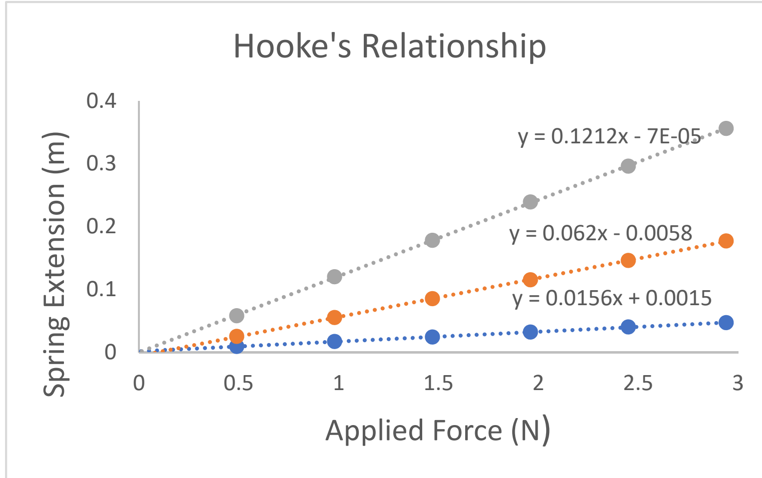
$$F_{Applied} = k\,\Delta X \ .$$

The symbol k represents the *spring constant* of the spring, the number of newtons required to stretch (or compress) the spring one meter, and is given in N/m. A high value of k means that the spring is stiff, while a low value implies the spring is flexible. Notice that I am using a capital X to describe the position of the end of the spring; the reason for this should become apparent later in the discussion.

DISCUSSION 6-7

HookesLaw.mp4

In the figure, we've applied known forces to three springs over an admittedly small range of stiffness. Is the amount each spring is stretched indeed proportional to the applied force? What property of the data in the graph would indicate that? How is the spring constant k found for each curve? Which is the independent variable and which the dependent variable in this experiment? How should Hooke's relationship be arranged to match the equation of a line?




We need to be a bit careful about signs. The relationship above is the force which needs to be applied to the spring to stretch (compress) it, and that force needs to be in the direction of the displacement of the end of the spring. We do not expect this force to be conservative, as it may be provided by a hand or other such agency. However, we are often interested in the force applied by the spring to some object to which it is attached. By the third law, this spring force would be in the opposite direction:

$$F_{Spring} = -\,k\,\Delta X\ .$$

However, this is further complicated by our habit of writing down the magnitudes of forces and adding in the appropriate directional signs as necessary. As a result, I shall write this relationship this way,

$$F_{Spring} = (-)\,k\,\Delta X$$

with the minus sign there in parentheses to remind you that the force exerted by the spring is in the direction opposite to that in which the spring is stretched, but not to be taken literally. You must determine the correct sign for each specific problem encountered. Occasionally, the 'delta' is dropped as well, if it is understood that the relaxed position is at X = 0:

$$F_{Spring} = (-)k\,X\,.$$

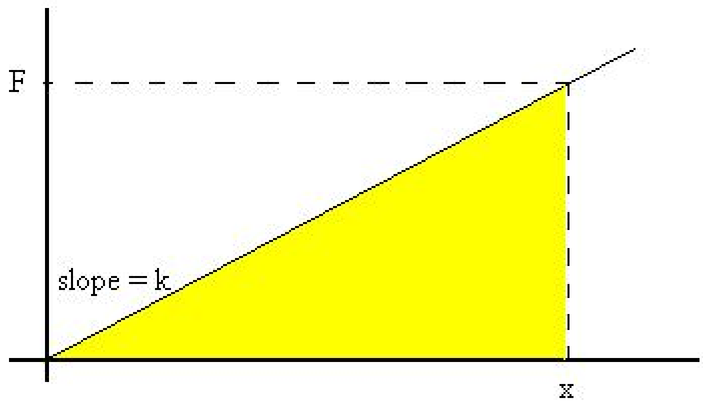
One last assumption: unless told otherwise, springs, like strings, will be considered to be massless.

DISCUSSION 6-8

Is the spring force conservative? Can you make a quick argument that it is? Suppose that we were to stretch the spring from X = 0 to point A. Repeat from X = 0 to A to a point B beyond point A, then back to B?

Since the force exerted by the spring depends only on the position of the end of the spring X (we'll assume that the other end is fixed), reversing the displacement back over already covered ground simply undoes the work done the first time (by flipping the sign of the cosine term), so that the net work done depends only on the initial and final positions of the end of the spring.

DERIVATION 6-5




How much work is necessary to stretch (or compress) a spring distance X from its relaxed position? We can use the graphical representation showing $F_{on\ spring}$ as a linear function of X with slope k:

We showed above that the work done by any variable force is represented by the area under the force *vs* position curve. Since this is a triangle, the area is one-half the base times the height:

$$W_{\text{on Spring}} = \frac{1}{2}\,X\,F = \frac{1}{2}\,X\,(kX) = \frac{1}{2}\,kX^2$$

Now we have to do a couple of flip-flops. The work done on the spring is $^1/_2kX^2$, the work done by the spring is $-^1/_2kX^2$ (the forces are in opposite directions), and the change in the potential energy of the spring is the negative of that, or

$$\Delta U_{\text{Spring}} = -W_{\text{by Spring}} = -\left(-W_{\text{on Spring}}\right) = +\frac{1}{2}\,kX^2\ ,$$

$$U(X) - U(0) = +\frac{1}{2}\,kX^2\ .$$

It would seem extremely convenient to make U(0) = 0, so that

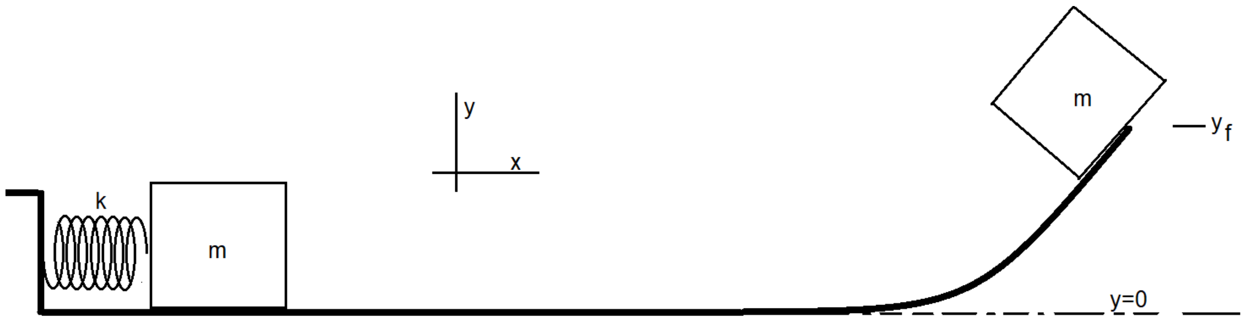
$$U_{\text{Spring}}(X) = +\frac{1}{2}\,kX^2\ .$$

Note that in this version, we have given up some freedom again. We will be assuming the spring's potential energy is zero when the spring is relaxed. Do we have to do this? No, but things will be much easier if we do. We will also see that maintaining this zero of potential energy supersedes our choice of where to make the gravitational energy zero, again for mathematical exigency.

Also, note that the potential energy of a spring depends on the square of the extension or compression. That is, for U(X), it really doesn't matter if we make compression or extension positive or negative; the potential energy increases either way.

EXAMPLE 6-5

Consider the frictionless surface shown. On the left is an ideal spring of constant k = 30 N/m. A mass of 5 kg is pushup against the spring, compressing it 0.2 m. When the mass is released, it is pushed to the right, slides across the surface, and travels up the incline. What is the mass's altitude $y_f$ when it stops?




There are three forces acting on the mass at one time or another. Use the work-energy theorem.

$W_N = 0$ (the normal force is always perpendicular to the path).
$W_{Sp}$ – conservative
$W_g$ – conservative

Let's define y = 0 to be at the bottom of the problem. Then,

$$W_{NC} = \Delta K + \Delta U_g + \Delta U_{Sp} \quad .$$

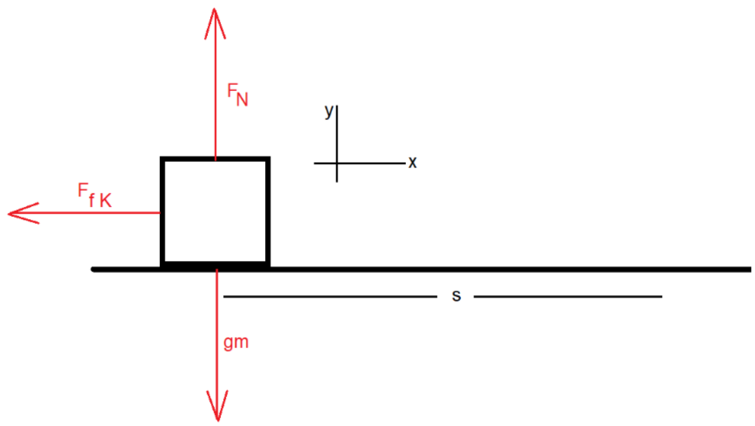
$$0 = \underbrace{\tfrac{1}{2}\mathrm{m}\,\mathrm{v}_\mathrm{f}^2}_{\text{stops}} - \underbrace{\tfrac{1}{2}\mathrm{m}\,\mathrm{v}_\mathrm{i}^2}_{\text{starts from rest}} + \mathrm{gmy}_\mathrm{f} - \underbrace{\mathrm{gmy}_\mathrm{i}}_{\mathrm{y}_\mathrm{i}=0} + \underbrace{\tfrac{1}{2}kX_f^2}_{\text{spring is relaxed}} - \tfrac{1}{2}kX_i^2$$

$$\mathrm{gmy}_\mathrm{f} = \tfrac{1}{2}\mathrm{kX}_\mathrm{i}^2 \quad \rightarrow \quad \mathrm{y}_\mathrm{f} = \frac{\mathrm{kX}_\mathrm{i}^2}{2\mathrm{gm}} = \frac{30(0.2^2)}{2(10)5} = 0.012\ \mathrm{m}$$

This illustrates why I used X; it represents the location of the end of the spring, not the location of the mass.

EXAMPLE 6-6

Show that an object of mass m moving at speed $v_o$ across a rough horizontal floor will slide a distance $s = (v_o)^2/2\mu_K g$.




There are three forces acting on the object. Consider the work each does.

$W_N = 0$ (the normal force is always perpendicular to the path).

Wg – conservative

$Wf = F_f\, s \cos\theta F{,}s$. We need to find the frictional force, which unfortunately requires a trip back to Newton's second law land. Although the result here may seem obvious to you, it is important to actually show the effort. From NII in the y-direction,

$$+F_N - gm = ma_y = 0 \quad \rightarrow \quad F_N = gm$$

$$F_{fK} = \mu_K F_N = \mu_K gm \ .$$

Since the displacement and frictional forces are in opposite directions, we'll be taking the cosine of 180°, so

$$W_f = (\mu_K gm)\, s\, (-1) = -\mu_K gms \ .$$

The work-energy theorem then results in

$$-\mu_K gms = \underbrace{\tfrac{1}{2} m\, v_f^2}_{\text{comes to a stop}} - \tfrac{1}{2} m\, v_i^2 + \underbrace{gmy_f - gmy_i}_{y_f = y_i}$$

Finish up with

$$\mu_K gs = \tfrac{1}{2} v_o^2$$

$$s = \frac{v_o^2}{2\mu_K g} \ .$$

HOMEWORK 6-7

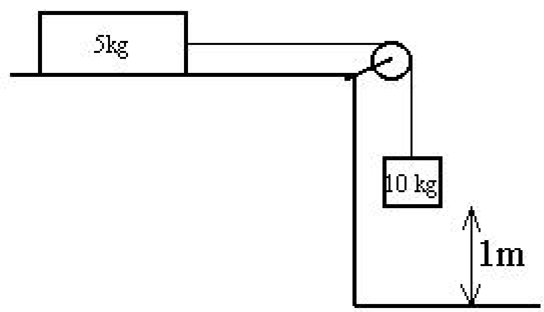
Two identical massless springs of constant k = 400 N/m are fixed at opposite ends of a level track, as shown. A 12 kg block is pressed against the left spring, compressing it by 0.3m. The block is then released from rest. The entire track is frictionless except for the region of length 0.2 m between points A and B, where $\mu_K = 0.08$. What is the maximum compression of the spring on the right?


HOMEWORK 6-8

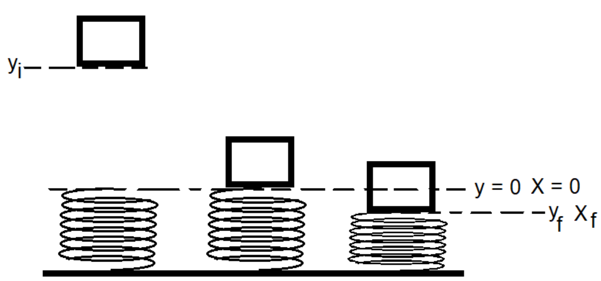
Consider a 13 kg block sitting at rest on a rough surface. The coëfficient of friction between the block and surface is 0.7. The block is barely touching a relaxed spring of constant k = 300 N/m. How much work would a hand or other such agency need to do to push the block very slowly 0.2 m against the spring?


HOMEWORK 6-9

The 10 kg mass is released from rest at a height of 1m above the floor. If the coëfficient of kinetic friction between the 5 kg mass and the table is 0.8, what will be the speed of the 10 kg mass just before it hits the floor?




EXAMPLE 6-7




Let's look at a problem where it is not a good idea to make the lowest point the zero of gravitational potential energy. Let's drop a box of mass M onto a vertical spring. How far is the spring compressed when the box comes to a stop? The figure shows three points in the process. Because the spring's potential energy is quadradic while the gravitational potential energy is linear, things will go much easier mathematically if we set the springs relaxed position to be the zero for both. Note then that the final values for y and for X will be the same. Let's pick some numbers: $y_i$ = 12 m; M = 8 kg; k = 120 N/m.

Use the work-energy theorem:

There are two forces acting on the box, the weight and the spring force.

$W_g$ – conservative
$W_{Spring}$ – conservative

So, we have that happy situation when $W_{NC} = 0$.

$$0 = \frac{1}{2}m\,v_f^2 - \frac{1}{2}m\,v_i^2 + gmy_f - gmy_i + \frac{1}{2}kX_f^2 - \frac{1}{2}kX_i^2$$

The box begins and ends at rest the spring is initally relaxed

Remember that $y_f = X_f$.

$$0 = gmy_f - gmy_i + \frac{1}{2}k\,y_f^2$$

This is a quadratic equation, so let's insert the values now and re-arrange for solution. We'll also divide both sides by 20 to make the numbers smaller.

$$0 = 10(8)y_f - 10(8)y_i + \frac{1}{2}(120)\,y_f^2$$

$$3\,y_f^2 + 4y_f - 4y_i = 0$$

$$y_f = \frac{-4 \pm \sqrt{4^2 - 4(3)(-4)}}{2(3)} = +0.67\text{m or } -2\text{ m} \quad .$$

This time, we want the negative root because we know the final position will be below the y = 0 level.

DISCUSSION 6-9

Why does the equation give us two values? What condition did we impose on the locations of the box and the end of the spring? What situation does the other root correspond to?

HOMEWORK 6-10

A 0.85 kg bunch of bananas depresses the pan of a spring balance at Wegman's 3.0 cm when resting on it. If the bananas were dropped onto the pan from a height of 0.3 m above the empty pan, how far will the pan be depressed before starting to return upward?

---

## Power

Sometimes, we're interested in the rate at energy is put into, or removed from, an object, or the rate at which work is done, the *power*:

$$P_{AVE} = \frac{W}{t} \ .$$

The instantaneous power is of course

$$P_{INST} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} \ .$$

If we consider forces acting on an object during a short interval of time (and displacement), we obtain an interesting result:

$$P_{INST} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} = \lim_{\Delta t \to 0} \left( \frac{F\,\Delta x \cos(\theta_{F,\Delta x})}{t} \right) = F \lim_{\Delta t \to 0} \left( \frac{\Delta x}{t} \right) \cos(\theta_{F,\Delta x}) = F\,v\,\cos\left(\theta_{F,v}\right) = \vec{F} \cdot \vec{v} \ .$$

If one joule of work is performed in one second, we say that the power is one watt (symbol W). There is an alternate unit for power that is still commonly used in the U.S., the horsepower (hp). The hp has been redefined as exactly 750 W.

DISCUSSION 6-10

What is the power rating of a typical incandescent light bulb?[11] What is the power rating of a corresponding diode light bulb? What is the power rating of the engine in your car? How many light bulbs could your car engine presumably light up at once?

EXAMPLE 6-8

Suppose you're late for your next class. You need to run up a flight of stairs as quickly as possible. What power output is required?

The result depends on the values picked and will of course vary from person to person. Let's assume that the floors of your building are 6 m apart (fairly typical for an office building). The average adult male American masses in at 90 kg. The part no one ever agrees on is the amount of time required to run up a flight of stairs. Let's call it twelve seconds. If we can agree that all of the work goes into increasing the person's potential energy (he's running the same speed at the top and at the bottom), then we have

$$P = \frac{W}{t} = \frac{\Delta U}{t} = \frac{gmH}{t} = \frac{10(90)6}{7} = 771\ \mathrm{W} = 1.03\ \mathrm{hp}.$$

DISCUSSION 6-11

The horsepower was originally defined, loosely, as the power a draught house could supply while drawing a plough. From the previous result, it appears you could do this job. What's the difference between your ability and the horse's ability to draw a plough?

[11] This is not the same power that we discussed; it is the rate of conversion of electrical energy to thermal energy. But, let's continue anyway.

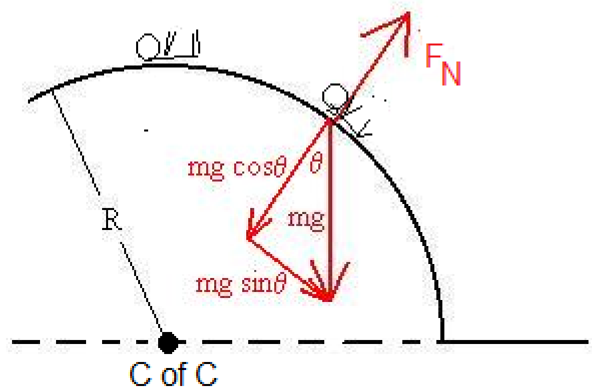
EXERCISE 6-1 Solution

Start as usual with a free body diagram. There are two forces acting on the child, his weight and the normal force.

$W_N = 0$ (the normal force is always perpendicular to the path).
$W_g$ – conservative

So, we have again that $W_{NC} = 0$, and

$$0 = \tfrac{1}{2}\,m\,v_f^2 - \tfrac{1}{2}\,m\,v_i^2 + gmy_f - gmy_i \;.$$




The child starts from rest (or close to it) and we will set ground level as y = 0. The problem ends not at the ground, but when the child leaves the igloo surface at y = H. The original altitude is y = R, the radius of the igloo.

$$0 = \tfrac{1}{2}\,m\,v_f^2 + gmH - gmR \;.$$

The trouble we have here is that there are two unknowns. We need more information. We might notice that the child is moving in a circle, and we know a lot about things moving in circles. Let's return to Newton's second law:

$$+\,gm\cos(\theta) - F_N = ma_C = \frac{mv^2}{r} \;.$$

The radius of the circle is of course R. The normal force goes to zero when the child loses contact with the surface, and at that moment, $v = v_f$.

$$gm\cos(\theta) = \frac{mv_f^2}{R} \;.$$

We may notice that the cosine of theta is H/R, so that

$$gm\frac{H}{R} = \frac{mv_f^2}{R} \quad \rightarrow \quad gmH = mv_f^2 \;.$$

Returning to the energy equation,

$$0 = \tfrac{1}{2}gmH + gmH - gmR \;;$$
$$\frac{3}{2}\,H = R\,;$$

$$H = \frac{2}{3}R\ .$$

## SECTION 7 – THE THIRD PICTURE

We've looked at the motions of objects using two outwardly different, but ultimately identical, points of view: forces and accelerations, and work and energy. We know that they are the same, since we derived the work-energy theorem using Newton's Second Law and a couple of definitions. Now, we'll introduce yet another picture which we may, or may not, find convenient to use on certain classes of problems.

CHEESY EXPERIMENT 7-1

Consider a small toy car sitting on a table at a spot marked 'X'; we'll assume the wheels make its contact with the table frictionless. Observes the car closely. Now, observe the car as it travels through point X. Is it fair to say that the car possesses some quality or property when it's moving through X that it lacks when stationary? How did the car acquire that property?

DISCUSSION 7-1

Impulse.mp4

You may notice that the car had the property only after a force acted on it. Indeed, I can also remove the property by applying a force opposite to the motion of the car. This is sounding awfully familiar. In fact, we will approach this in a manner very much like the one we used for work and energy. After the experiment, we agreed to the following:

- There is some quality or property the object possesses when it's moving through X that it lacks when it's stationary at X. Of course, we know that the object possesses kinetic energy when it's moving, but we are measuring the transfer of this new property differently, so we must be transferring something else as well. For want of a better name, let's call this new property *momentum* (symbol p)
- Momentum is transferred into the object by applying a force. However, the force must act for some period of time. That means that the property is not energy, although clearly energy was also being transferred.
- Transferring momentum into (or out of) an object is a process; let us call the <u>transfer</u> of momentum the *impulse* (J) done on the object. Impulse is <u>not</u> momentum; it is the transfer of momentum.
- The bigger the force, the more momentum is transferred: as F↑, J↑. We might even speculate that J is proportional to F. That would certainly be the simplest relationship consistent with our observations. We could be wrong, of course; perhaps $J \sim F^2$ or $F^3$. We'll make the simplest assumption and see if there is a contradiction somewhere in our subsequent experiments.
- The greater the time interval over which the force acted, the more impulse is provided: that is, as Δt ↑,J↑ .We might speculate that J is proportional to Δt.

We may perhaps further speculate that, in the simplest possible scenario, $J = F\ \Delta t. = \Delta p$.

DISCUSSION 7-2

Let's look at the center term of the hypothesis formula above. Which kind of a quantity is force? Then, what about impulse and momentum? Is the bank account analogy appropriate here? Can you think of one that may be more appropriate?

As in Section 6, we know that there may be more than one force acting on an object at once, each with its own effect on the momentum. Let's start with our guess above and see it we can figure out what momentum is. This derivation works for constant forces, and as we'll see, in three dimensions.

DERIVATION 7-1

$$\vec{J}_{TOTAL} = \sum_n \vec{J}_n = \sum_n \vec{F}_n \, \Delta t = \left(\sum_n \vec{F}_n\right) \Delta t = (m\vec{a})\,\Delta t = m(\vec{a}\,\Delta t) = m(\vec{v}_f - \vec{v}_i)$$
$$= m\vec{v}_f - m\vec{v}_i = \Delta(m\vec{v}) \;.$$

Keeping in mind that we required that $\vec{J}_{TOTAL} = \Delta\vec{p}$, we may perhaps jump to the conclusion that $\vec{p} = m\vec{v}$. [1]

Now, because we wrote this derivation in terms of vectors, and because we know that if two vectors are equal, then their x, y, and z components must independently be equal, we can treat these problems with momentum as three separate problems, one with the x-components, one with the y-components, and one with the z-components.

DISCUSSION 7-3

In the derivation above, we stipulated that the forces should be constant in time. This allowed us to make use of kinematic equation 1: $v_f = v_i + at$. How would you deal with a situation in which forces are not constant in time?

DISCUSSION 7-4

It is useful to remember that these quantities, such as momentum or energy, are ones that we have defined. Is momentum a real thing? What about energy?

HOMEWORK 7-1

What is the ratio of the magnitude of the momentum of a 3kg mass moving at 3 m/s to that of a 2 kg mass moving at 4 m/s? What is the ratio of the respective kinetic energies? NOTE: You should find that one of the objects has more momentum, while the other has more kinetic energy. How is that possible?

---

[1] As always, we could be wrong and perhaps $\vec{p} = m\vec{v} + \vec{A}$, where $\vec{A}$ is some constant that subtracts out when we find the change in $\vec{p}$. For now, let's assume the simplest case that $\vec{A} = 0$.

HOMEWORK 7-2

Show that the kinetic energy of an object can be written in terms of the magnitude of the momentum as $K = p^2/2m$.

EXAMPLE 7-1

A constant force of 14 N acts on an initially stationary object (mass 6 kg) for 8 seconds. If this was the only force acting on the object, what is the impulse? What is the final speed of the object?

Call the direction of the force the +x direction.

$$\vec{J}_{TOTAL} = \vec{F}\,\Delta t = 14(8) = \text{112 Ns in the + x direction} \quad .$$

Note that, unlike for work/energy, there is no special unit for impulse/momentum. Typically, impulse is newtons seconds, while momentum is kilogram meters/second.

Then,

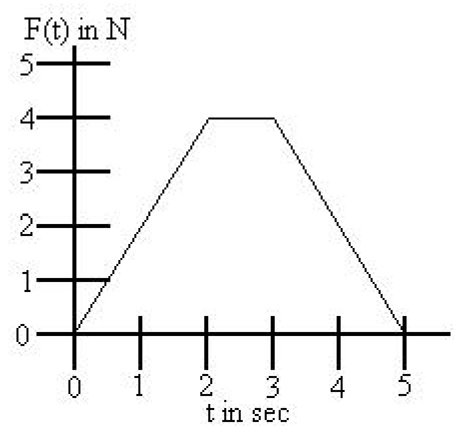
$$\vec{J}_{TOTAL} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i \ ,$$

$$\vec{v}_f = \vec{v}_i + \frac{\vec{J}_{TOTAL}}{m} = 0 + \frac{112}{6} = \text{18.7 m/s in the + x direction} \ .$$

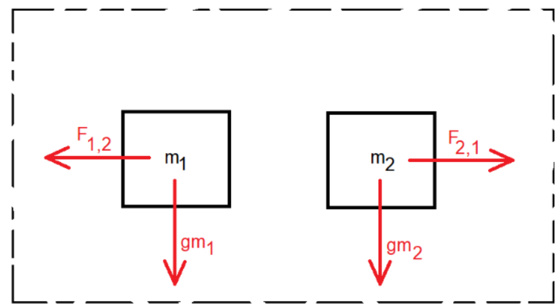
HOMEWORK 7-3

Suppose F(t) shown in the figure is the net force acting in the +x direction on a particle of mass 2kg. Find

a) the impulse imparted to the object by the force.
b) the final velocity of the object if it had been originally at rest.
c) the final velocity of the object if its initial x velocity had been -2 m/s.




As mentioned several times, each of our 'pictures' is especially well suited to solving a particular type of problem. Newton's second law and the kinematic equations were useful when the forces were constant. The work energy theorem was useful when there were no non-conservative forces doing work (mechanical energy was conserved), and had the advantage of not requiring us necessarily to know the path taken by the object of interest or the time that the trip required. In some special cases, the momentum picture is very useful for examining *collisions*. A collision is when two or more objects interact with one another. They do not need actually to touch one another, as we may think about, say, automobile collisions. They may exert other forces on each other, whether gravitational, electric, nuclear, *et c*.




Before we start, let's define a *system*. You may be familiar with this term from chemistry. A system is just the collection of objects in which we are interested. A *closed system* is one for which the only forces acting on the objects are due to other objects in the system; these are called *internal forces*. An *open system* is one for which some force or forces are due to agencies not included in the system; these are of course *external forces*. We can mentally draw an imaginary box around the system; any force that crosses the box's boundary will be an external force. In the figure, the system comprises mass 1 and mass 2. The force exerted on 1 by 2 and the force exerted on 2 by 1 are internal forces. Their weights, however, are exerted by the earth, which is not in the box; the weights are therefor external forces.

As usual, let's start with a simple case, then generalize.

DERIVATION 7-2

Consider two objects, $m_1$ and $m_2$, each with its proper initial velocity, $\vec{v}_{1i}$and $\vec{v}_{2i}$. When the objects interact, they exert forces on each other that obey Newton's third law:

$$\vec{F}_{1,2} = -\vec{F}_{2,1} \ .$$

By the second law,

$$m_1\vec{a}_1 = -m_2\vec{a}_2 \ .$$

Careful here. Because the statement above is true instant by instant, it must also be true when averaged over the duration of the interaction between the masses, so we can write

$$m_1\vec{a}_{AVE\,1} = -m_2 a_{AVE\,2} \ .$$

Note that at this stage, we lose a lot of information about the forces involved. Remember the definition of the average acceleration and substitute:

$$m_1\frac{\Delta\vec{v}_1}{\Delta t_1} = -m_2\frac{\Delta\vec{v}_2}{\Delta t_2} \ .$$

We can make an argument using the third law that the two time intervals must be the same; if they were not, then there would be a time when one object would be exerting a force on the other while the other would not be exerting a force on the one.

$$m_1 \Delta \vec{v}_1 = -m_2 \Delta \vec{v}_2$$

$$m_1 \vec{v}_{1f} - m_1 \vec{v}_{1i} = -m_2 \vec{v}_{2f} + m_2 \vec{v}_{2i}$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \ .$$

At this point, we might recognize a relationship similar to one we saw in Section 6. The total momentum of the system before the interaction is equal to the total momentum of the system after the interaction. Momentum may well have been transferred from one object to the other, but the total momentum was conserved. Unlike energy, though, momentum does not change from one form to another, *e.g.*, from kinetic to potential.

For future reference, let's write this last expression as

$$m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} - m_1 \vec{v}_{1i} - m_2 \vec{v}_{2i} = \Delta \vec{p}_{TOTAL} = 0 \ .$$

Before we continue, a number of comments.

What if, in our derivation, there were an external force? Let's redo the work with an external force acting on mass 1 as an example. It would still be true that

$$\vec{F}_{1,2} = -\vec{F}_{2,1} \ .$$

But by the second law,

$$\vec{F}_{1,EXT} + \vec{F}_{1,2} = m_1 \vec{a}_1 \ ,$$

so that

$$-\vec{F}_{1,EXT} + m_1 \vec{a}_1 = -m_2 a_2 \ .$$

Following through to the end, we see that

$$-\vec{F}_{1,EXT} + m_1 \vec{a}_{AVE\,1} = -m_2 \vec{a}_{AVE\,2}$$

$$-\vec{F}_{1,EXT} + m_1 \frac{\Delta \vec{v}_1}{\Delta t_1} = -m_2 \frac{\Delta \vec{v}_2}{\Delta t_2}$$

$$-\vec{F}_{1,EXT} \Delta t + m_1 \Delta \vec{v}_1 = -m_2 \Delta \vec{v}_2$$

$$-\vec{F}_{1,EXT} \Delta t + m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \ .$$

$$\vec{F}_{1,EXT} \Delta t = \Delta \vec{p}_{TOTAL} \neq 0$$

and the total momentum is not conserved. We do not expect the total momentum of a system to be conserved if there are external forces.

However, there are two loopholes. The first, in analogy with the work energy theorem, is that momentum will still be conserved if the net external force is zero, that is, if the external forces happen to cancel to zero.[2] The other loophole is more common. Because we used vector notation, the derivation is valid for three dimensions. But remember that when two vectors are equal, their x, y, and z components are also independently equal. This means that the result can be written as three separate equations:

$$m_1 v_{1xi} + m_2 v_{2xi} = m_1 v_{1xf} + m_2 v_{2xf}$$

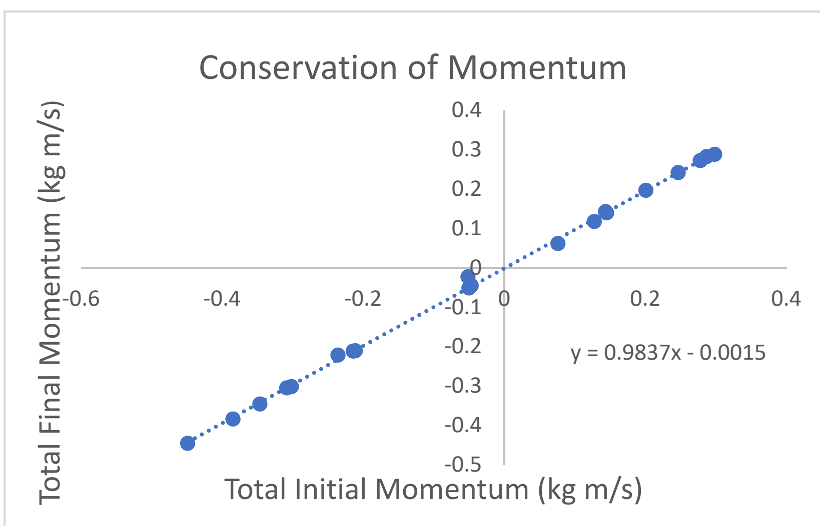
$$m_1 v_{1yi} + m_2 v_{2yi} = m_1 v_{1yf} + m_2 v_{2yf}$$

$$m_1 v_{1zi} + m_2 v_{2zi} = m_1 v_{1zf} + m_2 v_{2zf} \quad .$$

Suppose that there are some external forces in the x direction, but none in the y or z directions. Then, we can still use conservation of momentum in those two directions. As an example, think of two skydivers falling toward the earth. If the system is the two divers, there are external forces in the vertical direction (their weights), but none in the horizontal directions. Momentum will still be conserved horizontally in any collision the divers may suffer.

O.K., we are actually in a position to test the idea of conservation of momentum, and Newton's third law as well (remember that we skipped on that in Section 5) since our notion was based on Newton's second law (already tested) and the third law.

EXPERIMENT 7-1




Let's look at the results of an experiment to give us some confidence this is true. A system of two masses were placed on an airtrack to reduce friction (an external force) and collided together under different conditions. The vertical forces of weight and air from the track should not affect the horizontal motions. The velocities before and after were measured, and the total momentums before and after calculated and plotted. If conservation of momentum is true, a line of slope one through the origin should be seen. In these

[2] A simple analogous situation for work-energy might be to push a crate parallel to the surface on which it moves while balancing a kinetic frictional force.

results, the intercept is very small compared to the values measured, and the slope is very close to one. This gives us some confidence that linear momentum is conserved, and indirectly, that the third law of motion is supported.

DERIVATION 7-3*

What if there are more than two masses in the system? Well, suppose that there are q masses. For each mass n, add up the k impulses acting on it.

$$\sum_{\mathrm{k}} \vec{\mathrm{J}}_{\mathrm{n,k}} = \Delta\vec{\mathrm{p}}_{\mathrm{n}}$$

Add up these terms for all q masses:

$$\sum_{n=1}^{q}\sum_{\mathrm{k}} \vec{\mathrm{J}}_{\mathrm{n,k}} = \sum_{n=1}^{q} \Delta\vec{\mathrm{p}}_{\mathrm{n}} = \Delta\vec{\mathrm{p}}_{\mathrm{TOTAL}} \ .$$

We can divide the impulses on the left into two categories, internal and external, in the same way we divided forces into conservative and non-conservative forces. Keep in mind that any object in the system won't exert a force on itself.

$$\sum_{n=1}^{q}\sum_{\substack{\mathrm{k=1} \\ \mathrm{k\neq n}}}^{q} \vec{\mathrm{J}}_{\mathrm{INT\ n,k}} + \sum_{n=1,}^{q}\sum_{\mathrm{k}} \vec{\mathrm{J}}_{\mathrm{EXT\ n,k}} = \Delta\vec{\mathrm{p}}_{\mathrm{TOTAL}} \ .$$

The second term on the left side is just the sum of the external impulses, so let's concentrate on the first term. Since $\vec{\mathrm{J}} = \vec{\mathrm{F}}\,\Delta\mathrm{t}$,

$$\sum_{n=1}^{q}\sum_{\substack{k=1 \\ k\neq n}}^{q} \vec{F}_{n,k}\,\Delta t_{n,k} + \vec{J}_{EXT\ TOTAL} = \Delta\vec{\mathrm{p}}_{\mathrm{TOTAL}} \ .$$

Now, $\Delta t_{n,k} = \Delta t_{k,n}$ is the time interval during which objects n and k interact; by our third law argument above, these time intervals are equal and indeed occur simultaneously. From the third law, $\vec{F}_{n,k} = -\vec{F}_{k,n}$, and so we see that all of the terms in the summation cancel in pairs, leaving

$$\vec{J}_{EXT\ TOTAL} = \Delta\vec{\mathrm{p}}_{\mathrm{TOTAL}} \ .$$

## Brief Review

Let's review the three pictures:

- The velocity of an object will remain constant unless the object is acted on by a force, in which case $\vec{F}_{TOTAL} = m\vec{a}$.
- The kinetic energy of an object will remain constant unless the object has work performed on it, in which case $W_{TOTAL} = \Delta K$.
- The momentum of an object will remain constant unless the object is acted on by an impulse, in which case $\vec{J}_{TOTAL} = \Delta\vec{p}$.

These last two we can re-write for systems of objects:

- The total mechanical energy ($E = K + U$) of a system will remain constant unless the system has work performed on it by non-conservative forces, in which case $W_{NC} = \Delta E_{TOTAL}$.
- The total momentum of a system will remain constant unless the system is acted on by an external impulse, in which case $\vec{J}_{EXT} = \Delta\vec{p}_{TOTAL}$.

## Collisions

As was noted above, conservation of momentum is particularly useful in analyzing collisions. You may have noticed that, in Derivation 7-X, the details of the forces acting between the objects disappeared, which is one of the strengths of this method. We don't even need to know what kind of force acted on the objects! It is somewhat along the lines of having an object slide down along a curved frictionless surface; the details of the path were not necessary to find the speed of the object at the bottom.

We're going to consider only the extremes of the spectrum of collisions. The easier of the two is the *completely inelastic collision,* one in which the objects stick together after the collision. Let's start by considering a simple situation in one dimension in which there are no external forces (that is, the only forces are those that each object exerts on the other):

EXAMPLE 7-2

An object of mass 5 kg is moving at 7 m/s along the +x axis when it strikes a stationary object of mass 3 kg. If they stick together, what is their common final velocity?

First of all, for a problem like this one, it's convenient to revert to the notation we used in Sections 2 through 5: a vector in the +x direction will carry a positive value, while one in the -x direction will carry a minus sign.

Assuming the two masses form a closed system, conservation of momentum seems appropriate.

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi} = m_1\vec{v}_{1xf} + m_2\vec{v}_{2xf}\ .$$

They have a common final velocity, or, if you prefer, it's as if they are now one object of mass $m_1 + m_2$ moving at velocity $v_f$.

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi} = (m_1 + m_2)\vec{v}_{xf}\ .$$

$$\vec{v}_{xf} = \frac{m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi}}{m_1 + m_2} = \frac{5(+7) + 3(0)}{5 + 3} = +4.38\,\mathrm{m/s}\,.$$

EXAMPLE 7-3

An object of mass 12 kg is moving at 5 m/s along the +x axis has a *rear-end collision* with an object of mass 3 kg travelling at 4 m/s. If they stick together, what is their common final velocity?

First, what does rear-end collision mean? It's an expression from automobile collisions meaning that the two objects were moving in the same direction. A *head-on collision* would be one in which they were heading in the opposite directions.

Assuming the two masses form a closed system, conservation of momentum seems appropriate. Also, because they have a common final velocity,

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi} = (m_1 + m_2)\vec{v}_{xf}\ .$$

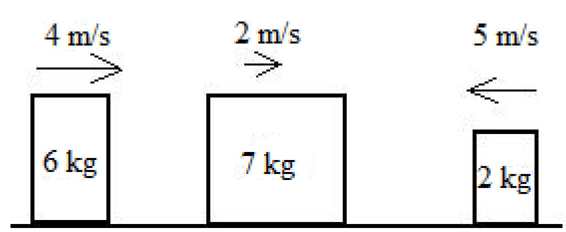
$$\vec{v}_{xf} = \frac{m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi}}{m_1 + m_2} = \frac{12(+5) + 3(+4)}{12 + 3} = +4.8\,\mathrm{m/s}\,.$$

EXERCISE 7-1

An object of mass 6 kg is moving at 4 m/s along the +x axis has a *head-on collision* with an object of mass 3 kg travelling at 8 m/s. If they stick together, what is their common final velocity?

HOMEWORK 7-4

Three masses of 6 kg, 7 kg, and 2 kg move on a frictionless horizontal surface with initial speeds of 4 m/s, 2 m/s, and 5 m/s, respectively, as shown in the figure. If the masses all stick together after the collisions, what will be the final velocity of the combined mass?




HOMEWORK 7-5

Two railcars have a head-on collision, couple together, and stop dead. If Car A was moving four times as quickly as Car B was, and the total mass of both cars together is 90,000 kg, what are the masses of each car individually?

DISCUSSION 7-5

Consider the result of Exercise 7-1. We ended up with no momentum because we happened to start with no momentum. Sure, each object had some momentum of its own to start, but the total was zero. You may notice, however, that there is something else that we ended with zero of, but started with a positive amount of. What is it? Where did it go? Consider an automobile accident. What do the cars look like afterward and what was necessary to make them that way?

In Exercise 7-1, the objects started with 144 joules of kinetic energy, and ended with none. One of the characteristics of totally inelastic collisions is that kinetic energy is lost.

EXERCISE 7-2

Find the total initial and final kinetic energies in Example 7-x.

DERIVATION 7-4

Show that kinetic energy is always lost during the special case of a totally inelastic collision in one dimension when one of the objects is initially at rest. We've already shown that

$$\vec{v}_{xf} = \frac{m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi}}{m_1 + m_2} \xrightarrow[m_2 \text{ starts at rest}]{} v_{xf} = \frac{m_1 v_{1xi}}{m_1 + m_2} .$$

Now we need to show that

$$\tfrac{1}{2}m_1v_{1xi}^2 + \tfrac{1}{2}m_2v_{2xi}^2 > \tfrac{1}{2}(m_1 + m_2)v_{xf}^2 \xrightarrow[m_2 \text{ starts at rest}]{}$$

$$\tfrac{1}{2}m_1v_{1xi}^2 > \tfrac{1}{2}(m_1 + m_2)\left(\frac{m_1v_{1xi}}{m_1 + m_2}\right)^2.$$

$$m_1v_{1xi}^2 > \frac{m_1}{m_1 + m_2}m_1v_{1xi}^2$$

This leads us to the true statement that

$$1 > \frac{m_1}{m_1 + m_2} ,$$

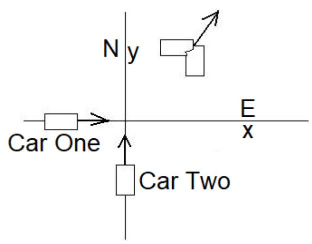
which is equivalent to saying that kinetic energy is always lost in this very special case.

DISCUSSION 7-6

The previous derivation was done for a very special case of one of the masses being initially at rest. After completing Section 7, you should be able to return here and make an argument that kinetic energy is lost in <u>any</u> totally inelastic collision regardless of the initial motions of the two masses.

Let's examine a particular situation. Suppose that a bullet is fired into a block of wood. The bullet penetrates a given distance into the block, and the block of course moves a bit in the direction the bullet was moving. The bullet applied a force to the block and the block applied an equal though opposite force to the bullet. The bullet did positive work on the block, and the block did negative work on the bullet. However, the displacements of each object were not the same during this process, and so more negative work was done on the bullet than positive work done on the block. As a result, kinetic energy was lost.

EXAMPLE 7-4




Let's try a two-dimensional example. Suppose you are an insurance accident investigator. Two cars collided at an icy intersection as shown, and the wreckage moved off at an angle 59° north of east. You know that Car One (1500 kg) was moving eastward at 30 m/s just before the collision, because it was caught on a speed camera. The question is, how quickly was Car Two (2000 kg) moving?

We'll let the system comprise the cars. The road is icy, or frictionless, so there are no external horizontal forces. The vertical normal forces and weights will not prohibit conservation of momentum in the horizontal directions. Let east be the +x direction and north be the +y direction. Use conservation of momentum separately in each direction.

$$\text{x: } m_1 v_{1xi} + m_2 v_{2xi} = (m_1 + m_2)\, v_{xf}$$

$$\text{y: } m_1 v_{1yi} + m_2 v_{2yi} = (m_1 + m_2)\, v_{yf}$$

In this solution, $v_{1yi}$ and $v_{2xi}$ are both zero, and $v_{xf} = v_f \cos(\theta)$ and $v_{yf} = v_f \sin(\theta)$.

$$m_1 v_{1xi} = (m_1 + m_2)\, v_f \cos(\theta)$$

$$m_2 v_{2yi} = (m_1 + m_2)\, v_f \sin(\theta)$$

Divide the second equation by the first to obtain

$$\frac{m_2 v_{2yi}}{m_1 v_{1xi}} = \tan(\theta) \ .$$

Then,

$$v_{2yi} = \frac{m_1}{m_2}\, v_{1xi}\, \tan(\theta) = \frac{1500}{2000}(30)\tan(59^o) = 37.4\ \mathrm{m/s} .$$

Now, let's consider a *totally elastic collision*, by which we mean no kinetic energy is lost during the collision (although, it can be transferred from one object to the other). Think of the objects as having springs on them; instead of kinetic energy being used to deform the objects, some kinetic energy is stored as potential energy, then re-released as kinetic. For reasons that will be discussed later, this derivation will be applicable to problems in <u>one dimension only</u>.

DERIVATION 7-5

We will write one equation representing conservation of momentum (in one dimension only) and another representing the fact that the total kinetic energy is the same before and after the interaction.

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi} = m_1\vec{v}_{1xf} + m_2\vec{v}_{2xf}$$

$$\frac{1}{2}m_1 v_{1xi}^2 + \frac{1}{2}m_2 v_{2xi}^2 = \frac{1}{2}m_1 v_{1xf}^2 + \frac{1}{2}m_2 v_{2xf}^2$$

Typically, we are given the masses and initial velocities and are asked to find the final velocities. Since we have two independent equations and two unknowns, we should be good. One solution should be obvious: $v_{1xi} = v_{x1f}$ and $v_{2xi} = v_{x2f}$; the equations require merely that K and **p** be conserved, which is certainly the case if no collision actually occurs. However, finding the other, more interesting, solution requires about two pages of effort. So, what we're going to do is what physicists often do when a problem is too difficult; we'll look at a special case. Here, we'll simplify the problem to require that mass two is initially at rest. Of course, the results we obtain will be valid for only that situation. Our two equations become

$$m_1\vec{v}_{1xi} = m_1\vec{v}_{1xf} + m_2\vec{v}_{2xf}$$

$$\frac{1}{2}m_1 v_{1xi}^2 = \frac{1}{2}m_1 v_{1xf}^2 + \frac{1}{2}m_2 v_{2xf}^2$$

Reverting to our Section 2 notation, this first equation can be rewritten as

$$m_2 v_{2xf} = m_1 v_{1xi} - m_1 v_{1xf} = m_1(v_{1xi} - v_{1xf})$$

and the second as

$$\frac{1}{2}m_2 v_{2xf}^2 = \frac{1}{2}m_1 v_{1xi}^2 - \frac{1}{2}m_1 v_{1xf}^2 = \frac{1}{2}m_1\,(v_{1xi} - v_{1xf})(v_{1xi} + v_{1xf})\ .$$

Dividing the second equation by the first and multiplying through by two results in

$$v_{2xf} = v_{1xi} + v_{1xf}\ ,$$

which we substitute into the original momentum equation.

$$m_1 v_{1xi} = m_1 v_{1xf} + m_2(v_{1xi} + v_{1xf})$$

Solving for the final velocity of mass one gives us

$$v_{1xf} = \frac{m_1 - m_2}{m_1 + m_2} v_{1xi}\ .$$

If instead, we substitute to solve for the final velocity of mass two, we get

$$\mathrm{v}_{2xf} = \frac{2m_1}{m_1 + m_2}\mathrm{v}_{1xi} \ .$$

Once again, remember that these solutions are only valid if one mass had been initially at rest, the collision was totally elastic, and motion was restricted to one dimension. To be clear, you should label whichever mass was not initially moving as mass two. If you encounter this type of problem in a homework or exam question, you may move directly to these two relationships as your starting point.

EXAMPLE 7-4

An object of mass 12 kg moving at 5 m/s along the +x axis has a totally elastic collision with a stationary object of mass 3 kg. What are their final velocities?

As allowed above, we will start with the two relationships we derived. The solution becomes 'plug-and-chug.'

$$\mathrm{v}_{1xf} = \frac{\mathrm{m}_1 - \mathrm{m}_2}{\mathrm{m}_1 + \mathrm{m}_2}\mathrm{v}_{1xi} = \frac{12 - 3}{12 + 3}5 = 3\ \mathrm{m/s}\,.$$

$$\mathrm{v}_{2xf} = \frac{2\mathrm{m}_1}{\mathrm{m}_1 + \mathrm{m}_2}\mathrm{v}_{1xi} = \frac{2(12)}{12\ 3}5 = 8\ \mathrm{m/s}\,.$$

EXERCISE 7-3

An object of mass 7 kg is moving at 10 m/s along the +x axis has a totally elastic rear-end collision with an object of mass 4 kg travelling at 3 m/s. What are their final velocities?

What if neither mass had been at rest? Well, we could go back and re-do the derivation with the two extra terms, but here is a neat trick: we can make use of the material of Section 4 (relative motion) and pick a new *frame of reference* (indicated below by a *prime*) in which mass 2 <u>is</u> initially at rest, calculate the final velocities in that frame, then convert back to the original frame.

Let's use the previous Exercise as an example. For the observer who described the problem, Mass 1 is moving in the +x direction at 10 m/s before it hits Mass 2 moving the same way at +3 m/s. If we were passengers riding alongside Mass 2, we would of course think that Mass 2 is stationary and see Mass 1 approaching us from behind at 7 m/s. From our point of view, the relationships derived above would be perfectly O.K. to use. Then, we need only calculate what the original observer sees.

EXAMPLE 7-5

I like to keep track of this process with a chart. It also makes the process somewhat mechanical, and thereby less susceptible to mistakes. The information for each mass runs horizontally in the rows. The first column contains the original values for each mass's initial velocity. The third column contains the initial velocities in the new frame of reference; the initial velocity of Mass 2 here <u>must</u> be zero. The second column is the process that changes the values. We ask, what must be done to $M_2$'s initial velocity to make it zero? In this case, we must subtract 3 m/s.

| | $v_{initial}$ | convert to new frame in which $v_{2xi}' = 0$ | $v_o'$ |
|---|---|---|---|
| $M_1$ | +10 m/s | | |
| $M_2$ | + 3 m/s | Subtract 3 m/s | 0 m/s |

Of course, if we subtract 3m/s from $M_2$'s velocity, we must do the same for $M_1$:

| | $v_{initial}$ | convert to new frame in which $v_{2xi}' = 0$ | $v_o'$ |
|---|---|---|---|
| $M_1$ | +10 m/s | Subtract 3 m/s | + 7 m/s |
| $M_2$ | +3 m/s | Subtract 3 m/s | 0 m/s |

Now we have a problem we can solve. Use the relationships derived, we can find the final velocities in the new frame of reference.

| | $v_i$ | convert | $v_i'$ | Find $v_f'$ |
|---|---|---|---|---|
| $M_1$ | +10 m/s | -3 | +7 m/s | $v_{1xf}' = \frac{m_1 - m_2}{m_1 + m_2} v_{1xi}' = \frac{7 - 4}{7 + 4} 7 = 1.91 \text{ m/s}$. |
| $M_2$ | +3 m/s | -3 | 0 m/s | $v_{2xf}' = \frac{2m_1}{m_1 + m_2} v_{1xi}' = \frac{2(7)}{7 + 4} 7 = 8.91 \text{ m/s}$. |

We're not done, because we need to convert back to the original frame. We do that by reversing the transformation that we did previously, in this example, by adding 3 m/s to the results.

| | $v_i$ | convert | $v_i'$ | Find $v_f'$ | convert back to original frame by reversing the previous transformation | $v_f$ |
|---|---|---|---|---|---|---|

| | | | | | | |
|---|---|---|---|---|---|---|
| $M_1$ | +10 m/s | -3 | +7 m/s | $v'_{1xf} = \frac{m_1 - m_2}{m_1 + m_2} v'_{1xi} = \frac{7-4}{7+4} 7 = 1.91\text{ m/s}.$ | +3 | 4.91 m/s |
| $M_2$ | +3 m/s | -3 | 0 m/s | $v'_{2xf} = \frac{2m_1}{m_1 + m_2} v'_{1xi} = \frac{2(7)}{7+4} 7 = 8.91\text{ m/s}.$ | +3 | 11.91 m/s |

Before I give you an exercise to try, let's do another short derivation. To be honest, I have never found the result of this to be useful, except as a quick check of my results for the chart solution. I'll show you what I mean in a moment.

DERIVATION 7-6*

Here is an additional interesting derivation for a totally elastic collision. Here, we do not need to assume that $m_2$ is initially at rest. That is, the result is valid for any one-dimensional totally elastic collision.

We start with the conditions for conservation of momentum and kinetic energy:

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi} = m_1\vec{v}_{1xf} + m_2\vec{v}_{2xf}$$

$$\tfrac{1}{2}m_1 v_{1xi}^2 + \tfrac{1}{2}m_2 v_{2xi}^2 = \tfrac{1}{2}m_1 v_{1xf}^2 + \tfrac{1}{2}m_2 v_{2xf}^2 \ .$$

Let's re-arrange each:

$$m_1(v_{1xi} - v_{1xf}) = m_2(v_{2xf} - v_{2xi})$$

$$m_1(v_{1xi} - v_{1xf})(v_{1xi} + v_{1xf}) = m_2(v_{2xf} - v_{2xi})(v_{2xf} + v_{2xi})$$

Dividing the second equation by the first leaves

$$v_{1xi} + v_{1xf} = v_{2xf} + v_{2xi} \ .$$

So if the chart solution was done correctly, we find that the sums of the masses' initial and final velocities should be the same. For example, 10 + 4.91 = 3 + 11.91. You can use this as a quick check on your answers. An agreement won't guarantee you're correct, but a failure will tell you if you're wrong.

HOMEWORK 7-4

A 10 kg object initially moving to the right at 20 m/s makes a totally elastic head on collision with a 15 kg object which was initially moving to the left at 5 m/s. Find the final velocities of each object.

HOMEWORK 7-5

A 10 kg object initially moving to the right at 20 m/s has a totally elastic rear-end collision with a 15 kg object which was initially moving to the right at 5 m/s. Find the final velocities of each object.

JUSTIFICATION OF ASSUMPTIONS*

In the method discussed above, that is changing to a new frame of reference to solve our problem, we assumed that if momentum and kinetic energy are conserved in one frame, that they are conserved in the other frame. We need to justify those assumptions. It's not too difficult for momentum. Let's bite the bullet and do it for three dimensional collisions. For the original frame, we can rewrite the momentum equation as

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} - m_1\vec{v}_{1f} - m_2\vec{v}_{2f} = 0 \ .$$

Let $\vec{u}$ be the velocity of the first frame relative to the second frame. Then in that new frame we ask if,

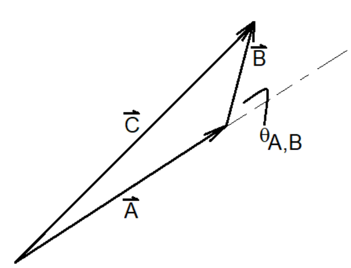
$$m_1(\vec{v}_{1i} + \vec{u}) + m_2(\vec{v}_{2i} + \vec{u}) - m_1(\vec{v}_{1f} + \vec{u}) - m_2(\vec{v}_{2f} + \vec{u}) = 0 \ .$$

Re-arranging a bit results in

$$(m_1\vec{v}_{1i} + m_2\vec{v}_{2i} - m_1\vec{v}_{1f} - m_2\vec{v}_{2f}) + (m_1 + m_2 - m_1 - m_2)\,\vec{u} = 0 \ .$$

The first term is zero from our knowledge of the initial frame, and the second term is clearly zero, and so momentum is conserved in the new frame. Since there was no restriction put on $\vec{u}$, momentum is conserved in every possible frame if it is conserved in any one frame, regardless of the type of collision.

Kinetic energy is a bit more difficult because we deal with the objects' speeds, not their velocities. Let's review a bit. Suppose we add two vectors, $\vec{A}$ and $\vec{B}$, that are not in the same (or opposite) directions and want to know the magnitude of the sum, $|\vec{A} + \vec{B}|$. Draw $\vec{A}$, $\vec{B}$, and their sum $\vec{C}$ so as to form a triangle. The *law of cosines* tells us that[3]




$$C^2 = A^2 + B^2 + 2AB\cos\theta_{A,B} = A^2 + B^2 + 2\,\vec{A}\cdot\vec{B} \,.$$

[3] The angle is defined differently here than is usual. It is the exterior angle rather than the interior angle, which leads to the difference in sign.

Note that this is a general statement that reduces to the Pythagorean theorem when theta is 90°. Now, in our original frame of reference, let's assume that kinetic energy is conserved during the collision. We can write

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 - \tfrac{1}{2}m_1v_{1f}^2 - \tfrac{1}{2}m_2v_{2f}^2 = 0 \ .$$

In the new frame, we'd like to know if

$$\tfrac{1}{2}m_1|\vec{v}_{1i} + \vec{u}|^2 + \tfrac{1}{2}m_2|\vec{v}_{2i} + \vec{u}|^2 - \tfrac{1}{2}m_1|\vec{v}_{1f} + \vec{u}|^2 - \tfrac{1}{2}m_2|\vec{v}_{2f} + \vec{u}|^2 = 0 \ .$$

After multiplying it all out and re-arranging a bit,

$$\left(\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 - \tfrac{1}{2}m_1v_{1f}^2 - \tfrac{1}{2}m_2v_{2f}^2\right) + (m_1\vec{v}_{1i} + m_2\vec{v}_{2i} - m_1\vec{v}_{1f} + m_2\vec{v}_{2f}) \cdot \vec{u} \\ + \tfrac{1}{2}(m_1 + m_2 - m_1 - m_2)\,u^2 = 0 \ .$$

The contents of the first and second sets of parentheses are zero from our knowledge of the original frame of reference ($K_{TOTAL}$ and $\vec{p}_{TOTAL}$ were conserved), and that of the third is clearly zero, and so if the kinetic energy is conserved in any frame, then it is conserved in every frame. We can return to Derivation 7-4 and generalize the result: if kinetic energy is lost during a collision in one frame, some is lost in any frame, including any in which both objects were initially moving.

This is going to be an important point in Physics Three.

While we're here, let's think about a couple of other considerations that students have asked about over the years. What about impulse and work in different frames? Suppose a mass has a force acting on it for a given duration of time. What is the force in a new frame of reference?

$$\vec{F}\Delta t = m(\vec{v}_f - \vec{v}_i) = m(\vec{v}_f - \vec{v}_i) + m(\vec{u} - \vec{u}) = m\big((\vec{v}_f + \vec{u}) - (\vec{v}_i + \vec{u})\big) \\ = m(\vec{v}_f' - \vec{v}_i') = \vec{F}'\Delta t \ .$$

So, a force of a certain magnitude in one frame of reference has the same magnitude in any other frame of reference, as does the impulse.[4] Knowing that, what can we say about the work done by a force in two different frames? We certainly expect that the work <u>could</u> be different because the displacements could be different. The work-energy theorem in the original frame will be

$$W = \vec{F} \cdot \Delta\vec{r} = \tfrac{1}{2}m\left(v_f^2 - v_i^2\right) \ .$$

In a new frame moving at velocity $\vec{u}$ with respect to the original frame, we have that

[4] That is, if the time intervals in each frame are the same, which is a characteristic of Galilean transformations. The problem comes about in relativistic transformations, which we'll discuss in Semester Three.

$$W' = \vec{F}' \cdot \Delta\vec{r}' = \vec{F} \cdot (\Delta\vec{r} + \vec{u}\,\Delta t) = \vec{F} \cdot \Delta\vec{r} + \vec{F} \cdot \vec{u}\,\Delta t = W + \vec{F} \cdot \vec{u}\,\Delta t \ .$$

So, fun fact, if the new frame is moving perpendicularly to the force, the work in each frame is the same. Continuing,

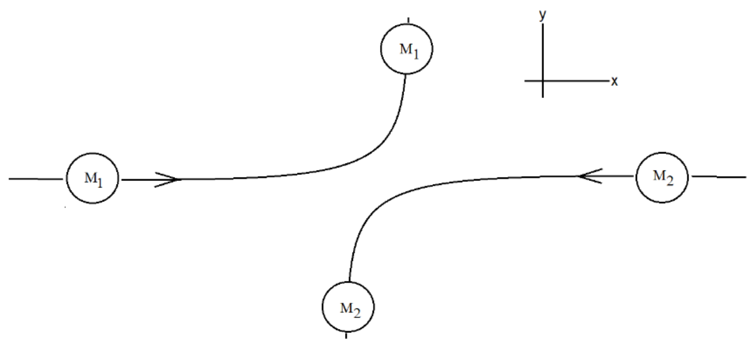
$$W' = \tfrac{1}{2}m\left(v_f^2 - v_i^2\right) + \left(\vec{F}\,\Delta t\right) \cdot \vec{u} = \tfrac{1}{2}m\left(v_f^2 - v_i^2\right) + m(\vec{v}_f - \vec{v}_i) \cdot \vec{u}$$
$$= \tfrac{1}{2}m\left(v_f^2 - v_i^2 + 2(\vec{v}_f - \vec{v}_i) \cdot \vec{u} + u^2 - u^2\right) =$$

$$\tfrac{1}{2}m(|v_f + u|^2 - |v_i + u|^2) = \tfrac{1}{2}m\left(v_f'^2 - v_i'^2\right) = \Delta K' \ .$$

So, in any frame, the work done in that frame is the change in kinetic energy in that frame, but certainly not necessarily the same change as in another frame, as we expected.

ADMONITION*

When we discussed totally inelastic collisions, we made the point that we could treat a three-dimensional problem as three separate one-dimensional problems. You were warned, however, not to treat totally elastic problems that way. Let's discuss briefly why we can not simply use the chart method above three times, one for each direction.




The derivation that resulted in those relationships for the final velocities required the total kinetic energy to be conserved. To split the solution up into three separate parts would require that the contributions to the kinetic energy due to motion in any one of the directions would also need to be conserved, which is a much stricter requirement. Here is an illustration of a two-dimensional situation in which this strict requirement would not be met. Consider two masses heading toward each other that undergo a glancing collision, as shown. Before the interaction, there is kinetic energy due to the motions in the x-direction and none due to the y-motion. After, however, the situation is reversed. So the kinetic energy overall is conserved, but it is not conserved independently in each direction. Consequently, the relationships we have been using are not valid for anything other than a one-dimensional collision.

DISCUSSION 7-6

Does this mean that it is impossible to solve two-dimensional totally elastic collision problems? What is the general rule for solving algebraic systems of equations?

We can solve any of these problems so long as we have enough information and patience, although the solution may be difficult algebraically. Let's look at two situations.

Consider two masses that collide totally inelastically in three dimensions. Given the masses and the initial velocities, can we find the final velocities?

$$m_1 v_{1xi} + m_2 v_{2xi} = (m_1 + m_2)\, v_{xf}$$

$$m_1 v_{1yi} + m_2 v_{2yi} = (m_1 + m_2)\, v_{yf}$$

$$m_1 v_{1zi} + m_2 v_{2zi} = (m_1 + m_2)\, v_{zf}$$

Three equations and three unknowns; we're good. In fact, we did a two-dimensional example earlier.

Consider two masses that collide totally elastically in three dimensions. Given the masses and the initial velocities, can we find the final velocities?

$$m_1 v_{1xi} + m_2 v_{2xi} = m_1 v_{1xf} + m_2 v_{2xf}$$

$$m_1 v_{1yi} + m_2 v_{2yi} = m_1 v_{1yf} + m_2 v_{2yf}$$

$$m_1 v_{1zi} + m_2 v_{2zi} = m_1 v_{1zf} + m_2 v_{2zf}$$

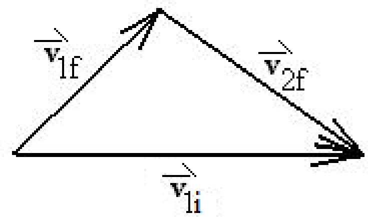
$$\begin{aligned} &\tfrac{1}{2} m_1\left(v_{1xi}^2 + v_{1yi}^2 + v_{1zi}^2\right) + \tfrac{1}{2} m_2\left(v_{2xi}^2 + v_{2yi}^2 + v_{2zi}^2\right) \\ &\qquad = \tfrac{1}{2} m_1\left(v_{1xf}^2 + v_{1yf}^2 + v_{1zf}^2\right) + \tfrac{1}{2} m_2\left(v_{2xf}^2 + v_{2yf}^2 + v_{2zf}^2\right) \end{aligned}$$

Here, unfortunately, we have six unknowns, but only four independent equations. We need more information.

EXAMPLE 7-6

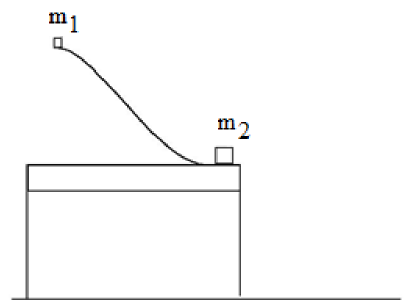
> Let's look at a very special case, that of the masses being equal and mass two initially at rest. This can be made into a two-dimensional problem, since all of the momentum vectors line in a plane (you had a question on Sample Exam One along these lines). We'll write the equations for conservation of momentum (in vector form) and kinetic energy.
>
> $$m_1 \vec{v}_{1i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \quad \rightarrow \quad \vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$$
>
> $$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \quad \rightarrow \quad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

The first equation says we can make a triangle with the velocity vectors like the one at right, and the second, which looks a lot like the Pythagorean theorem, is only going to be true if the triangle is a right triangle, so that $\vec{v}_{1f}$ and $\vec{v}_{2f}$ are at right angles to one another, a nice result. Notice however, that this does not give us the actual directions or magnitudes of the velocities; to know those, we need more information.




EXERCISE 7-4

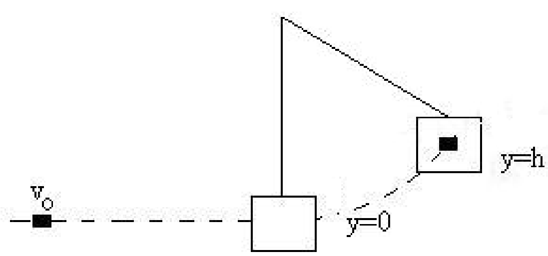
Here is a nice synthesis problem. It requires you to choose which of the three 'pictures' we have developed to use in each section. Keep in mind that the three pictures are essentially identical, but that one may be much more convenient to use than the other two in a given situation.




A 0.2 kg block ($m_1$) is released from rest at the top of a frictionless, curved track 1.5 meters above the top of a 1.1 meter high table. At the bottom of the track, where it is horizontal, this mass collides elastically with a 0.8 kg mass ($m_2$) that is initially at rest. How far from the base of the table does the 0.8 kg mass land?

HOMEWORK 7-6

The *ballistic pendulum* is a device used to measure the muzzle velocity of a bullet. A block of wood of mass M is suspended by a string from the ceiling, and the bullet of mass m is fired horizontally into it. As the block moves backward with the embedded bullet, it swings upward to some maximum height.[5] If the bullet has mass 2 g, the block has mass 2.5 kg, and the block/bullet combination rises through a vertical distance of 6 cm, find the initial speed $v_o$ of the bullet.



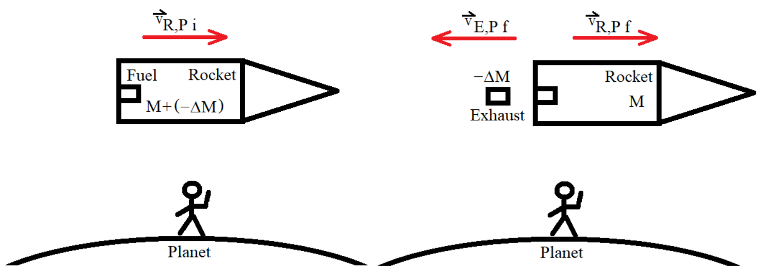

## Outside the Safety Zone*

Calculus-based textbooks often wrap this section up with *the rocket equation*. But we won't need calculus because we've already solved the pertinent equation in Section 5. So, let's concentrate on the physics instead.

DERIVATION 7-6*

[5] When I was much younger, I taught at a school out west where this was actually done with a .22 in lab class.

Consider a rocket of mass M travelling past a planet at velocity $\vec{v}_{R,P}$. At that time, it has just ejected some exhaust (burnt fuel, mass = -ΔM) at a velocity $\vec{v}_{E,R}$ relative to itself and at velocity $\vec{v}_{E,P} = \vec{v}_{E,R} + \vec{v}_{R,P}$ relative to the planet.




Next comes what I think is the really tricky part: the mass of the rocket changes by amount ΔM, which is a negative quantity, but the mass of the ejected exhaust is must be positive, so -ΔM. Let's make use of conservation of momentum, while ignoring gravity, from just before fuel ejection to just after. During that process, the rocket's velocity increases by amount $\Delta\vec{v}_{R,P}$. Let's make 'to the right' in the figure be positive.

$$(M + (-\Delta M))v_{R,Pi} = Mv_{R,Pf} + (-\Delta M)v_{E,Pf}$$

$$(M + (-\Delta M))v_{R,Pi} = M\left(v_{R,Pi} + \Delta v_{R,P}\right) + (-\Delta M)\left(v_{R,Pi} - v_{E,R}\right)$$

We can cancel out quite a few terms, and let's drop the 'initial' subscript:

$$0 = M\Delta v_{R,P} + \Delta M\, v_{E,R}\ ,$$

$$\frac{\Delta M}{\Delta v_{R,P}} = -\frac{1}{v_{E,R}}\, M\ .$$

We would like to make this process happen continuously, so we'll take the limit as $\Delta v_{R,P} \rightarrow 0$, which if nothing else would give a smooth ride to the rocket's crew.[6] Remember that we worked out that the solution to such an equation,

$$\lim_{v_{R,P}\rightarrow 0}\frac{\Delta P}{\Delta q} = CP\ ,$$

is

$$P(q) = P_o e^{Cq}\,,$$

so that

$$M(v) = M_o e^{-\frac{v - v_o}{v_{E,R}}}\ .$$

Here, $M_o$ is the mass of the rocket and all of its fuel at the start of the problem when its speed is $v_o$, and M is the mass of the rocket and its <u>unexpended</u> fuel when the speed is v. It looks a bit strange perhaps because there is no explicit time dependence.

EXAMPLE 7-6*

[6] Unlike Project Orion.

The spaceship HMCSS Clark is 'at rest' and fully fueled at Space Station TALC. Her mass is $2x10^7$ kg, with all but 2 per cent of it fuel. Her engine expels exhaust at 3 km/s. What maximum speed can she attain relative to Station TALC?

Starting with our previous result, re-arranging, and setting M = 0.02 $M_o$,

$$M(v) = M_o e^{-\frac{v-v_o}{v_{E,P}}} \quad \rightarrow \quad v = v_o + v_{E,R} \ln\frac{M_o}{M} = 0 + 3000 \ \ln\frac{1}{0.02} = 11{,}736 \text{ m/s} \ .$$

HOMEWORK 7-7*

In a severe pinch, Space Force decides to utilize *Lenkflugkörper NG* missiles to defend earth from the Jovian attackers during a deep space battle. The missiles themselves have a mass of 3 kg and contain an additional 22 kg of fuel with an exhaust velocity of 465 m/s. They must reach a speed of 700 m/s relative to the launching space vessel. What is the largest payload that could be attached to one?

---

EXERCISE 7-1 Solution

Assuming the two masses form a closed system, conservation of momentum seems appropriate. Also, because they have a common final velocity,

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi} = (m_1 + m_2)\vec{v}_{xf} \ .$$

$$\vec{v}_{xf} = \frac{m_1\vec{v}_{1xi} + m_2\vec{v}_{2xi}}{m_1 + m_2} = \frac{6(+4) + 3(-8)}{6 + 3} = 0 \text{ m/s} .$$

EXERCISE 7-2 Solution

$$K_i = \tfrac{1}{2}m_1 v_{1i}^2 + 0 = \tfrac{1}{2}5\,(7^2) = 122.5 \text{ J}$$

$$K_f = \tfrac{1}{2}(m_1 + m_2)v_f^2 = \tfrac{1}{2}8\,(4.38^2) = 76.7 \text{ J}$$

EXERCISE 7-3 Solution

Are your answers 3 m/s and 8 m/s? Did you confirm that the problem meets the criteria for using the two relationships derived in class? Does it?

EXERCISE 7-4 Solution

This problem has three parts. There is $m_1$ sliding down the incline. There is the collision. There is the trajectory of $m_2$ as it travels toward the floor. Each of these is best treated with one of the three pictures we have discussed.

As $m_1$ slides down the ramp, it is acted on by a normal force and by its weight. There is no friction. We have no details about the actual shape of the ramp, and apparently we do not care how much time it takes for the mass to reach the bottom of the ramp. This looks like a job for work-energy!

$W_N = 0$ (the normal force is always perpendicular to the path)
$W_g$ – conservative

$$0 = \tfrac{1}{2}m_1 v_f^2 - \underbrace{\tfrac{1}{2}m_1 v_i^2}_{\text{starts from rest}} + gm_1y_f - gm_1y_i$$

Let's put y = 0 at the foot of the table. We want to find $v_f$, the speed of $m_1$ just before the collision.

$$0 = \tfrac{1}{2}v_f^2 + gy_f - gy_i$$

$$v_f = \sqrt{2g(y_i - y_f)} = \sqrt{2(10)(2.6 - 1.1)} = 4.47\,\text{m/s}\,.$$

The second part of the problem is a collision, and that screams for conservation of momentum. During the interaction between the masses, they are moving horizontally with no external horizontal forces acting on them. There are vertical external forces (the weights and the normal forces from the ramp), but that doesn't preclude conservation of momentum in the horizontal direction. Because it's a totally elastic collision in one dimension with mass 2 initially at rest, we can jump right to the relationships we derived for just such a situation:

$$v_{2xf} = \frac{2m_1}{m_1 + m_2}v_{1xi} = \frac{2(0.2)}{0.2 + 0.8}4.47 = 1.79\,\text{m/s}\,.$$

The last part of the problem is projectile motion. Let's put the origin at the foot of the table, with +x to the right and +y upward. Our inventory is

$x_i = 0$ m
$x_f = ?$ ←
$v_{xi} = +1.79$ m/s
$v_{xf} = +1.79$ m/s
$a_x = 0$ m/s$^2$
$t = ?$

Since there is not enough information on the x-side, we need to look to the y-side and try our 80% Rule.

$y_i = 1.1$ m
$y_f = 0$ m
$v_{yi} = 0$ m/s (the ball was travelling horizontally as it left the table)
$v_{yf} = ?$
$a_y = -10$ m/s$^2$ (we chose upward to be positive)

$t = ?$

KEq. 3:

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

This will become a quadratic equation in t. Inserting the numbers and re-arranging to the standard format leaves us with

$$(5)t^2 + (0)t + (-1.1) = 0\,,$$

which, it turns out, we can solve directly:

$$t = \pm\sqrt{\frac{1.1}{5}} = +0.47 \text{ seconds}\ .$$

Take this back to the x-side to find $x_f$.

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 = 0 + 1.79(0.47) + 0(0.47^2) = 0.84 \text{ m}\ .$$

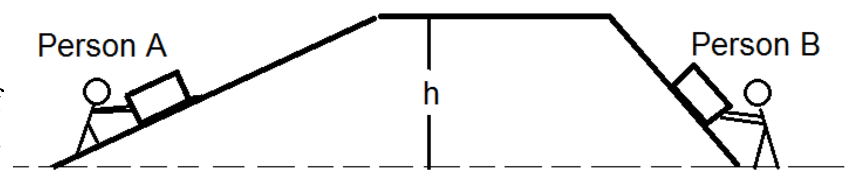
Sample Exam III

**MULTIPLE CHOICE** (4 pts each)

1) According to statistical data, the probability that an occupant of an automobile suffers a lethal injury during an accident is proportional to the square of the speed of the car (*i.e.*, to the KE!). If the probability of death is 3% at 50 mph, what is the probability of death at 75 mph?

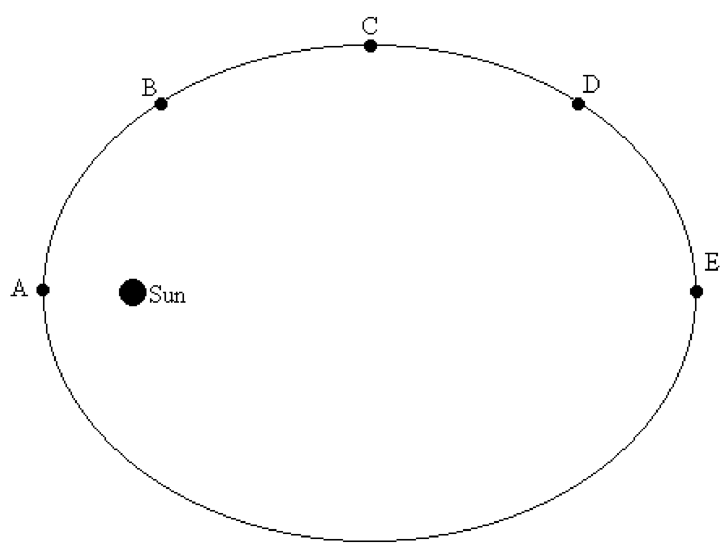
   A) 3%  B) 4.5%  C) 6.8%  D) 9%  E) 14.3%

2) Consider two identical boxes being pushed *very slowly* up frictionless ramps, one gentle, the other steep. Compare the amount of work each person does in pushing his box up his ramp. Assume the applied forces are each applied parallel to the respective ramp.




   A) A does more work than B.
   B) B does more work than A.
   C) A and B do the same amount of work.
   D) It's impossible to know who does more work.
   E) There is no Choice E.

3) The earth orbits the sun on a path that is not circular, but elliptical, as shown with great exaggeration in the figure. The gravitational force from the sun on the earth keeps the earth in orbit. At which of the labeled points will the earth's speed be least?




   A) A
   B) B
   C) C
   D) D
   E) E

4) Magnetic fields exert forces on electrical charges that are always perpendicular to the velocity of the charge and perpendicular to the field itself. Therefore, the power transferred by the magnetic field to a charged particle

A) is always positive
B) is always negative
C) is always zero
D) depends on the sign of the charge.
E) depends on the speed of the particle.

5) Consider two flowerpots in windows of an apartment building. Pot A is knocked off the third floor window ledge by Mr Smith's cat, and it hits the pavement below. Pot B, which has half the mass of Pot A, is knocked of its twelfth floor ledge by Mrs Jones's goldfish and it also hits the pavement. Which of these statements is correct?

A) Pot B will hit the pavement with twice the speed and twice the kinetic energy as does Pot A.
B) Pot B will hit the pavement with the same speed and twice the kinetic energy as does Pot A.
C) Pot B will hit the pavement with twice the speed and the same kinetic energy as does Pot A.
D) Pot B will hit the pavement with twice the speed and four times the kinetic energy as does Pot A.
E) Pot B will hit the pavement with four times the speed and twice the kinetic energy as does Pot A.

**PROBLEM I** (20 pts)

Derive the Work-Energy Theorem, $W_{TOTAL} = \Delta KE$. Consider the problem in only one dimension. Make use of the following relationships:

$KE = ½ m v^2$ $W = F \Delta x$ (forget the cosine term for this derivation)

Show all work and justify any assumptions.

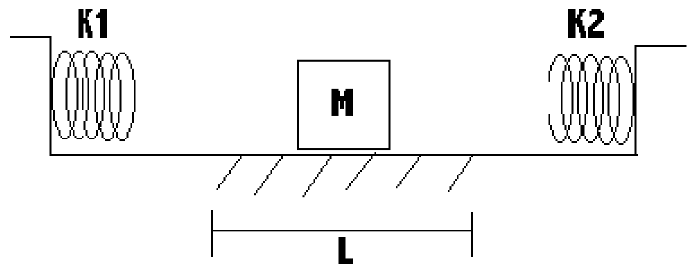
**PROBLEM II** (20 pts)

Andy and Bonnie fire identical caliber rifles (of reasonable lengths) using identical shells. The barrel of Andy's rifle is 2 cm longer than the barrel of Bonnie's rifle. The force of the expanding gases in the barrel accelerate the bullets.

Which bullet (if either) will have a higher muzzle velocity than the other? Be sure to explain fully (or at least sufficiently).

**PROBLEM III** (20 points)

Consider two cars on in an icy road that undergo a completely elastic *head-on* collision. Car 1 has mass $M_1$ = 3500 kg and initial speed $v_{1i}$ = 15 m/s. Car 2 has mass $M_2$ = 2000 kg and initial speed $v_{2i}$ = 10 m/s. Use the technique of relative velocities to determine the velocities of each car after the collision.




**PROBLEM IIII** (20 pts)

Consider two massless springs with *different* spring constants, $k_1$ = 300 N/m and $k_2$ = 500 N/m. A mass M=4 kg is pressed against the left spring, compressing it by 0.2 m. The mass is released from rest, slides across a rough portion of the floor (length L=0.4m and coëfficient of kinetic friction $\mu_K$ = 0.25) and hits the spring on the right. By how much is the second spring compressed?

## Section 9 – Rotation in a Plane

Rotation is generally fairly complicated to study and usually involves yet another type of quantity beyond scalars and vectors, called *tensors*. However, we will restrict ourselves to rotation in a plane, usually the x-y plane, which should make things a bit easier.

## The Center of Mass

Up to now, we have been treating objects as point masses. That is, if we were asked the location of an object, we could respond with as little as a single number, such as x = 3.576 meters. Extended objects, on the other hand, occupy many locations. A car may be said to be between x= 4.582m (at the tip of the front bumper) and x = 8.935 m (at the rear end bumper). Even that doesn't give many details. So, we often speak of the average position of a car. We don't mean that in the sense of a trip from Baltimore to Philly, where the average position is in Wilmington, but average in the sense of examining each particle composing the car and averaging their positions. We call this position the *center of mass*[1] of the object, and it is calculated in much the same way that the average on an exam is found.

DISCUSSION 9-1

> How do we find the average on an exam? If nineteen students each earn one hundred and one earns a zero, is the average fifty? If not, what is?

For an exam, we take each possible grade ($G_n$) from zero to one hundred and weight the importance of each with the number of students who earned that grade ($N_n$), then divide by the total number of students. Or, if we were to share all the earned points equally among all students, how many would each get?

$$G_{AVE} = \frac{\sum_n N_n G_n}{\sum_n N_n}$$

Instead, we look at every possible position and weight each by how much mass is located there. Clearly, there are many positions at which there is no mass, and we usually just skip them:

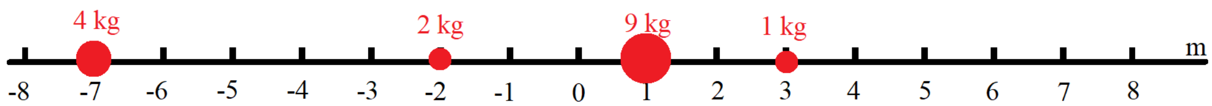
$$x_{CM} = \frac{\sum_n m_n x_n}{\sum_n m_n} \quad .$$

Because we like to ease into new things, let's start by finding the center of mass of a bunch of point masses.

EXAMPLE 9-1

[1] You may also hear the term *center of gravity*. So long as the gravitational field is uniform, these points are the same.

Find the center of mass of these four point masses.




$$x_{CM} = \frac{\sum_n m_n x_n}{\sum_n m_n} = \frac{4(-7) + 2(-2) + 9(+1) + 1(+3)}{4 + 2 + 9 + 3} = \frac{-20}{18} = -1.11\ \text{m}\ .$$

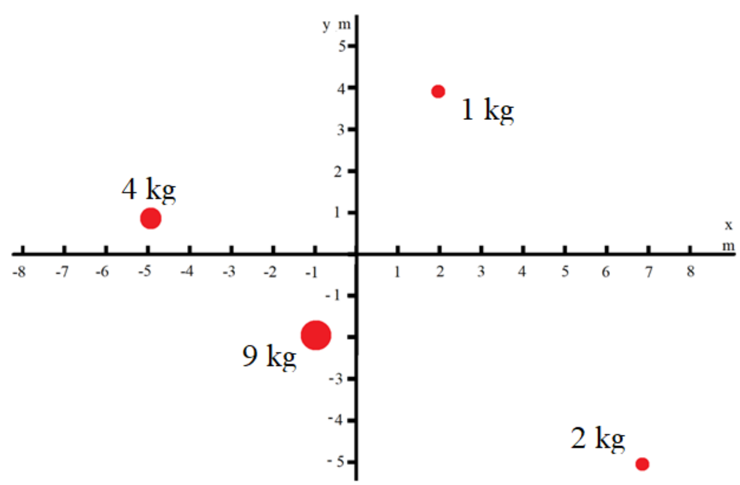
Of, course, not every object is composed of a linear arrangement of masses. In three dimensions, we can represent the location of each object by the location vector $\vec{r}$, so that

$$\vec{r}_{CM} = \frac{\sum_n m_n \vec{r}_n}{\sum_n m_n}\ ,$$

but in reality, this is just a way of writing three separate equations for $x_{CM}$, $y_{CM}$, and $z_{CM}$.

HOMEWORK 9-1

Find the center of mass of the four masses shown in the figure. The coördinates are the integers they appear to be.




What about objects that are not made up of discrete point masses, but rather a continuous structure? In principle, it's the same process, although the process of performing the calculations may be quite difficult. Luckily, we don't need many shapes to get out points across in this course.

DISCUSSION 9-2

Consider, for example, a thin uniform rod of length L and mass M, like a meterstick. Where do you suppose is the center of mass? Can you make an argument to support your contention?

We can make use of the notion of *symmetry* to make an argument about a few simple shapes. If we initially suppose that the center of the rod is at its physical center, we see that for every bit of mass on the left side, there is an equal amount of mass on the right side at exactly the same distance out. The positions of these two masses will average out to the center of the rod, the average of the averages will of course also be at the center. We can use this argument for other shapes, such as a uniform hoop, uniform disk, and uniform sphere.

DISCUSSION 9-3

> What if the rod above were half made of maple and half made of pine? Where would the center of mass be?

One last thing: although we will be dealing with extended objects for the next few sections, we'll restrict ourselves to *rigid objects*, where the individual pieces always maintain the same distances to all of the other pieces in the object. So, a hammer would be considered a rigid object, but a bucket's worth of water thrown across the room would not. We should probably be a bit more careful with that definition, but it's good enough to get the idea across.

The center of mass has one especially remarkable property that makes life a bit easier for us.

DERVIATION 9-1

> Consider a system of masses $m_n$ with total mass $M = \Sigma_n m_n$ :

$$\vec{r}_{CM} = \frac{\sum_n m_n \vec{r}_n}{\sum_n m_n} \ ,$$

$$\left(\sum_n m_n\right)\vec{r}_{CM} = \sum_n m_n \vec{r}_n \ ,$$

$$M\,\vec{r}_{CM} = \sum_n m_n \vec{r}_n \ .$$

> Now, find the ITRC[2] twice for each side. The first makes the positions into velocities and the second makes the velocities into accelerations.

$$M\,\vec{a}_{CM} = \sum_n m_n \vec{a}_n \ .$$

> Now, let's consider all of the forces acting on any one of the masses. For the $n^{th}$ one, we have

$$\sum_m \vec{F}_{n,m} = m_n \vec{a}_n \ .$$

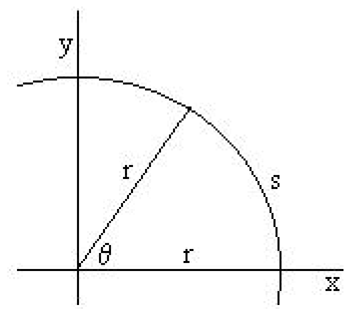
> If we add up all of those equations, we get

$$\sum_n \sum_m \vec{F}_{n,m} = \sum_n m_n \vec{a}_n = M\,\vec{a}_{CM} \ .$$

[2] ITRC is the instantaneous time rate of change.

In other words, the forces acting on any of the parts of the collection of masses will accelerate the center of mass as if it were a single point particle of mass M. This is how we got away with the first half of the course.

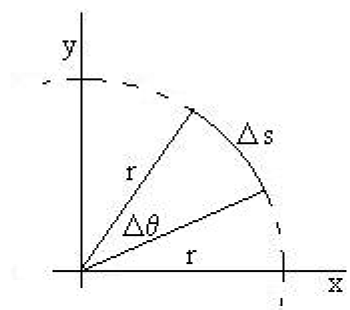
## Rotational Kinematics



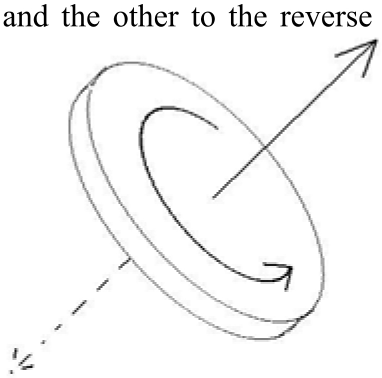

Before we proceed, let's review a bit from Section 3. Consider a point mass m free to move about a circle of radius r. First, we need to be able to specify the object's position. For this, we will return to our convention of measuring angles CCW from the x-axis. However, we will for now on think of such angles in *radians*, not degrees. A *radian* is the angle such that the *arclength* s subtended is equal to the radius r, or about 57.3°. Clearly, if we halve the angle, we also halve the distance along the arc, so that theta and s are proportional by the factor r:

$$\mathbf{s = r\,\theta}\ .$$

So, there are then $2\pi$ radians in a circle, since the circumference is $2\pi r$.




We should next find a way of describing changes in the position, or the *angular displacement*, $\Delta\theta = \theta_f - \theta_i$, so that $\Delta s = \Delta\theta\ r$. Since linear displacement was a vector, we should require the angular displacement to be as well. The magnitude of $\Delta\vec{\theta}$ will of course indicate how much the object has turned. The direction of $\Delta\vec{\theta}$ will tell us two things: the plane in which the object rotated, and the direction in which it rotated. A plane can be defined by a vector that is perpendicular to the plane, and luckily, there are two directions, one of which we'll assign to rotation in one direction and the other to the reverse direction. The choice is arbitrary, but we'll want to match what everyone else does. We can remember which is which by using our right hands. Curl your fingers like little arrows in the direction of rotation, and your thumb will point in the direction of the vector $\Delta\vec{\theta}$. Be sure to use your right hands. Most of the problems you'll encounter here are with objects rotating in the plane of the page; in that case, $\Delta\vec{\theta}$ out of the paper (motion is CCW) is considered to be positive, and $\Delta\vec{\theta}$ into the page (CW motion) is considered to be negative. Of course, as always, you can change this for your convenience so long as you are clear and consistent.



We're going to work our way through the analogs of all the quantities we discussed in terms of linear motion. We continue with the *angular velocity*, the angular displacement *per* unit time:

$$\vec{\omega}_{AVE} = \frac{\Delta\vec{\theta}}{\Delta t} \;\; ; \;\; \vec{\omega}_{INST} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} \;\; ; \text{ The direction of } \vec{\omega} \text{ is the same as for } \Delta\vec{\theta} \;\; .$$

Looking back to Section 3, a point on the rotating object will possess a speed tangent to its path given by

$$v_T = \omega r \;\; .$$

Likewise, we can define the angular acceleration as the time rate of change of the angular velocity:

$$\vec{\alpha}_{AVE} = \frac{\Delta\vec{\omega}}{\Delta t} \;\; ; \;\; \vec{\alpha}_{INST} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} \;\; .$$

A piece of this rotating object will have a tangential acceleration, $a_T$, given by

$$a_T = \alpha r \;\; .$$

Determining the direction of the angular acceleration alpha is sometimes difficult. Remember what we said back in Section 2: if an object (moving in one dimensions) is speeding up, $\vec{v}$ and $\vec{a}$ are in the same direction, while if it is slowing down, $\vec{v}$ and $\vec{a}$ are opposite. Do the same for $\vec{\omega}$ and $\vec{\alpha}$. We'll leave problems when this is otherwise until your junior year Physics class.

Of course, there is also the angular jerk, the angular kick, and the angular lurch, *et c*.

If we assume that there are situations where the angular acceleration is constant, we can derive some kinematic relationships. Since θ, ω, and α share the same relationships as x, v, and a, we need not actually perform these derivations, but simply replace each linear quantity with the analogous rotational quantity:

$$\vec{\omega}_f = \vec{\omega}_i + \vec{\alpha}t$$

$$\vec{\omega}_{AVE} = \frac{\vec{\omega}_f + \vec{\omega}_i}{2}$$

$$\Delta\vec{\theta} = \vec{\omega}t + \tfrac{1}{2}\vec{\alpha}t^2$$

$$\omega_f^2 = \omega_i^2 + 2\vec{\alpha} \cdot \Delta\vec{\theta}$$

If we restrict ourselves to rotation in a single plane, we can use the same notation (+ or -) for the direction of each vector and drop the dot product in KEq 4.

EXAMPLE 9-2

A wheel starts from rest and starts to spin with an angular acceleration of 2.5 rad/s$^2$. After 34 seconds, what is the angular speed and through what angle has it turned?

We treat the problem the same as we did linear kinematic problems, by constructing a table. We weren't told which way the wheel is spinning, so let's just make that the positive direction.

$\theta_i = 0$ (make that the origin)
$\theta_f = ?$ ←
$\omega_i = 0$ (starts from rest)
$\omega_f = ?$ ←
$\alpha = +2.5$ rad/s$^2$
$t = 34$ sec

KEq 1 gives us the final velocity directly:

$$\omega_f = \omega_i + \alpha t$$

$$\omega_f = 0 + 2.5(34) = 85\,\frac{\text{rad}}{\text{s}}.$$

KEq 3 gives us the displacement directly:

$$\Delta\theta = \omega t + \tfrac{1}{2}\alpha t^2$$

$$\Delta\theta = 0(34) + \tfrac{1}{2}(2.5)(34^2) = 1445 \text{ radians}.$$

Seem familiar? Try this one.

EXERCISE 9-1

A wheel is turning at 30 rad/s but slows and reverses direction to 40 rad/s. It does so while turning through a net 500 revolutions. How much time did this take and what as the acceleration?

HOMEWORK 9-2

An object initially rotating at an angular speed of 1.8 rad/sec turns through 50 revolutions during the time it experienced an angular acceleration of 0.3 rad/s$^2$. For how much time did the acceleration last and what was the final angular speed?

## Torque

DISCUSSION 9-4

Suppose you go home this evening, open the fridge, and take out a container of your favorite beverage. How will you open it? For some of you, applying a force will be sufficient, but would that work for everyone? What must the rest of you do?

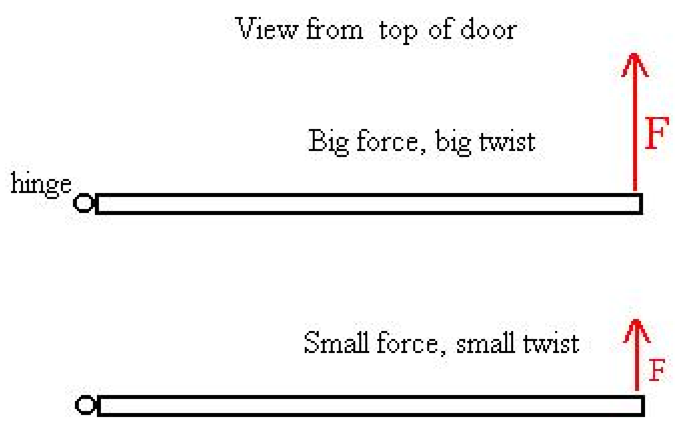
Back in Section Five, we saw that Newton's second law of motion says that a net force is necessary in order for an object to have an acceleration. We might expect a similar necessary condition in order for an object to have an angular acceleration. Instead of a 'push' or 'pull,' it requires a 'twist.' Physics talk for a twist is *torque*, represented by the Greek letter $\tau$ (tau). So, we might guess that, in analogy with NII, that the net torque and the acceleration are proportional, and in the same direction:

$$\sum_n \vec{\tau}_n \sim \vec{\alpha} \quad .$$

Let's choose a symbol to make this an equation,

$$\sum_n \vec{\tau}_n = I\,\vec{\alpha} \quad ,$$

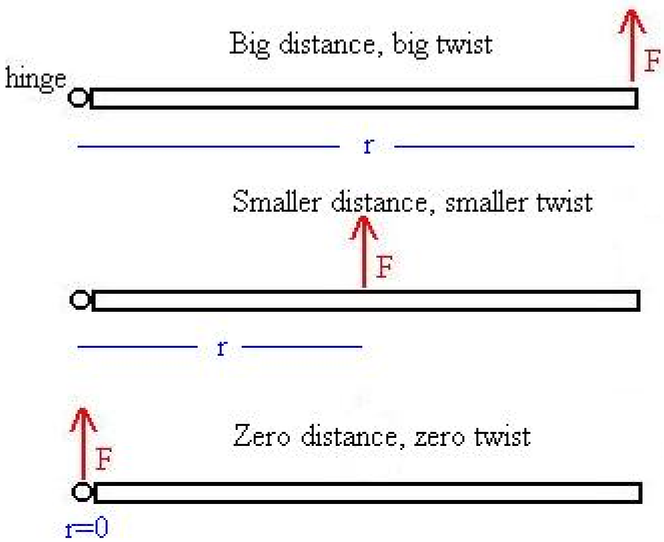
where I is some measure of the object's rotational inertia (how hard it is to accelerate rotationally) in the same way that the mass m is a measure of its translational inertia (how hard it is to accelerate linearly). This quantity I may well be the mass, but let's not jump to any conclusion too soon.




Before we try to justify this relationship, let's see if we can work out exactly what we mean by torque. Remember that we are the ones who get to define things, and if we're clever, what we define might actually be useful.

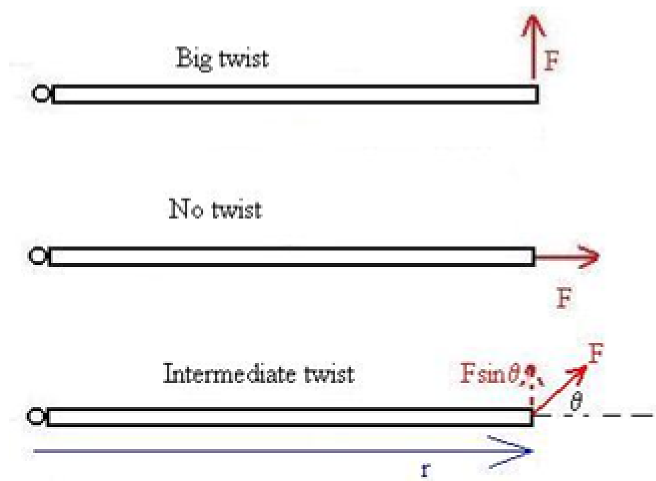
Get yourself a meter stick to play with, if you like, or better yet, walk over to a convenient door. Consider an object free to rotate around a particular axis, such as a door about its hinges. To get the door to begin to accelerate rotationally, it seems clear that a force must be applied. The larger the force, the bigger the twist applied. So, we might guess that $\tau \sim F$.

Where the force is applied also seems to matter. Try pushing the door near the end, then with the same force near the center. See how the former results in more twist than the latter. Pushing near the hinge (axis) results in no twist at all. So, now we might think that $\tau \sim Fr$, where r represents the distance from the axis of rotation to the point of application of the force.




Lastly, we see that there is a dependence on the orientation of the force with respect to the door. Namely, if we pull or push along the length of the door, there is no twist, and we obtain the maximum twist when the force is at right angles to the door. At intermediate angles, it seems clear that we need to take the component of the force which is perpendicular to the r-vector, namely F sinθ, where θ is the angle as shown between the force vector and the r vector. So, perhaps $\tau \sim F\, r \sin\theta$.

We also need to define a direction for the torque (after all, we can twist a bottle cap on or off). Assume that the door in the figures starts from rest in the example above, then starts to turn CCW as a result of the applied force shown. Then, $\Delta\vec{\theta}$ is out of the page, $\vec{\omega}_{ave}$ is out of the page, and $\vec{\alpha}$ is out of the page. Since for Newtonian translational motion, the net force and the acceleration point in the same direction, we will require the net torque and the angular acceleration to do so as well. We see that we can get this result by defining the torque as the *cross-product* of $\vec{r}$ and $\vec{F}$:




$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{or} \quad |\vec{\tau}| = |\vec{r}|\left|\vec{F}\right| \sin\theta_{r,F} \quad \text{(RHR)}\,.$$

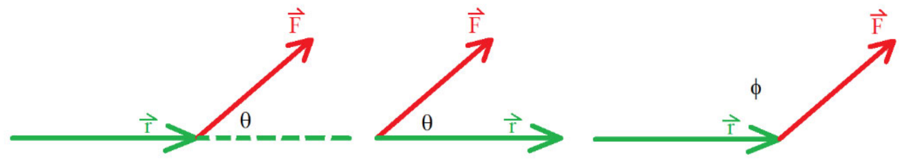
Writing this definition as a cross product is really just shorthand; the second version above reminds you of what you actually need to do. Review the discussion of the right-hand rule (RHR) in Section 1. The order of the subscripts on theta tells you which finger to use for each vector.

EXAMPLE 9-3

Use the right-hand rule to confirm that it gives you the desired direction of the torque in the very last figure above.

Your index finger should be pointed exactly to your right, your middle finger should be at an angle to the right and towards the top of the page, and your thumb should be pointing up out of the page.

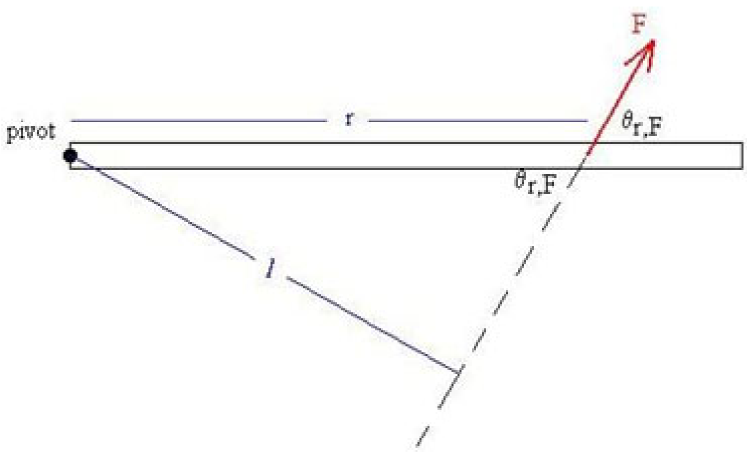
Let's take a moment to clarify something that seems to confuse. Notice that I described the angle theta in the diagram above as being




between $\vec{r}$ and $\vec{F}$. Doesn't look it though, does it? You have to remember that r and F there are not drawn tail to tail, but tail to tip. When drawn correctly, it becomes clearer, as in the middle drawing. What some students do is use the angle labeled phi, which is not the correct angle, although you can see that it more closely meets the expectation of being the angle 'between' the vectors. But here's the thing. Theta and phi are supplementary, and the sines of supplementary angles are the same. So, it really doesn't matter which angle is used, the numerical result will be the same. Since we've now discussed it, it's O.K. with me, just be clear in your solutions.

Now, like everything else we've discussed so far in this course, this result is still tentative, since although we think we know on what factors the torque depends, we don't know the exact dependence. Only testing of the usefulness of this definition will vindicate our work here.

There is no special unit for torque; it's clear then that we can write it in terms of newton-meters. In the U. S. Customary System., the unit is the *pound-foot*, distinct from the *foot-pound*, the unit of work. Which is an interesting point: the dimension of work and of torque are the same, but they quantities are themselves very different (vector *v*. scalar).




Occasionally, the torque will be expressed as the product of a force and its *lever arm*, *l*. A little bit of trigonometry shows that $l = r\sin\theta$, so this definition is equivalent to, and sometimes more useful than, the one given above. The lever arm is found by extending the line of action of the force and finding the perpendicular distance (the lever arm) from the pivot to this line. You must be careful when reading other sources; some define r as the lever arm and it is not.

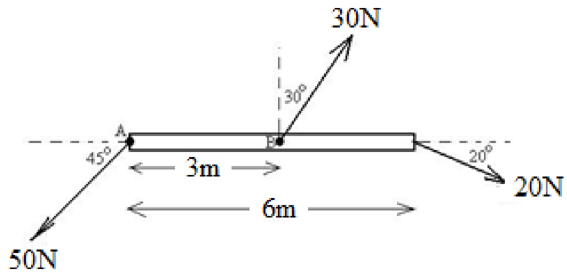
$$\vec{\tau} = \vec{l} \times \vec{F} \quad \rightarrow \quad |\vec{\tau}| = |\vec{l}||\vec{F}| \quad \text{(RHR)} \ .$$

HOMEWORK 9-2

A pendulum consists of a 2 kg bob at the end of a 1.2 m long light string, suspending the bob from a pivot. Calculate the net torque on the bob about the pivot when the string makes a 6° angle with the vertical. Indicate the direction of this net torque. Your answer to the direction will depend on how you draw your figure.

HOMEWORK 9-3

Calculate the net torque of these forces about an axis through Point A that is perpendicular to the length of the rod. Repeat for an axis through Point B.




## The Second Law for Rotation and the Moment of Rotational Inertia

Following up on the notion of the existence of an analogy between linear and rotational motion, we might suspect that that there is a relationship similar to Newton's second law,

$$\sum_n \vec{F}_n = m\,\vec{a} \quad ,$$

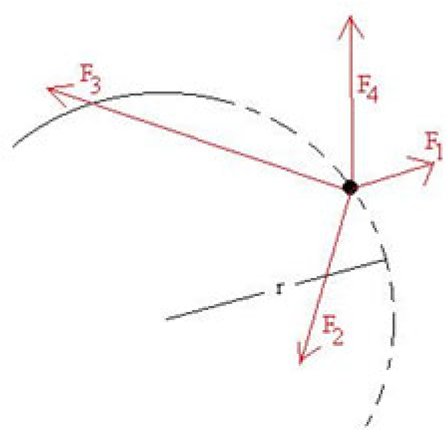
perhaps of the form

$$\sum_n \vec{\tau}_n = I\,\vec{\alpha} \quad ,$$

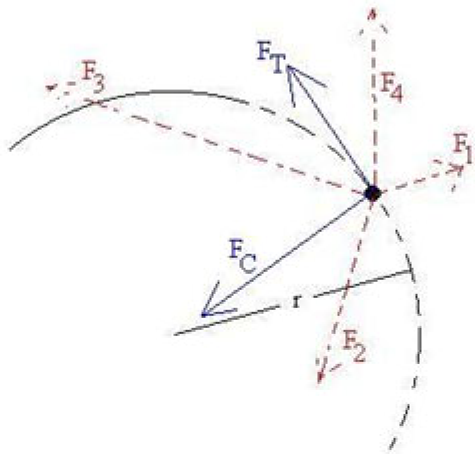
where I is a constant whose meaning we still need to divine, but which we suspect might be a measure of how difficult it is to accelerate rotationally some object, in the same way that an interpretation of the mass is as a measure of the difficulty of altering an object's linear velocity.

DERIVATION 9-2

Consider an object (point mass) constrained (for now) to move along a circular path, to which forces are applied. However many forces are applied, they can be added and resolved into components which are either centripetal or tangential, resulting in net force components as shown in the figure below. The centripetal component is what keeps the object moving in a circle and is of no particular interest to us just now. The tangential component, however, will accelerate the object <u>along</u> the circle, that is, tangentially:




$$\sum_n F_{Tn} = m\,a_T \quad .$$




Let's multiply both sides of the relationship by the radius of the circle, because, well why not?

$$r \sum_n F_{Tn} = m\, a_T\, r$$

Distribute the r to get

$$\sum_n r\, F_{Tn} = m\, a_T\, r \ .$$

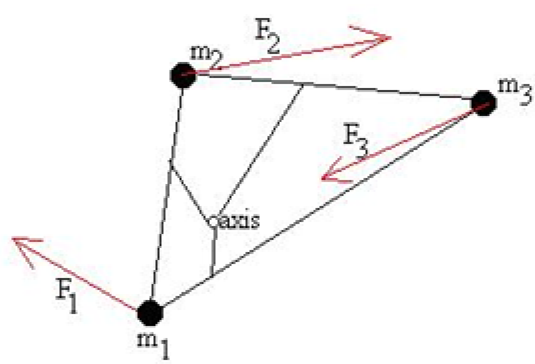
Since every tangential force component is (by definition) perpendicular to the radius r, we recognize the terms in the sum to be the torques exerted by each of the forces, and we remember that $a_T = \alpha\, r$, so that

$$\sum_n \tau_n = m\,(\alpha r)\, r = mr^2\,\alpha \ .$$

So, <u>in this very special case</u>, we see that a rotational form of Newton's second law holds true if the proportionality constant is

$$I_{\text{Point Mass}} = mr^2 \ .$$

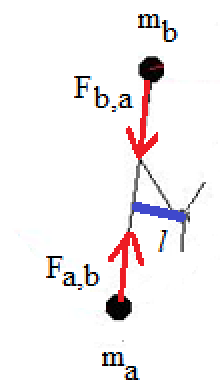
Note that this quantity depends not only on the mass, but on the <u>distribution</u> of the mass. This last comment should become clearer after the next discussion. Suppose we have an object that comprises several point masses which are somehow connected, perhaps with light rigid rods. I drew three, which is enough to make the point, but there could be as many as you like. Without bothering to calculate each torque explicitly, we can safely assume that there will be some torques applied to each object, including external torques due to the forces from other objects (make each the net force, if more than one force is desired), and also internal torques from the other objects, mediated through the rods. For each mass $m_n$, we can write that




$$\sum_m \vec{\tau}_{\text{EXT}\,n\,m} + \sum_m \vec{\tau}_{\text{INT}\,n\,m} = m_n r_n^2\, \vec{\alpha}_n \ .$$

If the objects rotate as a single object about a common axis, then all the $\alpha_n$ 's are the same. Let's add the equations.

$$\sum_n \sum_m \vec{\tau}_{\mathrm{EXT}\,n\,m} + \sum_n \sum_m \vec{\tau}_{\mathrm{INT}\,n\,m} = \left(\sum_n m_n r_n^2\right)\vec{\alpha} \quad .$$




Let's concentrate on the internal torques term. We know from NIII that masses a and b exert forces on each other that are equal in magnitude and opposite in direction. If those forces act along the line between $m_a$ and $m_b$,[3] then the lever arms associated with those forces about the axis are the same, and so the torques generated by those forces will also be equal but opposite in direction. Therefore, the sum of all the internal torques should be zero. The sum of the external torques is just the sum of the torques exerted on the masses as a unit, so we now have that

$$\sum_q \vec{\tau}_{\mathrm{EXT}\,q} = \left(\sum_n m_n r_n^2\right)\vec{\alpha} \quad .$$

from which we see that <u>the moment of inertial of an extended, rigid object is the sum of the moments of its constituent parts</u>:

$$I_{\mathrm{TOTAL}} = \sum_n I_n = \sum_n m_n r_n^2 \quad .$$

Because the value for the moment of inertia depends not only on the mass, but also on the <u>distribution</u> of the mass in an object, the value for I for a given object may well (and probably will) be different for different axes of rotation.

DISCUSSION 9-5 VIDEO

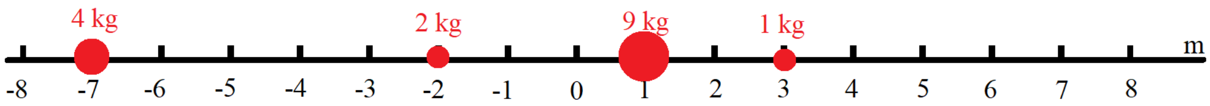
Consider the two cylinders in the video. They have the same mass, but the blue one is much harder to twist around than the red one. Can you explain why?

DISCUSSION 9-6

Pick up a meter stick at its center and try to twist it back and forth. Now try the same thing, but while holding the stick near the end. Which was harder to do? Why?

EXAMPLE 9-4

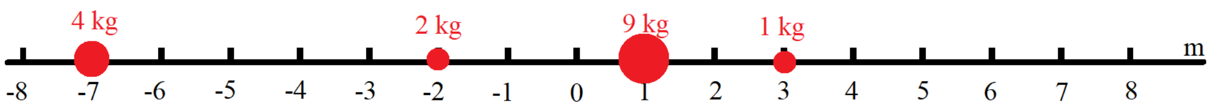
[3] This is a necessary condition for this argument to work. Looking ahead to Semester Two, the forces between atoms in real objects do indeed act along the line of connection, so we should be alright.




Find the moment of inertia of these masses about an axis passing through x = -4 m.

$$I = \sum_n m_n r_n^2 = 4(3^2) + 2(2^2) + 9(5^2) + 1(7^2) = 318 \text{ kg m}^2 \, .$$

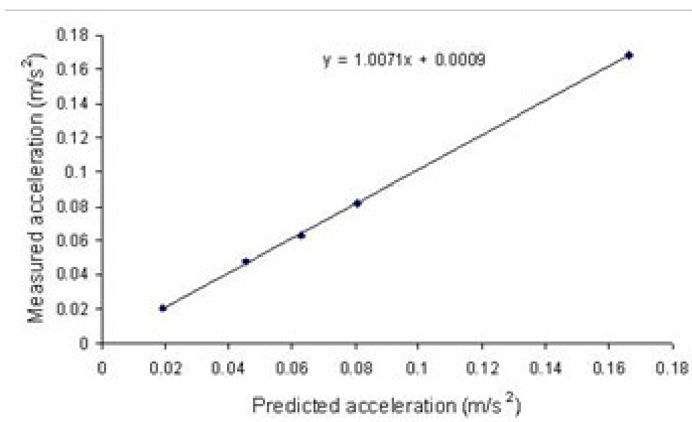
HOMEWORK 9-4




Find the moment of inertia of these masses about an axis passing through x = +1 m. What do you notice about your answer and the answer to Example 9-x?

EXPERIMENT 9-1

Here are the results of an experiment that should give us some confidence that the second law for rotation is true. Similarly to the experiment in Section 5, a hanging mass pulled a string wrapped around a horizontal disc of moment I. The data here are presented a bit differently. The linear acceleration of the falling mass is plotted against the predicted acceleration, based on the concepts discussed above. The results vary from prediction by less than 1 %.

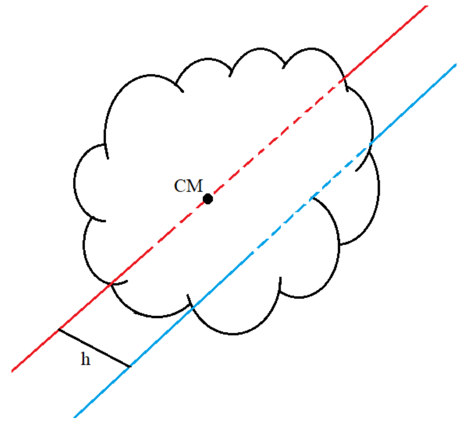



Finding the moment of inertia of an object may be conceptually easy, $I = \Sigma_n m_n r_n^2$, but actually performing this calculation can be quite difficult and is usually accomplished with calculus. As a result, we will restrict ourselves to some common shapes; even then we will need to be fairly clever to determine I for each. Before we start, let's add to our toolbox with two derivations.

The *parallel axis theorem* states that, if one knows the moment of inertia of an object of mass M about an axis passing through the center of mass of an object ($I_{CM}$), then the moment about any other axis parallel to that one is given by

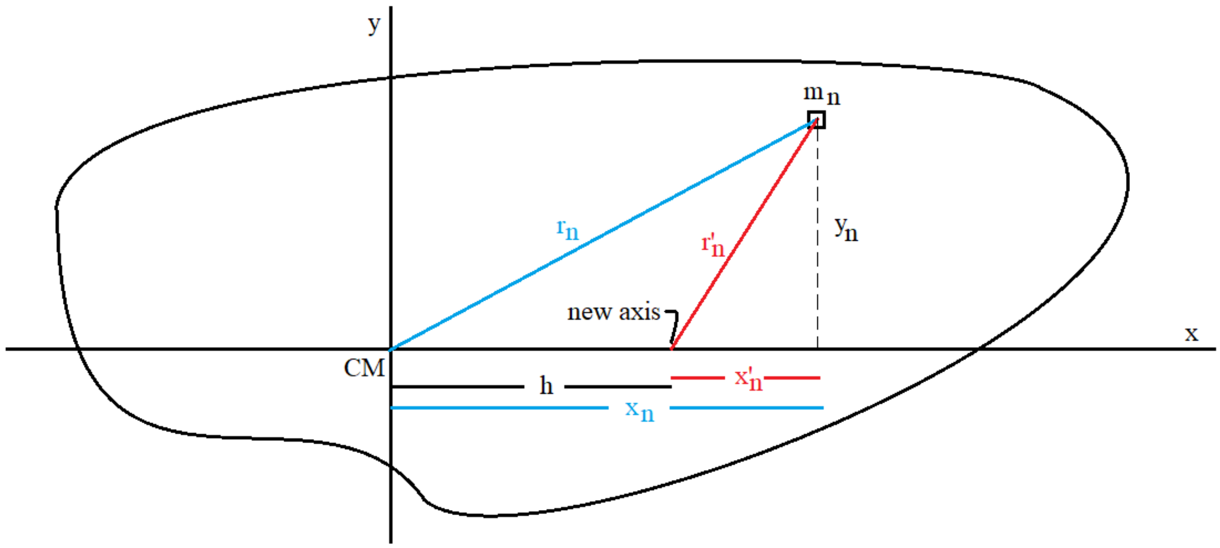
$$I_{PARALLEL} = I_{CM} + Mh^2 \quad ,$$

where h is the distance the second axis is displaced from the first.




DERIVATION 9-3

For simplicity of calculation, place the origin at the center of mass, let the original axis of rotation be the z-axis, and align the x axis along the direction of the displacement of the axis of rotation. That way, $y_n = y_n'$.




The moment about the center of mass is

$$I_{CM} = \sum_n m_n r_n^2 = \sum_n m_n (x_n^2 + y_n^2) \ .$$

The moment about the new axis is

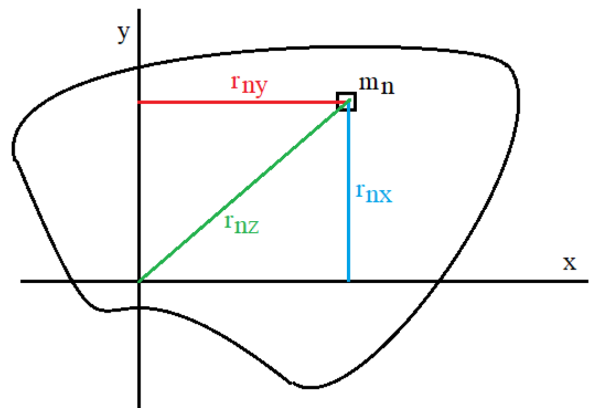
$$I_{PARALLEL} = \sum_n m_n (r_n')^2$$
$$= \sum_n m_n (x_n'^2 + y_n'^2) = \sum_n m_n ((x_n - h)^2 + y_n^2)$$
$$= \sum_n m_n (x_n^2 - 2x_n h + h^2 + y_n^2) =$$
$$= \sum_n m_n (x_n^2 + y_n^2) - 2h \sum_n m_n x_n + \left(\sum_n m_n\right) h^2.$$

The first term we recognize as $I_{CM}$, the third is $Mh^2$, and the second is 2hM times the x coördinate of the center of mass, which we specified was at the origin, so that term is zero. So,

$$I_{PARALLEL} = I_{CM} + Mh^2 \quad .$$

To sum up, the parallel axis theorem is valid for any rigid object of any shape. One should know the moment of inertia about an axis through the center of mass, but that could be any such axis. The moment about any axis parallel to that original axis can be found with the relationship above.

The *perpendicular axis theorem* is valid for conditions very different than for the parallel axis theorem. The object must be infinitesimally thin and flat. The location of the center of mass is irrelevant here. Choose a perpendicular set of x- and y- axes in the plane of the object; these axes do not even need to pass through the object. Suppose that we know the moments of inertia about each of the x- and y-axes, $I_x$ and $I_y$. The moment of inertia about the z-axis, perpendicular to the plane of the object and intersecting the other two axes, is given by




$$I_z = I_x + I_y \quad .$$

DERIVATION 9-4

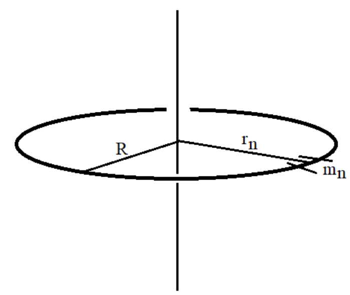
$$I_x = \sum_n m_n r_{nx}^2 = \sum_n m_n y_n^2 \qquad\qquad I_y = \sum_n m_n r_{ny}^2 = \sum_n m_n x_n^2$$

$$I_z = \sum_n m_n (r_{nz}^2) = \sum_n m_n (x_n^2 + y_n^2) = \sum_n m_n x_n^2 + \sum_n m_n y_n^2 = I_x + I_y \quad .$$

We'll return to these theorems later with some examples. Now, though, we're in a position to find the moments of inertia of some common shapes about specific axes. We'll be using a number of approaches, more as a demonstration of the wealth of possibilities for solving problems without using calculus. Some of these are actually easier than using calculus!

EXAMPLE 9-5

Find the moment of inertia of a thin ring of radius R and mass M about an axis through the center perpendicular to the plane of the ring.



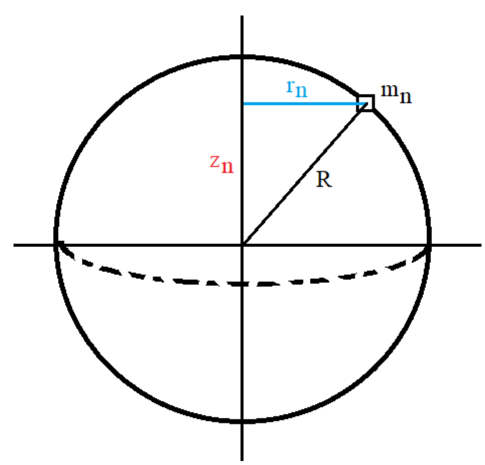

Let's break the ring up into very small masses $m_n$, so small that they seem like point masses. Each is a distance $r_n$ from the axis. We've shown that the moment of each point mass is $m_n r_n^2$ and that the moment of an extended object is the sum of the moments of the individual parts. In this situation, all of the $r_n$s are equal to R, so

$$I = \sum_n m_n r_n^2 = \sum_n m_n R^2 = \left(\sum_n m_n\right) R^2 = MR^2 \quad .$$

Remember, this result is good only for the axis described. Other axes will have different distributions of mass.

EXAMPLE 9-6

Find the moment of inertia of a very thin hollow spherical shell of mass M and radius R about any one of its diameters.




For this solution, let's make it the z-axis. Each small mass $m_n$ is a distance $r_n$ from the z-axis. From the Pythagorean theorem, $z_n^2 + r_n^2 = R^2$, and so we can make a substitution as follows:

$$I_z = \sum_n m_n r_n^2 = \sum_n m_n (R^2 - z_n^2) \quad .$$

Now, here's the trick. Let's repeat this calculation for rotation around the x axis,

$$I_x = \sum_n m_n r_n^2 = \sum_n m_n (R^2 - x_n^2) \quad ,$$

and the y-axis,

$$I_y = \sum_n m_n r_n^2 = \sum_n m_n (R^2 - y_n^2) \quad .$$

Next, two things. First, all of the points $x_n$, $y_n$, $z_n$, must be located where there is mass, *i.e*,. a distance R from the center of the sphere, such that $R^2 = x_n^2 + y_n^2 + z_n^2$. Second, each of the expressions above are, by symmetry, equal and individually what we're looking for: $I_x = I_y = I_z = I_{SPHERE}$. Let's add them together.

$$
\begin{aligned}
3I_{SPHERE} = {} & I_x + I_y + I_z \\
& = \sum_n m_n(R^2 - x_n^2 + R^2 - y_n^2 + R^2 - z_n^2\,) = \sum_n m_n(3R^2 - (x_n^2 + y_n^2 + z_n^2)\,) \\
& = \sum_n m_n(3R^2 - R^2\,) = 2R^2 \sum_n m_n = \ 2MR^2
\end{aligned}
$$

Finally,

$$I_{SPHERE} = \frac{2}{3}MR^2 \quad .$$

EXAMPLE 9-7*

Find the moment of inertia of a disc (mass M, radius R, and uniform areal density σ) about an axis through its center perpendicular to its plane.

We'll modify a procedure developed by Fermat.[4] Pick a number q between 0 and 1. Take the disc and split it up into concentric annuluses. The outermost has outer radius R and inner radius qR. The next has outer radius qR and inner radius $q^2R$, *et c*., so that the $n^{th}$ one has outer radius $q^nR$ and inner radius $q^{n+1}R$. Note that there will be an infinite number of such annuluses. The area of each annulus will be the area of a circle with the outer radius minus the area of a circle with the inner radius:

$$A_n = \pi(q^nR)^2 - \ \pi(q^{n+1}R)^2 \quad .$$

The mass $m_n$ of each is proportional to the its area $A_n$,

$$\frac{m_n}{M} = \frac{A_n}{\pi R^2} \quad ,$$

so,

$$m_n = \left(\frac{M}{\pi R^2}\right)((\pi q^nR)^2 - \ (\pi q^{n+1}R)^2) = \ M(q^{2n} - \ q^{2n+2}) \ .$$

For the moment, let's assume that the mass of each annulus is concentrated at its outer; we'll fix that later. The moment of inertia of each ring about the central axis is

[4] Uta C. Merzbach and Carl B Boyer, *A History of Mathematics 3rd ed*. (Hoboken: Wiley, 2011), 324-5.

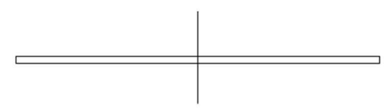
$$I \approx \sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} m_n\, r_n^2 = \sum_{n=0}^{\infty} M(q^{2n} - q^{2n+2})(q^n R)^2 = MR^2\left(\sum_{n=0}^{\infty} q^{4n} - \sum_{n=0}^{\infty} q^{4n+2}\right)$$
$$= MR^2 \sum_{n=0}^{\infty} (-q^2)^n = MR^2\, \frac{1}{1+q^2}\ .$$ [5]

Last, we want to make the annuluses as thin as possible, which also takes care of the problem of the mass being at the outside edge of each. Let $q \rightarrow 1$, which pushes the boundaries between rings outward toward the edge. The summation is then equal to ½, and we have

$$I_{DISC} = \frac{1}{2} MR^2.$$

EXAMPLE 9-8*

Find the moment of inertia of a uniform thin rod of mass M and length L about an axis through its center perpendicular to its length.[6]



Here's neat technique that makes use of the parallel axis theorem. We know from dimensional analysis that the moment of inertia will be some numerical coefficient times $ML^2$. Let's call that value gamma $\gamma$.

$$I_{CENTER} = \gamma ML^2\ .$$

We might think of the rod as two half rods rotating about their common ends. Consider one half of the rod rotating about its center; it will have the same value of gamma, but its mass and length will each be half as much as for the full rod:

$$I_{HALF\ CENTER} = \gamma \frac{M}{2}\left(\frac{L}{2}\right)^2 = \frac{\gamma}{8} ML^2\ .$$

Next, we'll use the parallel axis theorem to move the axis of rotation a distance L/4 to the end of the half rod:

$$I_{HALF\ END} = I_{HALF\ CM} + m_{HALF}h^2 = \frac{\gamma}{8} ML^2 + \left(\frac{M}{2}\right)\left(\frac{L}{4}\right)^2 = \left(\frac{4\gamma+1}{32} ML^2\right).$$

There are two halves, and the sum of their moments will be equal to the moment of the original full rod:

$$2\left(\frac{4\gamma+1}{32} ML^2\right) = \gamma ML^2$$

[5] Roger B. Nelson, *Proofs Without Words III* (Providence: MAA Press, 2015), 157. This book, along with its predecessors, presents a number of graphic proofs for this relationship, including one that makes use of the Fermat approach itself. You could also program Excel to calculate the sum for large n to check its validity.
[6] B. Oostra, "Moment of inertia without integrals," *Phys. Teach.* 44 (May 2006): 283–285.

$$\frac{4\gamma + 1}{16} = \gamma$$

$$4\gamma + 1 = 16\,\gamma$$

$$\gamma = \frac{1}{12}$$

The moment of inertial of a thin rod of mass M and length L about an axis through its center perpendicular to its length is $^{1}/_{12}ML^2$.

EXERCISE 9-2*

Find the moment of inertia of a uniform solid sphere of mass M and Radius R about a diameter. Use the Fermat method.

EXAMPLE 9-9

Suppose that we want to know the moment of inertia about the diameter of a hoop. We already know the moment about an axis through the center, perpendicular to the hoop, is $MR^2$. We'll make use of the perpendicular axis theorem 'in reverse' to solve this problem.

Let the x-axis be a diameter, and let the y-axis be the diameter perpendicular to the first. By symmetry, we can assert that $I_x = I_y$. Then,

$$I_{PERP} = I_z = I_x + I_y = 2I_x = 2I_{DIAMETER} \quad \rightarrow \quad I_{DIAMETER} = \tfrac{1}{2}I_{PERP} = \tfrac{1}{2}MR^2 \;\; .$$

EXAMPLE 9-10

Find the moment of inertia of a disk (mass M and radius R) about an axis in the plane of the disc, passing tangentially through the rim of the disc.

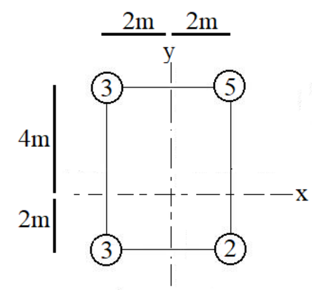
This time, we'll make use of both theorems. First, use the perpendicular axis theorem to find the moment of inertial about a diameter; the method is similar to that used in the preceding example. Then use the parallel axis theorem to slide the axis over to the edge.

$$I_{PERP} = I_z = I_x + I_y = 2I_x = 2I_{DIAMETER} \quad \rightarrow \quad I_{DIAMETER} = \tfrac{1}{2}I_{PERP} = \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)$$
$$= \tfrac{1}{4}MR^2 \;\; .$$

Since a diameter of a disc passes through the center of mass, we are O.K. with using the parallel axis theorem with h = R:

$$I_{TANGENTIAL} = I_{DIAMETER} + Mh^2 = \tfrac{1}{4}MR^2 + MR^2 = \tfrac{5}{4}MR^2 \;\; .$$

HOMEWORK 9-5

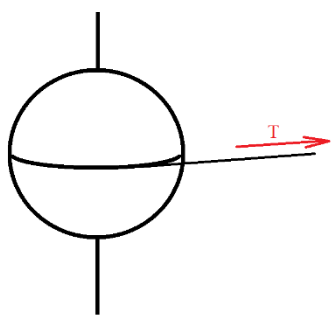



Four masses are connected by very light stiff rods, as shown in the figure. Find the moment of inertia of the four masses about the x-axis, then about the y-axis, then about the z-axis (out of the page, intersecting the other two). The masses are in kilograms. Are your results consistent with the perpendicular axis theorem?

HOMEWORK 9-6

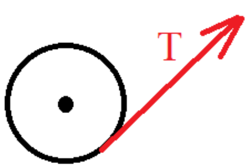
For the object in the preceding problem what magnitude toque must be applied to give it an angular acceleration of 3.5 rad/s$^2$ about the x-axis? The y-axis? The z-axis?

EXAMPLE 9-11




Consider a solid sphere with a light string wrapped around its 'equator.' The radius of the sphere is 3 kg and its radius 0.2 meters. If I pull the string in the plane of the equator with a force of 45 N, what will be the angular acceleration of the sphere?

The figure is as seen from above the sphere. There is the weight of the sphere is downward (into the page) and there is a normal force of some kind holding the sphere up. These forces exert no torque because they are exerted at the axis and so their rs are zero.



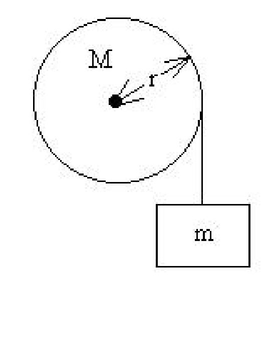

$$\sum_n \tau_n = I\alpha$$

$$(0)(gm)\sin(?) + (0)(F_N)\sin(?) + R(T)\sin(90^o) = \frac{2}{5}MR^2\alpha$$

I insert the question marks for a number of reasons. The angles themselves are undefined because there is no measurable angle between the force and a zero vector (r). Secondly, it maintains the format of the terms in the calculation, and so you are less likely to make an error. Lastly, this format tells me right away that you know that the torque term is zero and why its zero. Continuing,

$$T = \frac{2}{5}MR\alpha \quad \rightarrow \quad \alpha = \frac{5T}{2MR} = \frac{5(45)}{2(3)(0.2)} = 188\ \mathrm{rad/s^2}$$

HOMEWORK 9-7




A uniform disc (r = 0.6 m, M = 1.8 kg) is suspended vertically from a frictionless axle as shown in the figure. A string is wrapped around the wheel and is connected to a mass (m = 0.5 kg) as shown. If the mass m is released from rest, what is the linear acceleration of the mass and the tension in the string?

## Rotational Kinetic Energy

Continuing with the notion of there being quantities in rotational motion which are analogous to quantities in translational motion, we might expect that there is such a thing as *rotational kinetic energy*.

DISCUSSION 9-7

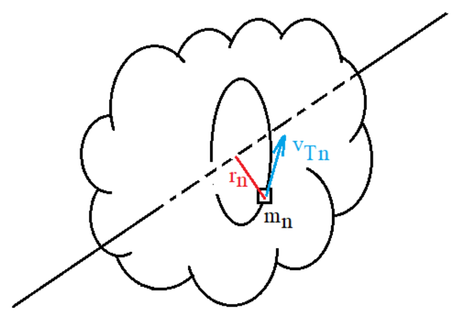
Can you guess the formula for rotational kinetic energy? In rotation, what takes the place of linear speed? What takes the place of the mass? Does your guess have the correct dimension?

DERIVATION 9-5

Consider a rigid object rotating about some stationary axis. That is, the object is rotating, but not translating. Each particle of the object, $m_n$, will have kinetic energy by virtue of its motion, and the total K will be the sum of the individual Ks:

$$K_{ROT} = \sum_n \tfrac{1}{2} m_n v_n^2 \quad .$$

As seen from the axis of rotation, these $\mathbf{v}_n$s are tangential velocities, $\mathbf{v}_{Tn}$. We saw previously that there is a relationship between the angular velocity and the tangential velocity,




$$v_{Tn} = \omega_n r_n \ ,$$

so we can substitute

$$K_{ROT} = \sum_n \tfrac{1}{2} m_n v_{Tn}^2 = \sum_n \tfrac{1}{2} m_n (\omega_n r_n)^2 \quad .$$

But all the ωs are the same, since it's a rigid body, so factor it (and the half) out of the sum:

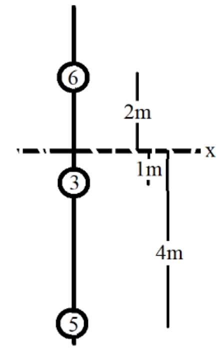
$$K_{ROT} = \sum_n \tfrac{1}{2} m_n (\omega r_n)^2 = \frac{1}{2}\left(\sum_n m_n {r_n}^2\right)\omega^2 \quad .$$

The quantity in parentheses we recognize as the moment of inertia for the object, and so

$$K_{ROT} = \frac{1}{2} I\omega^2 \quad .$$

as expected. The unit of rotational kinetic energy is still the joule. Note that, like many of the things we discuss, this is a bookkeeping thing; we can think of this energy as the sum of the translational kinetic energies of the individual particles, or as the rotational kinetic energy of the object as a whole. Don't double count!

HOMEWORK 9-7




In the discussion, we noted that rotational kinetic energy is just a convenient way of keeping track of the individual kinetic energies of all the small particles making up an object.

Three masses (labeled in kg) are connected in a line by strong light rods. They rotate at angular speed 6 rad/s$^2$. Find the following:

A) The moment of inertia about the x-axis
B) The kinetic energy using $^1/_2 I\omega^2$.
C) The tangential speed $v_T$ of each mass as it moves in its circle.
D) The kinetic energy from $\Sigma_n$ ½ $m_n v_{Tn}^2$.

How do the results from Parts B and D compare?

What happens when an object is rotating in addition to an overall translational motion? We'll consider a common case in which the axis of rotation maintains its orientation. In other words, the object rotates but doesn't tumble.

DERIVATION 9-6*

Each particle of mass $m_n$ will have a velocity vector $\vec{v}_n$, as seen by some outside observer, so that

$$K = \sum_n \frac{1}{2} m_n v_n^2 = \sum_n \frac{1}{2} m_n \vec{v}_n \cdot \vec{v}_n \quad .$$

Now, we can use the concept of relative velocities to write $\vec{v}_n = \vec{v}_{RA} + \vec{v}_{Tn}$, where $\vec{v}_{RA}$ is the velocity of the rotational axis as seen by our bystander and $\vec{v}_{Tn}$ is the tangential velocity of $m_n$ relative to an observer riding along with the rotational axis.

$$\begin{aligned} K &= \sum_n \frac{1}{2} m_n (\vec{v}_{RA} + \vec{v}_{Tn}) \cdot (\vec{v}_{RA} + \vec{v}_{Tn}) \\ &= \sum_n \frac{1}{2} m_n (\vec{v}_{RA} \cdot \vec{v}_{RA} + 2\vec{v}_{RA} \cdot \vec{v}_{Tn} + \vec{v}_{Tn} \cdot \vec{v}_{Tn}) \\ &= \frac{1}{2}\left(\sum_n m_n\right) v_{RA}^2 + \vec{v}_{RA} \cdot \left(\sum_n m_n \vec{v}_{Tn}\right) + \sum_n \frac{1}{2} m_n v_{Tn}^2 \quad . \end{aligned}$$

Let's work on the middle term, which is the hardest.

Remember that the velocity $\vec{v}$ is the instantaneous time rate of change (ITRC) of the position, $\vec{r}$. The masses of course do not change.

$$\sum_n m_n \vec{v}_{Tn} = \sum_n m_n \, \mathrm{ITRC}(\vec{r}_n) = \mathrm{ITRC}\left(\sum_n m_n \vec{r}_n\right) = \mathrm{ITRC}\left(M \vec{r}_{CM,RA}\right)$$
$$= M\,\mathrm{ITRC}\left(\vec{r}_{CM,RA}\right) = M \vec{v}_{CM,RA} \ .$$

This is the velocity of the object's center of mass relative to the rotational axis. Finally, we obtain

$$K = \tfrac{1}{2} M v_{RA}^2 + M \vec{v}_{RA} \cdot \vec{v}_{CM,RA} + \tfrac{1}{2} I_{RA} \omega^2 \ .$$

Now, let's consider a very common special case, that of an object which is translating while at the same time rotating about an axis passing through the center of mass. In that case, $\vec{v}_{CM,RA} = 0$ and $\vec{v}_{RA} = \vec{v}_{CM,RA}$, so that this reduces to:

$$K = \tfrac{1}{2} M v_{CM}^2 + \tfrac{1}{2} I_{CM} \omega^2 \ ,$$

that is, the total kinetic energy is the sum of the translational kinetic energy as if the object were not rotating and the rotational kinetic energy as if the object were not translating.

DISCUSSION 9-8

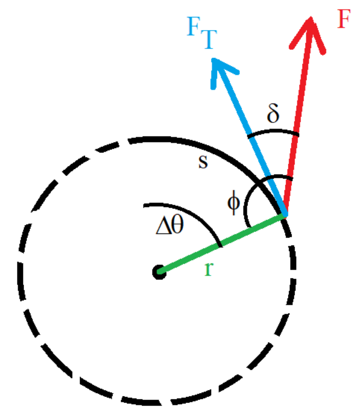
How do we transfer energy into or out of a rotating (or rotatable) object? What did we need to do to transfer energy in Section 6? Can you think of a relationship based on our analogies between linear and angular motions?

DERIVATION 9-7

We know that work involves forces, so let's apply a force F to an object at a distance r from the axis of rotation. The point of application of the force moves a distance s along a circle as the object rotates by an angle theta. We're interested in the component of the force tangent to the circle, that is, parallel to the motion of the point of application. See for example HOMEWORK 6-X.

$$W = F_{\parallel}\, s = F_T\, s = F \cos(\delta)\, s.$$

Since r and $F_T$ are perpendicular, the cosine of delta equals the sine of phi and $s = r\,\Delta\theta$. Substituting,




$$W = F\cos(\theta)\,s = F\sin(\phi)\,r\,\Delta\theta = \tau\,\Delta\,\theta = \vec{\tau}\cdot\Delta\vec{\theta}\,.$$

Since both the toque and the angular displacement are vectors, directions matter. If the torque and displacement are in the same direction, either into or out of the page, then the work is positive and if they are in opposite directions, then the work is negative (remember, we're dealing only with rotations in a plane).

The instantaneous power can be written as

$$P_{INST} = \vec{\tau}\cdot\vec{\omega}$$

We might also be able to define a potential energy associated with rotation. An example is that of a torsional spring. Consider a wire or string which exerts a torque proportional to the angle through which its end has been twisted and in the opposite direction of that angular displacement:

$$\vec{\tau}_{TORSION} = -\,\kappa\,\Delta\vec{\theta}\ \ .$$

Then we would without hesitation assume that there is a corresponding potential energy given by

$$U_{TORSION} = \frac{1}{2}\kappa(\Delta\theta)^2\ \ .$$

What about the units? Well, κ is in N m/radians (yet another quantity with the same dimension as energy!) and the $U_{TORSION}$ is in (Nm) $rad^2$ or Nm, so this looks O.K. dimensionally.

DISCUSSION 9-9

Now we have three types of potential energy and two types of kinetic energy. Can energy be redistributed from any of these to any other?

HOMEWORK 9-8

Consider the situation of Homework 9-x. Using conservation of mechanical energy, find the speed of the hanging mass after it has fallen a distance of 3 meters. Assume both masses are initially motionless.

A special example of an object translating and rotating is one which 'rolls without slipping.' In that case, there is a nice relationship between the angular velocity and the translational velocity of the center of mass. First, let's show that.

DERIVATION 9-8*

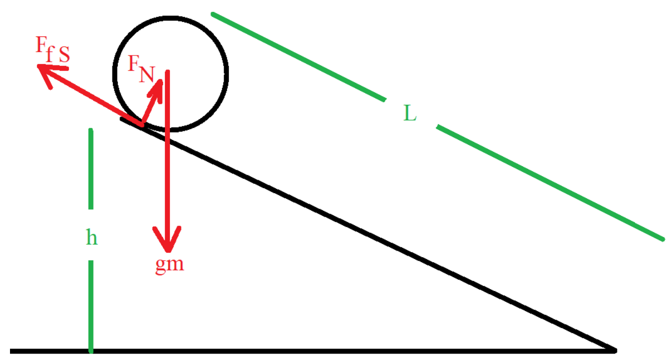
Consider a uniform circular wheel or something similar with radius R rolling without slipping across a horizontal floor. The center of mass has velocity $\vec{v}_{CM}$ as seen by an outside observer. Relative to the center of mass, a point on the outside edge where the object touches the floor will have an angular speed omega and a tangential velocity $\vec{v}_{T,CM}$. such that $v_{T,CM} = \omega R$. But, since that point is at the moment not moving,

$$\vec{v}_{CM} + \vec{v}_{T,CM} = 0 \quad \rightarrow \quad v_{CM} = \omega R \; .$$

Remember that if the object does slip in the surface, then this relationship is almost certainly invalid.

EXAMPLE 9-12

Let's repeat an example we've already done several times. A disk of mass M = 5 kg and radius R = 2 cm rests at the top of an incline (height h = 1.2 m, length L = 2 m). It's released and rolls without slipping down the incline. What is the disk's speed when it arrives at the foot of the incline? Will it be 4.9 m/s?




Let's try using conservation of mechanical energy. Set y = 0 at the bottom of the incline.

$$W_{NC} = \tfrac{1}{2}Mv_{CMf}^2 - \tfrac{1}{2}Mv_{CMi}^2 + \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2 + gMy_f - gMy_i$$

What forces act on the disk and how much work does each do?

$W_N = 0$ (force is perpendicular to the path)
$W_g$ – conservative
$W_f = 0$ – We're going to justify this after we're done. Be patient!

Then,

$$0 = \tfrac{1}{2}Mv_{CMf}^2 - \underbrace{\tfrac{1}{2}Mv_{CMi}^2}_{starts\ from\ rest} + \tfrac{1}{2}I\omega_f^2 - \underbrace{\tfrac{1}{2}I\omega_i^2}_{starts\ from\ rest} + \underbrace{gMy_f}_{y\,=\,0\ at\ bottom} - gMy_i$$

$$gMy_i = \tfrac{1}{2}Mv_{CMf}^2 + \tfrac{1}{2}I\omega_f^2$$

For a disk rotating about its central axis, I = ½ MR$^2$. Since it rolls without slipping, we can make use of the relationship $v_{CM} = R\omega$. Lastly, we'll replace $y_i$ with h to obtain:

$$gMh = \tfrac{1}{2}Mv_{CMf}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{CMf}}{R}\right)^2 .$$

Now some interesting developments. First, the mass drops out, so our answer is independent of the mass of the disk. Also, R drops out, so the result is independent of the size of the disk.

$$gh = \frac{1}{2}v_{CMf}^2 + \frac{1}{4}v_{CMf}^2 = \frac{3}{4}v_{CMf}^2$$

$$v_{CMf} = \sqrt{\frac{4gh}{3}} = \sqrt{\frac{4(10)1.2}{3}} = 4\ \mathrm{m/s}\,.$$

DISCUSSION 9-10

When we did this example for a block sliding down a frictionless incline, the result was 4.9 m/s. Why is this result different? The block and the disk started with the same potential energies. What happened to that energy? Which energy determines how quickly an object moves? Does one have more of that kind than the other and if so, where did the rest of the potential energy go? Would the result be different if this were a solid sphere instead of a disk?

In the Section 6 example, gravitational potential energy was converted into translational kinetic energy. Here, however, there are two types of kinetic energy, translational and rotational. The potential energy must be split between these two categories. How much goes into each category depends on the shape of the object. For example, repeating the example above with a solid sphere where the moment of inertia is $^2/_5MR^2$ changes the final velocity to 4.14 m/s. Less energy converted to rotational kinetic energy means more available for translational energy.

| Shape | Fraction before $MR^2$ | *Per cent* Translational K | *Per Cent* Rotational K |
|---|---|---|---|
| Hoop | 1 | 50% | 50% |
| Hollow Sphere | 2/3 | 60% | 40% |
| Disk | 1/2 | 66.7% | 33.3% |
| Solid Sphere | 2/5 | 71.4% | 26.6% |

The final velocities of these objects down the ramp depend only on the fraction in front of the moment of inertial term.

DISCUSSION 9-11

Let's run a race but placing two shapes at the top of an incline and releasing them simultaneously. Which will arrive first?

Shapes1.mp4

Consider a disk and a hoop with the same masses and radiuses. Which will win a race rolling down an incline?

Shapes2.mp4

Consider two disks of the same mass, but C has half the radius of A. Which will win a race rolling down an incline?

Shapes5.mp4

Consider two disks with D having both a radius and a mass much less than A. Which will win a race rolling down an incline?

Shapes4.mp4

Consider sphere F which has the same radius and mass as hoop E. Which will win a race rolling down an incline?

Consider Hoop G and Hoop E with G having both a radius and a mass much less than A. Which will win a race rolling down an incline?

Shapes7.mp4

Were any of the results seen in the film different than what you expected? Can you explain why the expected results were not obtained?

## JUSTIFICATION 9-1*

Let's clean up the question about work done by static friction. Consider a ball rolling on a flat horizontal surface. It has translational kinetic energy $\frac{1}{2} m v_{CM}^2$ and rotational kinetic energy $^1/_2 I\omega^2$. If it is simply rolling without slipping, not being driven by any agency, then there is a relationship between $v_{CM}$ and $\omega$, namely that $v_{CM} = \omega R$. The friction, if any, will be static, but due to the synchonization of the two types of motion, there is no tendency to slip, and the static frictional force, which is only as large as it needs to be, will be zero.

But what about an object on an inclined plane? Well, we know that the rolling object will have a lower speed at the bottom of the incline than will the frictionlessly sliding object, so friction must have done some negative work on the rolling object. It's easy enough to calculate for the example above:

$$W_{fS} = F_{fS}\, L \cos(180^o) = -F_{fS}\, L\,.$$

The frictional force also exerts a torque on the object about its center that points into the page,

$$\tau_{fs} = \mathrm{R}\,\mathrm{F}_{fs}\ \sin(90^o) = -\mathrm{R}\,\mathrm{F}_{fs}\ .$$

Since the angular displacement $\Delta\theta$ is also into the page, the work done in terms of rotation is

$$\mathrm{W}_{fs} = \tau_{fs}\,\Delta\theta\ \cos 0^o = +\mathrm{R}\,\mathrm{F}_{fs}\,\Delta\theta == +\mathrm{F}_{fs}\,(\mathrm{R}\,\Delta\theta) = +\mathrm{F}_{fs}\,\mathrm{L}\,.$$

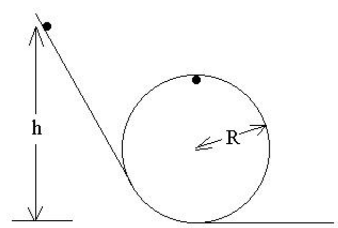
So the <u>total</u> work done by the friction is zero.

EXERCISE 9-3*

Here's a classic problem. Consider a bowling ball that is released with initial translational velocity $v_o$ sliding down the lane but not initially rotating. Calculate the velocity of the ball and how far down the alley it is when it begins to roll without slipping. HINT: What condition is met when the ball rolls without slipping?

HOMEWORK 9-14

Consider the loop-de-loop track. A small round object with radius r<<R is placed on the track at altitude h and released. It rolls without slipping along the track and just barely makes it around the top of the loop. Find h is the object were a




A) solid sphere.
B) hollow sphere.
C) disk.
D) hoop.

HINT: If you represent the fraction before the $mr^2$ by some symbol, you can do almost all of the problems at once.

---

## Angular Momentum

Again as an analogy with linear motion, we might suspect that there is such a thing as *angular momentum* ($\vec{L}$), and we might guess that it is defined as $I\vec{\omega}$ (analogous to $\vec{p} = m\vec{v}$). Let's see:

Starting from the rotational form of the Second Law,

$$\vec{\tau}_{EXT} = I\vec{\alpha}\quad ,$$

we'll substitute the definition of angular acceleration (and assume that I is constant!) to get

$$\vec{\tau}_{EXT} = I\frac{\Delta\vec{\omega}}{\Delta t}\quad ,$$

$$\vec{\tau}_{EXT}\,\Delta t = I\Delta\vec{\omega} = \Delta(I\vec{\omega}) = \Delta\vec{L}\ .$$

The left-hand side of the preceding relationship is the rotational equivalent of impulse, and we can see that, in the absence of any external rotational impulses, the total amount of angular momentum is constant, or conserved. Our result suggests that the angular momentum points in the same directions as does the angular velocity.[7]

Several observations. First, for linear momentum, we expected that the masses of objects could not change, so that any changes in momentum $\vec{p}$ were due to changes in velocity. For angular momentum, we see that a change in angular momentum can be effected by changing either $\vec{\omega}$ or I or both. Secondly, and more interestingly, we remember the constant, droning repetition that all three of the pictures we developed in linear motion (force and acceleration, work and kinetic energy, and impulse and momentum) were not only equally valid, but derivable from each other. We might expect the same from the three pictures developed for rotational motion, namely torque and angular acceleration, work and rotational kinetic energy, rotational 'impulse' and angular momentum. In the classical world we are studying this semester this is so, but in the real world, we find the suggestion that angular momentum is somewhat more fundamental as a concept than the other two. In your chemistry courses, you may have come across the notion that angular momentum is *quantized*, that is, that only certain numerical values are allowed; this can be true of energies also, but the values allowed depend on the exact system. Angular momentum may well be the most important topic we cover in this course, and the one we spend the least amount of time on.

DISCUSSION 9-11

VIDEO

> Rotating student with barbells. By pulling the barbells in towards his body, he reduces the moment of inertia, I. If there are no external torques, the angular velocity correspondingly increases. This is the same effect used by figure skaters and high divers.
>
> Student with bicycle wheel. A non-rotating student holds a wheel that is rotating so as to have (say) one unit of angular momentum, pointing upward (call this +1). Inverting the wheel causes the student to begin rotating. In the absence of external torques, the total angular momentum must remain +1. Inverting the wheel changes its angular momentum to -1, and the student then acquires angular momentum +2, so that the sum remains +1. How does the student magically acquire just the right amount of angular momentum? Inverting the wheel required that the student apply a torque, and so, by the third law, a torque equal in magnitude but opposite in direction was applied by the wheel on the student.

We can derive analogous relations for the final angular velocities for totally inelastic 'collisions' and for totally elastic 'collisions' by substituting moments of inertia for masses and angular

[7] Most of our derivations have worked out that way. For example, $\vec{J} = \Delta\vec{p} = \Delta(m\vec{v})$, so we assume that $\vec{p} = m\vec{v}$.

velocities for linear velocities, although there are some restrictions on when these will be valid (the Is should be constant, for example!).

EXAMPLE 9-13

A 10" LP of mass 110 grams is dropped down the spindle onto a freely turning 12" turntable platter of mass 1 kg initially turning at $33^1/_3$ revolutions *per* minute (rpm). What is the final speed of the turntable in rpm?

HINT: Assume that both the LP and the platter are discs.

This is like a totally inelastic collision in Section 7 since the two objects have a common final angular velocity. The two objects share a common axis, so the third law of motion is valid for torques. Since the platter is freely turning, there are no external torques and we can use conservation of angular momentum:

$$L_{TOTAL\,i} = I_R\omega_{Ri} + I_P\omega_{Pi} = I_R\omega_{Rf} + I_P\omega_{Pf} = L_f \quad ,$$

$$I_R\omega_{Pi} = (I_R + I_P)\omega_f \ ,$$

$$\omega_f = \frac{I_R\omega_{Pi}}{I_R + I_P} = \frac{\frac{1}{2}m_P r_P^2}{\frac{1}{2}m_R r_R^2 + \frac{1}{2}m_P r_P^2}\,\omega_{Ri} = \frac{(1)12^2}{(.11)7^2 + (1)12^2}33.3 = 32.1\text{ rpm} \ .$$

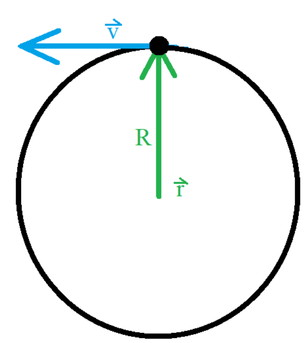
HOMEWORK 9-9

Many schools have a lab practical to cap off their physics courses. You are given a closed box with a shaft extending from one side. You are told that the shaft is attached to the center of a round symmetrical flywheel of mass 7 kg and radius 0.4 m. When you attach a constant torque motor (11.73 Nm), the system goes from rest to 600 rpm in 3 seconds. What shape or shapes could the flywheel be?

DERIVATION 9-9*

Is there a relationship between linear and angular momentum? Consider a special case of an object of mass m moving in a circle of radius r (location vector $\vec{r}$) with angular velocity $\vec{\omega}$. From our definition above,

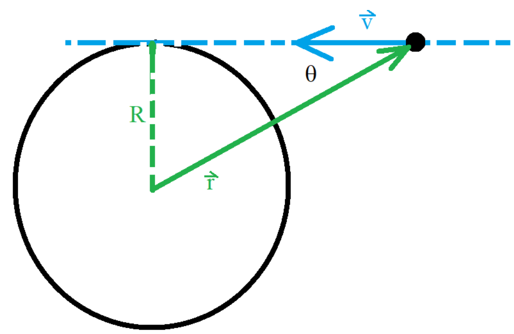
$$L = I\omega = mR^2\frac{v}{R} = Rmv \quad .$$




Magnitude-wise, this looks promising. Direction-wise, we want L to point out of the paper towards us, parallel to omega. We can do that with a cross product. We can try $\vec{v}\times\vec{r}$, but that points into the paper, so we'll make it $\vec{r}\times\vec{v}$. Since in our example, $\vec{v}$ and $\vec{r}$ are perpendicular, the angle between them is 90° and we have

$$\vec{L} = m\,\vec{r} \times \vec{v} = \vec{r} \times \vec{p} \quad \rightarrow \quad |\vec{L}| = m\,r\,v\sin\theta_{r,v} = r\,p\,\sin\theta_{r,p} \quad (\text{RHR})\,.$$

But what if $\vec{r}$ and $\vec{v}$ are not perpendicular? Well, since the object is moving in a plane, it's certain that at some time the situation will appear as above, but for most of the time, it will look like this figure. What does our proposed relationship give us in that situation? Since R = r sinθ, the result is the same, L = Rmv out of the page. So, it seems we have a nice way of writing angular momentum in terms of vectors.




HOMEWORK 9-x

Jimmy runs 2 m/s tangentially to a frictionless playground merry-go-round and jumps on. If Jimmy's mass is 30 kg and the platform has mass 100 kg and radius 2 m, what is the final angular speed of Jimmy and the platform?

HOMEWORK 9-10
VIDEO
A professor stands on a freely rotating platform like the one in the demonstration. With his arms outstretched, he has an angular speed of 2 radians/second. Once his arms are drawn inward next to his chest, his speed becomes 6 rad/sec. What is the ratio of his final kinetic energy to his initial kinetic energy?

---

EXERCISE 9-1 Solution

$\theta_i = 0$ (make that the origin)
$\theta_f = -500$ revolutions $= -\,3141.6$ radians  Why is it negative?
$\omega_i = +30$ rad/s
$\omega_f = -40$ rad/s  reversed direction
$\alpha = ?\ \leftarrow$
$t = ?\ \leftarrow$

Try KEq 4:

$$\omega_f^2 = \omega_i^2 + 2\vec{\alpha}\cdot\Delta\vec{\theta}$$

$$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta} = \frac{(-40)^3 - 30^2}{2(-3141.6)} = -0.11\ \text{rad/s}^2\,.$$

Then KEq 1:

$$\vec{\omega}_f = \vec{\omega}_i + \vec{\alpha}t \;\rightarrow\; t = \frac{\omega_f - \omega_i}{\alpha} = \frac{-40 - 30}{-0.11} = \text{636.4 seconds} \;.$$

EXERCISE 9-2 Solution

We'll set the parameter of interest as the z-axis and slice the sphere into many thin circular cylinders of height $\Delta z_n$, each with mass $m_n$, radius $r_n$, and volume $V_n$. The moment of each of these cylinders is already known to be ½ $m_n r_n^2$. We'll write everything in terms of z.

Define a number of positions $z_n$ along the z-axis between z = 0 and z = R as $z_n = q^n R$ with q<1 and n from zero to infinity. Then,

$$\Delta z_n = q^n R - q^{n+1} R = q^n\, R(1-q)\,,$$
$$r_n^2 = R^2 - z_n^2 = R^2 - (q^n R)^2 = R^2(1-q^{2n})\,,$$
$$V_n = \pi r_n^2\, \Delta z_n = \pi\left(R^2(1-q^{2n})\right)\left(q^n\, R(1-q)\right)\,,$$

and,

$$m_n = \rho V_n = \frac{M}{\frac{4\pi}{3}R^3}\pi\left(R^2(1-q^{2n})\right)\left(q^n\, R(1-q)\right) = \frac{3M}{4}(1-q^{2n})\, q^n\, (1-q)\;.$$

Then, the moment of inertia of this slice is

$$I_n = \frac{1}{2}m_n r_n^2 = \left(\frac{1}{2}\frac{3M}{4}(1-q^{2n})\, q^n\, (1-q)\right)\left(R^2(1-q^{2n})\right)$$
$$= \frac{3}{8}MR^2\left((1-q^{2n})^2\, q^n\, (1-q)\right)\;.$$

To find the moment of the entire sphere, we must sum this expression for all of the slices, and of course double the result to account for those slices for which $-R < z < 0$. Lastly, we will make $q \rightarrow 1$ to make the slices as thin as possible.

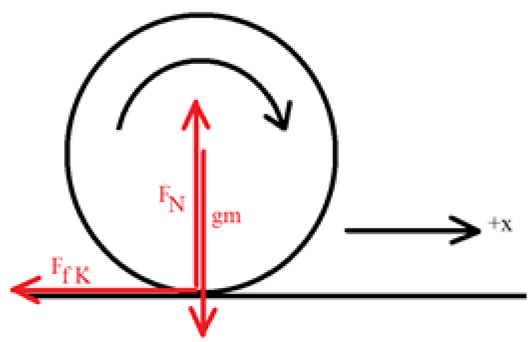
$$
\begin{aligned}
I = 2\frac{3}{8}MR^2 \lim_{q\to 1}&\left(\sum_{n=0}^{\infty}\left((1-q^{2n})^2\, q^n\,(1-q)\right)\right)\\
&= \frac{3}{4}MR^2 \lim_{q\to 1}\left(\sum_{n=0}^{\infty} q^n - q^{n+1} - 2q^{3n} + 2q^{3n+1} + q^{5n} - q^{5n+1}\right)\\
&= \frac{3}{4}MR^2 \lim_{q\to 1}\left((1-q)\left(\sum_{n=0}^{\infty} q^n - 2\sum_{n=0}^{\infty}(q^3)^n + \sum_{n=0}^{\infty}(q^5)^n\right)\right)\\
&= \frac{3}{4}MR^2 \lim_{q\to 1}\left((1-q)\left(\frac{1}{1-q} - 2\frac{1}{1-q^3} + \frac{1}{1-q^5}\right)\right)\\
&= \frac{3}{4}MR^2 \lim_{q\to 1}\left(\frac{1-q}{1-q} - 2\frac{1-q}{1-q^3} + \frac{1-q}{1-q^5}\right)\\
&= \frac{3}{4}MR^2 \lim_{q\to 1}\left(1 - 2\frac{1}{1+q+q^2} + \frac{1}{1+q+q^2+q^3+q^4}\right)\\
&= \frac{3}{4}MR^2\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{2}{5}MR^2 \;.
\end{aligned}
$$

EXERCISE 9-3 Solution

There are three forces acting on the bowling ball: the weight, the normal force from the floor, and the kinetic frictional force from the floor. Using the second law of motion,

$$+F_N - gm = ma_y = 0 \quad ; \quad -F_{fK} = ma_x \quad ; \quad F_{fK} = \mu_K F_N \quad \to \quad a_x = -\mu_K g \;.$$

In terms of torques about the center axis, $F_N$ has angle 180° (or 0°, depending on how you measure the angle) and the weight has a lever arm of zero, since we have previously demonstrated that it can be considered to be applied at the center of mass. That leaves the friction:




$$\tau = RF_{fK}\sin(90^o) = I\alpha$$

$$R\mu_K gM = \tfrac{2}{5}MR^2\alpha$$

$$\alpha = \frac{5\,\mu_K g}{2R}.$$

We'll decide that CW rotation is positive to match the positive x-motion to the right (that is, into the page will be positive). Now we <u>might</u> consider setting $\Delta x = R\,\Delta\theta$, but that's not true because of the skidding. It's also temping to think that, when the ball stops skidding, $a = \alpha R$; this is a true statement, but both are zero (the friction goes from kinetic to static!) and so that's not very useful.

We have to look at $v_{CM} = r\omega$ and use KEq 1 for both linear and rotational motions. Let t be the time from ball's launch to when it rolls without slipping.

$$v_f = R\omega_f$$

$$v_o + at = R(\omega_o + \alpha t)$$

$$v_o - \mu_K g\, t = R\left(\frac{5\,\mu_K g}{2R}\right) t$$

$$v_o = \frac{7\,\mu_K g}{2} t \quad \rightarrow \quad t = \frac{2v_o}{7\mu_K g}.$$

Then,

$$v_f = v_o + (-\,\mu_K g)\left(\frac{2v_o}{7\mu_K g}\right) = \frac{5}{7}\, v_o$$

and

$$x_f = x_i + v_i t + \tfrac{1}{2} a t^2 \quad \rightarrow \quad x_f = 0 + v_o\left(\frac{2v_o}{7\mu_K g}\right) + \tfrac{1}{2}(-\,\mu_K g)\left(\frac{2v_o}{7\mu_K g}\right)^2 \;=\; \frac{10}{49}\frac{v_o^2}{\mu_K g}$$

## Section 1-10 - Static Equilibrium

DISCUSSION 10-1

> Suppose you're constructing some structure, a bridge or office building. It's composed of many parts, none of which should move. What condition should be imposed on an object, such as a steel beam or a brick, so that it doesn't move? What else shouldn't the object do? What condition must be met for that motion to be avoided?

## Conditions for Static Equilibrium

*Statics* is a sub-topic of physics and engineering which is incredibly important; mechanical engineers study it for many semesters because of its applications to the design of buildings and other structures. What is needed are the conditions under which an object or assemblage of objects will not shift. These boil down to:

$$\sum_{n} \vec{F}_n = 0 \quad \text{and} \quad \sum_{n} \vec{\tau}_n = 0 \quad .$$

That is, we want the objects not to accelerate or rotate. And, that is it for Section 10.

I'm sure, however, that you'd like a few examples. Our convention for this chapter is that we shall write the magnitudes of the force and r vectors, then add the appropriate signs in front of each torque term; positive for torques out of the page (that is, those which would act to accelerate the object CCW), and negative for torques into the page (those which would act to accelerate the object CW).

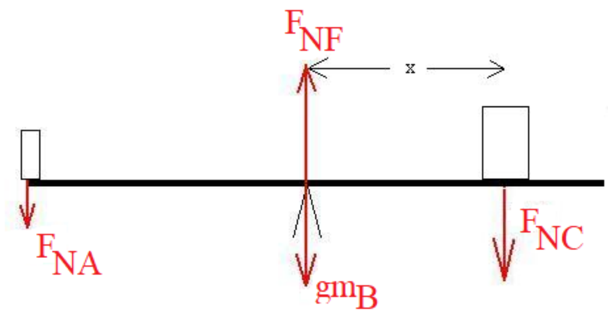
In the previous section, we considered rotations about specific axes. In this section, the object is not rotating about <u>any</u> axis, that is, the torque about any axis you may care to choose will be zero. This gives you a fair amount of freedom in solving problems. In this section, the point about which the torques are calculated is called the *pivot point*. My suggestion is to write the toque equation first, choosing if possible a pivot that makes the torques due to forces of unknown magnitude zero. In that way, many problems can be solved by considering only the torque equation without involving the force equations.

Remember from Derivation 9-1 that we can act as if all of the weight of an object is applied at the object's center of mass.

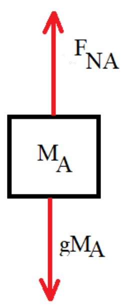
EXAMPLE 10-1

> Consider a see-saw of length L (6m) and mass m (m = 15 kg) which is pivoted at the center. Anna ($m_A$ = 20kg) sits right at the end of the board. Where should Carli ($m_C$ = 35 kg) sit so that the board is balanced?
>
> Much as we did back in Section 5, we need to choose an object (or objects) to analyze; in this

case, we choose the board. Let's draw a free body diagram (with +y upward and +x to the right) to inventory the forces:




$F_{NF}$ is the normal force exerted by the fulcrum of the see-saw on the board, and $gm_B$ is the weight of the board, which we assume can be thought to act as the center of the board. The forces labelled $F_{NA}$ and $F_{NC}$ are <u>not</u> the weights of Anna and Carli, but rather normal forces exerted by them on the board; remember that the weight of each child <u>acts on the child</u>. Through the use of NII and NIII, these forces are numerically equal to those weights.




For Anna, we have from NII that

$$+F_{NA} - \ gM_C = 0$$

$$+F_{NA} = \ gM_C = 10(20) = 200\ N\ \ .$$

The exact same calculation for Carli results in $F_{NC}$ = 350 N.

Let's do the problem twice to illustrate a point.

1) Choose the left end of the board as the pivot. The torque requirement is[1]

$$\sum_n \vec{\tau}_n = (0)F_{NA}\sin(?) + (3)(F_{NF})\sin(90^o) - (3)(gm_B)\sin(90^o)$$
$$- r_{LEFT\ END}(F_{NC})\sin(90^o) = 0.$$

We're going to need to know $F_{NF}$, so we'll go to the vertical force equation:

$$\sum_n \vec{F}_{nx} = 0 \ \text{ and } \ \sum_n \vec{F}_{ny} = +F_{NF} - \ F_{NA} - F_{NC} - gm_B = 0\ \ ,$$

$$F_{NF} = \ F_{NA} + F_{NC} + gm_B = 200 + 350 + 150 = 700\ N\ \ .$$

Returning to the torque equation,

$$(3)(F_{NF}) - (3)(gm_B) = r_{LEFT\ END}(F_{NC}).$$

---

[1] Note the question mark inserted as the angle. Since r = 0 for that force about that pivot, the angle is undefined. However, I think that keeping the format of each term uniform lessens the probability of making an error and at the same time, indicates to me that you know that that term is zero and why.

$$r_{LEFT\ END} = \frac{3F_{NF} - 3gm_B}{F_{NC}} = \frac{3(700) - 3(150)}{350} = 4.71 \text{ m from the left end.}$$

2) Choose the center of the board as the pivot. The torque equation is then

$$\sum_n \vec{\tau}_n = +(3)F_{NA}\sin(90^o) + (0)(F_{NF})\sin(?) - (0)(gm_B)\sin(?)$$

$$- r_{CENTER}(F_{NC})\sin(90^o) = 0\ ;$$

$$(3)F_{NA} = r_{CENTER}(F_{NC})\ ;$$

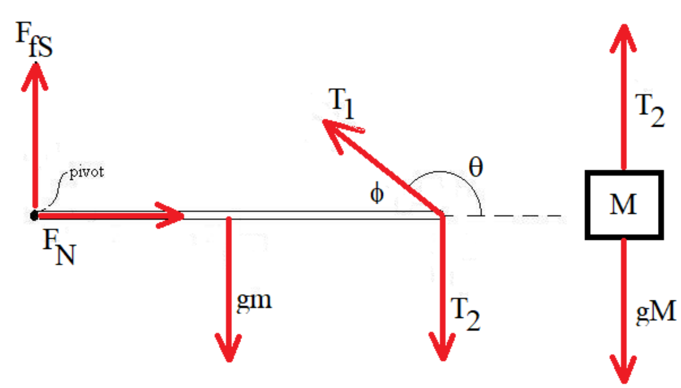
$$r_{CENTER} = \frac{(3)F_{NA}}{F_{NC}} = \frac{3(200)}{350} = 1.71 \text{ m to the right of the center},$$

which is of course the same answer.

What we see, then, is that a judicious choice of the pivot such as in the second solution can save a great deal of effort.

EXAMPLE 10-2

Consider a horizontal uniform beam of mass m and length L supporting a sign of mass M. The beam is attached to the wall with a wire which makes an angle θ with the beam. Its other end is supported by the friction between the end of the beam and the wall. How large would the co-efficient of static friction need to be to keep the beam from slipping?




Again using the usual coördinate system of +y up and +x to the right, and picking the left end of the beam as the pivot, we write that:

$$+T_2 - gM = 0\ ;$$

$$+F_N - T_1\cos\phi = 0\ ;$$

$$F_{fS} - gm + T_1\sin\phi - T_2 = 0\ ;$$

$$F_{fS} = \mu_S F_N\ \text{(crit. sit.)}\ ;$$

and

$$(0)F_{fS}\sin(?) + (0)F_N\sin(?) - \frac{L}{2}gm\sin 90^o - LT_2\sin 90^o + LT_1\sin\phi = 0\ .$$

Simplifying the torque equation and substituting for $T_2$ results in

$$T_1 \sin\phi = \frac{gm}{2} + T_2 = \frac{gm}{2} + gM \ .$$

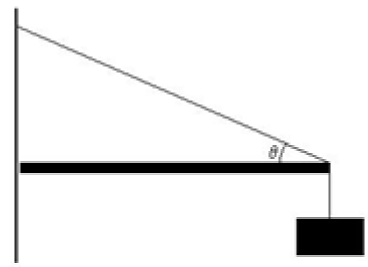
Now it's just substitution:

$$\mu_S = \frac{F_{fS}}{F_N} = \frac{gm - T_1 \sin\phi + T_2}{T_1 \cos\phi} = \frac{gm - \left(\frac{gm}{2} + gM\right) + gM}{\left(\frac{\frac{gm}{2} + gM}{\sin\phi}\right)\cos\phi} = \frac{m \tan\phi}{m + 2M}.$$

.

HOMEWORK 10-1

A sign weighing 300 N is suspended at the end of a massless beam 2 m in length, as shown. The beam is attached to the wall with a hinge, and its other end is supported by a wire. What is the tension in the wire if it makes an angle of 30° with the beam?

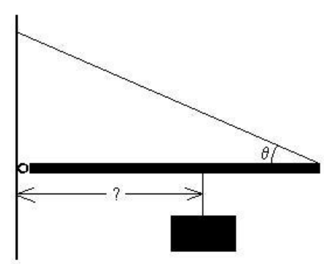



HOMEWORK 10-2

A sign weighing 500 N is suspended at the end of a uniform 300 N beam 5 m in length, as shown. The beam is attached to the wall with a hinge, and its other end is supported by a wire. What is the tension in the wire if it makes an angle of 25° with the beam? Use the same figure as for Homework 10-1.

HOMEWORK 10-3

A sign weighing 200 N is suspended under a uniform 300 N beam 4 m in length, as shown. The beam is attached to the wall with a wire which makes a 53° angle with the beam. Its other end is supported by a hinge connected to the wall. If the maximum tension permissible in the wire is 300 N, what is the range of distances from the wall that the sign can hang without causing the wire to snap?




HOMEWORK 10-4

An 6m long, 110N ladder rests against a smooth wall. The co-efficient of static friction between the ladder and the ground is 0.8, and the ladder makes a 53° angle with the ground. How far up the ladder can an 800N person climb before the ladder starts to slip?

Sample Exam IIII

**MULTIPLE CHOICE** (4 pts each)

1) Consider a solid sphere of radius R and mass M, and a disk of mass m (=½ M) and radius r (= ½ R). In a race rolling (without slipping) down an incline, the solid sphere wins. Which of the following is true?

A) The solid sphere and disk have the same moment of inertia.
B) The moment of inertia of the solid sphere is larger than that of the disk.
C) The moment of inertia of the disk is larger than that of the solid sphere.
D) There is no way to know which moment of inertia is larger.
E) There is no Choice E.

2) Consider a rigid body, not necessarily in equilibrium. Which of the following statements is always true?

A) If $\Sigma\vec{F}_n = 0$, then $\Sigma\vec{\tau}_n = 0$.
B) If $\Sigma\vec{\tau}_n = 0$, then $\Sigma\vec{F}_n = 0$.
C) If $\Sigma\vec{F}_n \neq 0$, then $\Sigma\vec{\tau}_n \neq 0$.
D) If $\Sigma\vec{\tau}_n \neq 0$, then $\Sigma\vec{F}_n \neq 0$.
E) None of these is always true.

3) Consider a solid sphere of mass M and radius R. What is the sphere's moment of inertia about an axis tangent to the surface of the sphere?

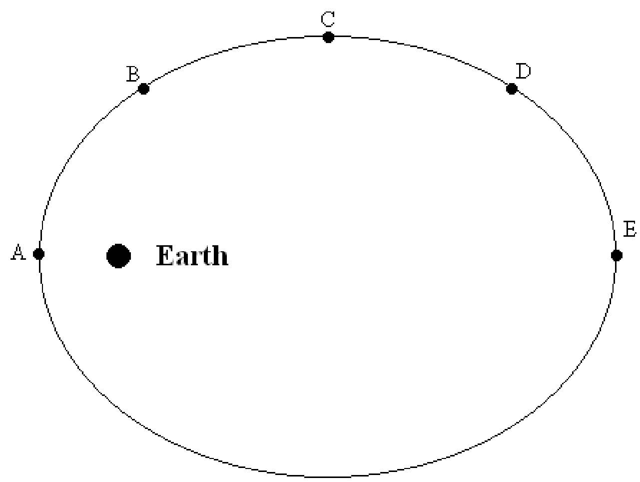
A) $2/5\ MR^2$
B) $3/5\ MR^2$
C) $7/5\ MR^2$
D) $3/2\ MR^2$
E) $2\ MR^2$

4) Consider two point masses on the x-axis. $M_1$ has mass 45 kg and is at x = 4m, while $M_2$ has mass 35 kg and is located at x = 9m. Where is the center of mass?

A) 1.0 m
B) 6.2 m
C) 6.5 m
D) 6.8 m
E) 38 m

5) The moon orbits the earth on a path that is not circular, but elliptical, as shown with great exaggeration in the figure. At which of the labeled points will the moon's speed be greatest?

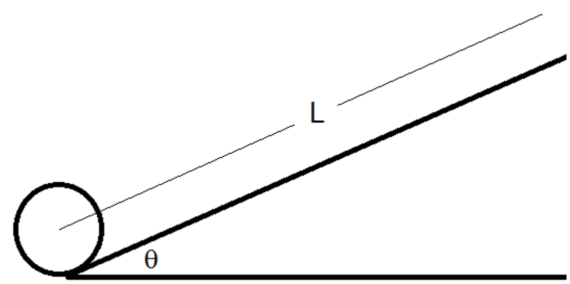
A) A
B) B
C) C
D) D
E) E




**PROBLEM I** (20 pts)

Suppose that a solid sphere (mass M and radius R) is launched up an incline, as shown, and subsequently rolls without slipping. How far up the incline (L) will the disk go before stopping?

Let $M = 4$ kg, $R = 0.02$ m, $v_i = 12$ m/s, and $\theta = 53^o$.

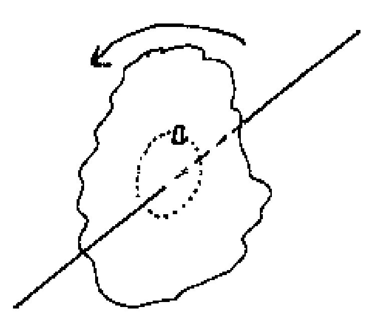



**PROBLEM II** (20 pts)

Consider a playground merry-go-round, a disk of mass 80 kg and radius 1.5 meters, rotating about a frictionless axis through its center. There are two twins, each of mass 30 kg, sitting at the edge of the disc. The angular speed of the ride is 7 rad/sec. Now, one of the twins moves to the center of the disk. What is the new angular speed?

**PROBLEM III** (20 pts)

Prove that the rotational kinetic energy of a rigid object as it turns about some axis with angular velocity $\omega$ is $KE_{rot} = \frac{1}{2}I\omega^2$, where $I$ ($=\Sigma_n m_n r_n^2$) is the moment of inertia of the object about that axis.



**PROBLEM IIII** (20 pts)

The blade of a circular saw of diameter 0.3 m accelerates uniformly from rest to 2800 rev/min in 32 seconds.

A) Convert the final angular velocity to radians/second.

B) What is the angular acceleration of the blade?

C) Through what angle did the blade turn in this process?

D) If the mass of the blade is 0.1kg, and the blade can be considered to be a disk, what net torque was applied to the blade during this process?

## Section 11 - Oscillations

Let's define a couple of terms. An *oscillator* is an object that moves repetitively through a given path in a given time period. In a sense, you are an oscillator, moving from home to school to work to home every day. A *simple harmonic oscillator* (SHO) is a very special case when an object moves through a cycle along a line (perhaps the x-axis) around a central point (we'll say at x = 0) such that its position x is given by

$$x(t) = A\cos(2\pi f t + \varphi)\ .$$

Here, t is time, A is the *amplitude*, the maximum excursion from the central point, and phi is the *phase angle* that allows us to change the cosine into a sine by shifting the curve in time. For this course, phi will always be set to zero. The symbol *f* is the *frequency*, the number of cycles completed *per* unit of time; one cycle per second is called one hertz (Hz). Note that we will often replace $2\pi f$ with the *angular frequency*, Ω (omega).[1] We will also define the *period* of oscillation, P, as the time to complete one cycle, and so necessarily, $P = 1/f$.

Since the object is not moving with constant velocity, there must be some force acting on it. More specifically, the force acts to return the object back towards the central point. Such a force is described as a *restoring force*. Let's see if we can suss out the nature of the force that causes simple harmonic motion (SHM). Force is related to acceleration, so let's find the acceleration of a SHO. Given that by definition the location is $x(t) = A\cos(2\pi f\, t)$, we can find the velocity function. Remember back to Section 3, when we found the instantaneous time rates of change of the sine and cosine functions:

$$\text{ITRC}\ (A\cos(\omega t)) = -\omega A\sin(\omega t) \text{ and } \text{ITRC}\ (A\sin(\omega t)) = \omega A\cos(\omega t).$$

In this case, we replace ω with $\Omega = 2\pi f$. The velocity of the SHO will be

$$v(t) = \text{ITRC}\big(x(t)\big) = \text{ITRC}\ (A\cos(2\pi f t)) = -2\pi f\ A\sin(2\pi f t)$$

and the acceleration is

$$\begin{aligned} a(t) &= \text{ITRC}\ \big(v(t)\big) = (-2\pi f)\text{ITRC}\ (\ A\sin(2\pi f t)) = (-2\pi f)\ (2\pi f)A\cos(2\pi f t) \\ &= -4\pi^2 f^2\ x(t)\ . \end{aligned}$$

The force is then, by NII,

$$F(t) = \frac{a(t)}{m} = -\left(\frac{4\pi^2 f^2}{m}\right) x(t)\ .$$

---

[1] The angular frequency is perhaps best explained during a differential equations course. For now, we'll use it as an abbreviation for $2\pi f$. In most textbooks, the symbol ω is used, but I am taking this opportunity to try to avoid confusion.

So, we see that the force required to produce this type of motion must be proportional to the displacement of the object from $x = 0$.

DISCUSSION 11-1

Can you remember a force we discussed that is opposite in direction and proportional to the displacement of an object from its equilibrium point? Is that necessarily the only such force that meets those conditions?

CHEESEY EXPERIMENT 11-1

Oscillations.mp4

We know from previous discussion that springs follow Hooke's relationship, $F = - kX$, the force applied to the mass is proportional to the displacement of the mass from the equilibrium point and is directed opposite to that displacement. If we compare Hooke's relationship to the requirement we developed above, we see that the spring constant would have to be given by

$$k = \frac{4\pi^2 f^2}{m} \ .$$

Turning it around, if a mass is attached to a spring and set into motion, the frequency of oscillation will be

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ .$$

This special frequency, at which a system 'prefers' to oscillate, is called the *natural frequency*, $f_o$. If I were to bop a mass on a spring, this is the frequency at which it will oscillate. If I do it again, the mass will again oscillate at the same frequency. Each of the five times a year I do this until I retire, it will oscillate at this same frequency. Notice that the frequency appears to be independent of the amplitude of oscillation, so it doesn't matter how far we pull the mass before we release it, the mass will oscillate with the same frequency.

DISCUSSION 11-2

To test this result, it is more convenient to measure the natural period of oscillation, $P_o = 1/f_o$:

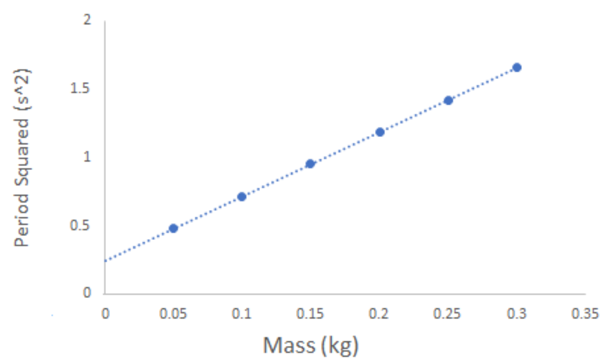
$$P_o = 2\pi\sqrt{\frac{m}{k}} \ .$$

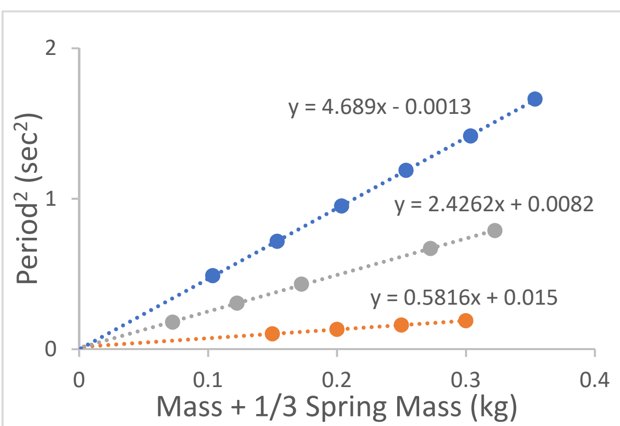
Would we expect that the period proportional to the mass and inversely proportional to the spring constant? How should the data be plotted to obtain a line? If we were to place 100 grams of mass at the end of a real spring, how much mass would be oscillating?

EXPERIMENT 11-1

Let's square both sides of the equation above with the intention of plotting the mass as the independent variable:

$$P_o^2 = \left(\frac{4\pi^2}{k}\right) m \ .$$




In theory, to obtain a line, the dependent variable plotted should be the square of the period. For the data shown in the graph, there is a serious problem. The theoretical relationship predicts that the intercept of the line should be zero. We asked a moment ago how much mass is actually oscillating. Since this is a real spring and not our abstract massless spring, we should take into account the parts of the spring that are also moving. We won't go into it here, but under these conditions, we should count one third of the mass of the spring.[2] Replotting and adding more springs result in lines with intercepts very close to zero. We see then that the square of the period for each is proportional to the mass plus 1/3 the mass of the spring. Now, what's different about these springs? These are the same springs we used for Hooke's relationship in Section 6. According to the relationship above, we can determine each spring constant from the slope of the associated line, since

$$\mathrm{y} = (\mathrm{slope})\mathrm{x} + (\mathrm{intercept}) \quad \leftrightarrow \quad \mathrm{P}_\mathrm{o}^2 = \left(\frac{4\pi^2}{\mathrm{k}}\right)\mathrm{m} + 0$$

$$\mathrm{slope} = \frac{4\pi^2}{\mathrm{k}} \quad \rightarrow \quad \mathrm{k} = \frac{4\pi^2}{\mathrm{slope}} \ .$$

Comparing results from this experiment and from those of section 6,

[2] J.G. Fox and J. Mahanty, "The Effective Mass of an Oscillating Spring," *Am. Jour. Phys.* 38 No 1 (January 1970): 98–100.

| Spring | Hooke's relationship experiment | Oscillation experiment |
|---|---|---|
| Nr 1 | 8.25 N/m | 8.42 N/m |
| Nr 2 | 16.13 N/m | 16.27 N/m |
| Nr 3 | 64.10 N/m | 67.88 N/m |

gives us a bit more confidence that this relationship is correct.

In the discussion above, we assumed that the only force acting on the mass was that of the spring. That might be appropriate way out in space, or if the mass were mounted horizontally on an airtrack. But often, springs are arranged vertically, and so gravity plays a part. Turns out, though, that this does not affect the results for the frequency of oscillation; the mass will simply hang at a lower equilibrium point as more mass is added.

JUSTIFICATION 11-1*

When the spring is horizontal, its motion is governed by the second law,

$$-kX = ma \ .$$

When a massless spring is hung vertically, its lower end sits at the equilibrium point, X = 0. If we add some mass to the end and allow the system to come to rest, the new equilibrium point will be obtained from the second law:

$$-kX_{EQ} - gm = 0 \quad \rightarrow \quad X_{EQ} = \frac{-gm}{k} \ .$$

Note that this is negative, because the new equilibrium point will be lower than the original one. Now, let the mass oscillate about this new equilibrium point. The second law equation will be

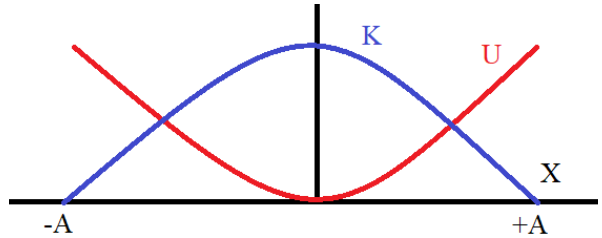
$$-kX - gm = ma \ .$$

Substitute:

$$-kX - kX_{EQ} = -k(X - X_{EQ)} = -kX' = ma \ .$$

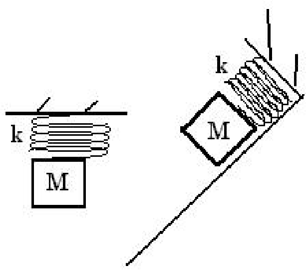
Here, X' is the displacement of the object from the new equilibrium point. This is the same force equation (and therefor the same motion) for the case of no gravity, except that the equilibrium point will be at $X_{EQ}$.

Let's have a quick discussion about the energy of a SHO. The kinetic energy is $K = \frac{1}{2}mv^2$ and the potential energy is $U = \frac{1}{2}kx^2$, and the total is the sum of these two.




$$
\begin{aligned}
E_{TOTAL} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kX^2 \\
&= \tfrac{1}{2}m(-2\pi f\ A\sin(2\pi f t))^2 \\
&+ \tfrac{1}{2}k(A\cos(2\pi f t))^2 \\
&= \tfrac{1}{2}m\frac{k}{m}A^2\sin^2(2\pi f t) \\
&+ \tfrac{1}{2}kA^2\cos^2(2\pi f t) = \tfrac{1}{2}kA^2\ .
\end{aligned}
$$

So, if we pull back the mass to x = A and release it, the energy will convert from potential to kinetic, then back to potential, *et c*.




HOMEWORK 11-1

An oscillator with a 0.23 second period is made from a mass M suspended from a spring of constant k. The mass is then placed on a frictionless surface which makes a 45° angle with the horizontal, and the spring is attached at the top of the incline as shown. What is the new period of the oscillation?

EXAMPLE 11-2

A 0.8 kg air-track car is attached to the end of a horizontal spring of constant k = 20 N/m. The car is displaced 12 cm from its equilibrium point and released. What is the car's maximum speed? What is the car's maximum acceleration? What is the frequency $f_o$ of the car's oscillation?

The frequency of oscillation is given by

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20}{0.8}} = 0.80\ \text{Hz}\,.$$

The velocity of the object[3] is given by

$$v(t) = -2\pi f_o A\sin(2\pi f_o t) = -2\pi(0.80)(0.12)\sin(2\pi f_o t) = -0.60\sin(2\pi f_o t)\ .$$

The speed is a maximum whenever the sine term equals ± 1. Maximum speed is 0.60 m/s.

[3] Remember that we're always assuming that the object is pulled in the positive direction to X = A and released at time = zero.

The acceleration is

$$a(t) = -(4\pi^2 f_o^2) A\cos(2\pi f t) = -4(9.87)(0.08^2)(0.12)\cos(2\pi f t) = -0.03\cos(2\pi f_o t).$$

The maximum magnitude acceleration is when the cosine term is ± 1, so $|a_{MAX}|$ = 0.03 m/s$^2$.

HOMEWORK 11-3

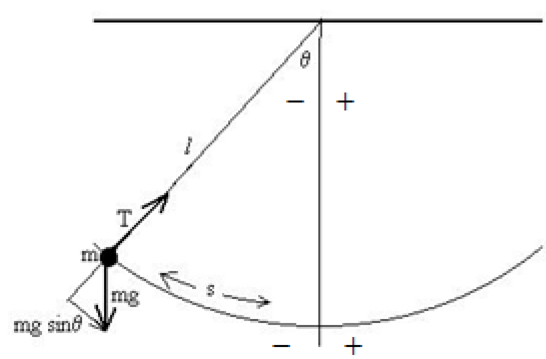
A mass (0.3 kg) and spring (k = 350 N/m) system oscillates with an amplitude of 6 cm. What is the total mechanical energy of the system? What is the maximum speed of the mass? What is the maximum acceleration of the mass?

## The Simple Pendulum

There are many other systems which exhibit simple harmonic motion (SHM), and even more that are close enough that we can make use of the results above for a reasonably correct approximate solution. One such system is the *simple pendulum*, which is a point mass m (the bob) at the end of a massless string or stick of length *l*. Let's look at the free body diagram for such an object.

DERIVATION 11-1

We are interested in the motion along the circular arc. Let us describe the bob's position with s (= $l$ θ), the displacement along the arc which we shall make positive to the right and negative to the left. Theta will follow the same convention. Break the forces into tangential and radial components. We aren't really interested in the radial components, but tangentially we have




$$-gm\sin\theta = ma_T \ .$$

The negative sign is necessary to get the direction of the force correct. When the bob is on the left side of the figure where s and theta are negative, we want the force to be in the positive direction. Similarly, when the bob is on the right side of the figure where s and theta are positive, the force must be to the negative direction.

We have two types of variables here, one tangential and the other angular. We need them to be the same type. Substitute s/*l* for theta:

$$-gm\sin\left(\frac{s}{l}\right) = ma_T \ .$$

Now, this is not the same as the for the mass/spring system, since the force F is proportional to the sine of the displacement, not to the displacement itself. In fact, this is a moderately difficult equation to solve, even using calculus. So, we will do what physicists often do, we will look at a special case, when the angle theta is 'small.' If an angle is small, the sine of the angle is approximately equal to the angle itself in radians.

MATHEMATICAL DIGRESSION

Put your calculator in radians mode. Take the sine of 0.0001 radians. How close it the result to 0.0001? Is 0.01% much of a difference? Repeat for 0.001, 0.01, and 0.1. Are the two values diverging slightly? Repeat for 0.5 radians and you will see that the result is about 4% off from the input. The art here is determine how much of a divergence is acceptable, or how small is 'small.' In a course like this one, we usually accept the approximation up to about 30°.

Continuing,

$$-\frac{gm}{l}\,s = ma_T\ .$$

Turns out, we've already solved this problem. The acceleration is negatively proportional to the position, same as for the mass on a spring. So, all of the steps we went through to solve that problem are the same steps we would go through here, except k is replaced with gm/$l$. Then,

$$f_{o\ \text{Mass on Spring}} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \rightarrow \quad f_{o\ \text{Simple Pendulum}} = \frac{1}{2\pi}\sqrt{\frac{\frac{gm}{l}}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\ .$$

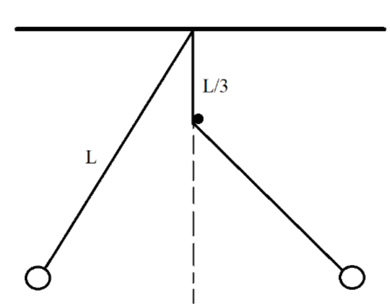
Some of you may have verified the relationship between the frequency and the length in lab.

HOMEWORK 11-4

Mary-Kate (m = 50 kg) is swinging on a tire tied by a (light) rope (L = 3 m) to a tree limb. Her twin Ashley comes along and squeezes into the tire with her. Assume that at all times the center of mass of the person(s) riding the tire is at the end of the rope. What was the period of oscillation for Kate alone? What is the period of oscillation for the twins together?

HOMEWORK 11-5

A pendulum bob on a light string of length L is arranged as shown in the figure. There is a peg stuck into the wall a distance L/3 below the point of suspension. What is the period of small oscillations for this system?



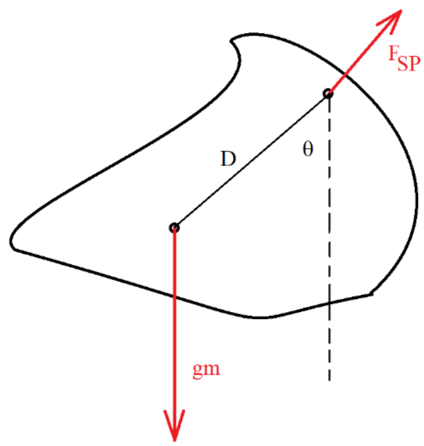

DISCUSSION 11-3

Because we're not solving the actual equation for the motion of the simple pendulum, the correct result differs to some degree from what we've derived. Do you think that actual period of a simple pendulum at large angles is larger or smaller that for small angles? Try writing the force equation in terms of torques instead:

$$\text{Actual equation: } -\frac{g}{l}\sin\theta = \alpha$$
$$\text{Approximation equation: } -\frac{g}{l}\,\theta = \alpha \quad .$$

## The Physical Pendulum

DERIVATION 11-2*




Consider an object of indeterminate shape hanging from an axis, as show. This is known as a *physical pendulum*. D is the distance between the point of suspension and the center of mass. The forces acting on the object comprise a force at the suspension point (the pivot) and the weight, which can be assumed to act at the center of mass. Consider the torques acting on the object when it has been displaced from equilibrium by angle theta:

$$-\mathrm{Dgm}\sin\theta + (0)F_{SP}\sin(?) = I\alpha \quad .$$

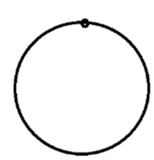
If we once again restrict ourselves to 'small' angles,

$$-\mathrm{Dgm}\,\theta = I\alpha \ ,$$

we see that this problem is the same as for the mass on a spring (alpha is to theta as a is to x) and that we have already solved it. The result is found by replacing k with Dgm and m with I:

$$f_{o\text{ Physical Pendulum}} = \frac{1}{2\pi}\sqrt{\frac{\mathrm{Dgm}}{I}} \quad .$$

EXAMPLE 11-1*



Find the frequency of small oscillations for a vertically suspended disk of radius R and mass M if it is attached to an axis at its top.

We can make use of the results above, but we will need to determine D and I. D is the distance between the suspension point and the center of mass, so, D = R. The moment of inertia of a disk about its center is $^1/_2MR^2$, but we've moved the axis a distance h = R, so we'll invoke the parallel axis theorem:

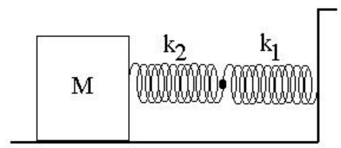
$$I_{EDGE} = I_{CM} + Mh^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2 \quad .$$

$$f_{0\ \text{Physical Pendulum}} = \frac{1}{2\pi}\sqrt{\frac{RgM}{\frac{3}{2}MR^2}} = \frac{1}{\pi}\sqrt{\frac{g}{6R}}\ .$$

HOMEWORK 11-6

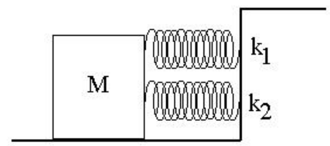
First, find the frequency of a simple pendulum with a point mass bob of mass M of a light string of length L. Then, find the frequency of a spherical bob of mass M and radius R = 0.1L at the end of that same light string. Calculate a *per cent* difference.

EXERCISE 11-1*




A mass is attached to two massless springs as shown in the figure. What is the natural frequency of oscillation $f_o$ if M = 7 kg, $k_1$ = 300 N/m, and $k_2$ = 900 N/m? Assume no friction.

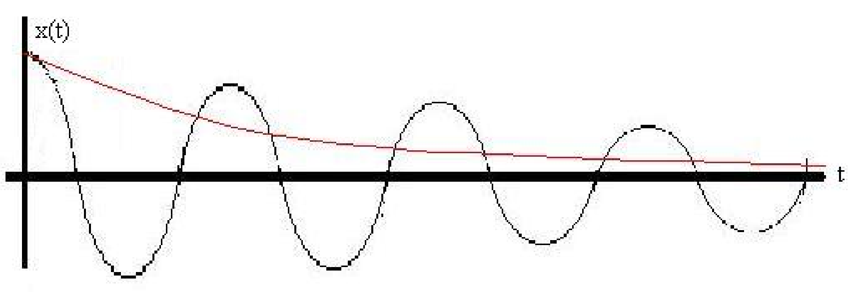
HOMEWORK 11-7




A mass is attached to two massless springs as shown in the figure. What is the natural frequency of oscillation $f_o$ if M = 7 kg, $k_1$ = 300 N/m, and $k_2$ = 900 N/m? Assume no friction.

## Damped Oscillations

We spoke briefly about *damped oscillations*. The discussion above suggest that, if one sets the mass/spring system into oscillation, the total energy of the system remains constant and the mass will vibrate forever with the same amplitude. In fact, we know that the mass will slow a bit on each pass due to friction with the air (usually assumed to be a *drag force* of the form $F_f$=-bv) or the table; energy is removed as friction performs negative work on the mass.

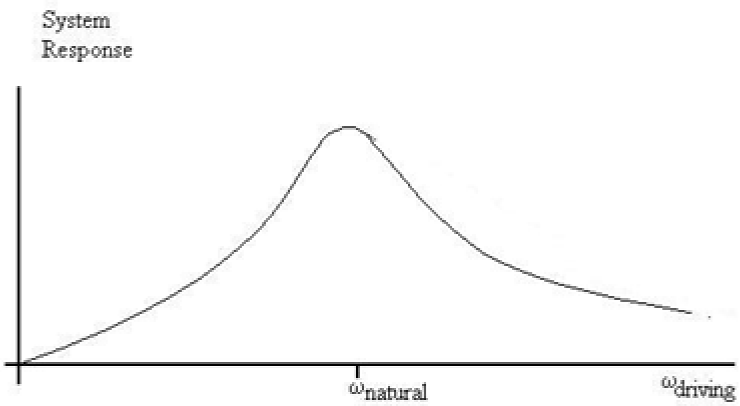



The figure shows a *lightly damped* system (black curve) and an *overdamped* system (red line), which loses so much energy so quickly that it never oscillates even once. A good example of the overdamped system is the car shock absorber. The car (m) is supported by springs (k), so that SHM is possible. If one were to drive over a bump with faulty shocks, the car would then continue to oscillate at about 1 Hz for several seconds. Shock absorbers dampen the system so that the ride smooths out without the oscillations.

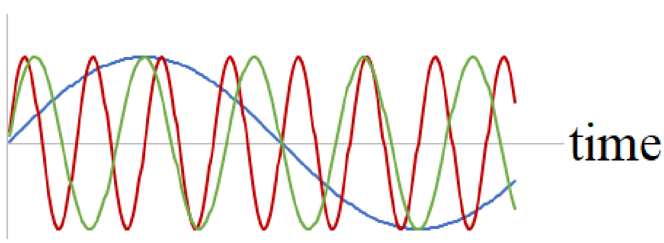
DISCUSSION 11-4

The natural frequency of a lightly damped oscillation is slightly lower than that of an undamped system. Can you give a brief, non-mathematical reason for this? Consider the very first swing of the bob. What does the retarding force do to the speed of the bob?

## Resonance




If we were to disturb the mass/spring system in some way and step back, the system will oscillate with natural frequency $f_o = [k/m]^{1/2}/2\pi$. If it's disturbed again in a different manner then left to itself, the system will again oscillate at that same natural frequency, until its energy is depleted. If we want the system to continue to oscillate, we must replace the energy lost to dissipative forces. Let's jiggle the other end of the spring, applying a force though a distance (*i.e.*, doing work), at some frequency $f$, which is then known as the *driving frequency*. Let us vary the driving frequency to see the effect on the system. If we jiggle the spring at a very low frequency, we see that the mass oscillates with the same frequency at which it is driven, but with a small amplitude. Changing to very high driving frequency, we see once again that the mass oscillates at the driving frequency, but with a very small amplitude. However, if we excite the system at a driving frequency very near to the natural frequency, we see that the response of the system, as demonstrated by the amplitude of oscillation, increases. If we plot this response as a function of the driving frequency, we see the curve shown in the figure. Consider this simplified situation:




The green line represents oscillation at the natural frequency. If we were to apply the force as the shown by the blue line at a frequency less than $f_o$, we would see that sometimes the force is acting in the direction of motion of the mass, but at other times, against the motion. On average, then, no work is done by that force. The red line indicates the force with frequency $> f_o$, and the argument is the same. When we vary the applied force at the same frequency as the natural frequency, we are always applying force in the direction of motion of the mass, so all work we do is positive. If the rate of doing work is greater than the rate of energy dissipation, the amplitude of the oscillation will increase. The condition when the system is driven at its natural frequency and delivers its greatest response is called *resonance*. Sometimes resonance is desirable, sometimes not. For example, if one wants to push a small child on a swing, the greatest amount of fun (or terror) is attained when one pushes the swing at its natural frequency. On the other hand, if the ground shakes at the natural frequency of a skyscraper, the building may respond with an amplitude beyond the limits of structural integrity. The Tacoma Narrows Bridge collapse occurred because the wind passing over the bridge excited one of the span's torsional oscillation modes, resulting in the collapse about three hours later. Are you surprised at the incredible elasticity of steel and concrete? Only a dog lost its life in the collapse, because the owner left it behind when he abandoned his car on the bridge (Hmm!). The bridge had exhibited strange effects for the three months it was open. There are films of the deck of the bridge oscillating in a vibrational mode much like waves in the ocean; cars could actually disappear from view behind the humps which rose and fell in the roadway. A related system is that of tall skyscrapers. Once again, if the wind were to gust at the natural frequency of the building, it might cause collapse; modern buildings often have a mechanism to 're-tune' the vibrational modes of the building away from the current driving frequency of the wind.

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Exercise 11-1 Solution

Suppose that Spring 1 is stretched from its equilibrium length a distance $X_1$. To do this, a force of $F_1 = k_1X_1$ is required. This force is applied by Spring 2. Suppose that Spring 2 is stretched a distance $X_2$ from its equilibrium length. This requires a force $F_2 = k_2X_2$. By the third law, this is the same magnitude force as $F_1$ and the force applied to the mass. We want to replace the two springs with a single spring of constant $k_{EFF}$ that will apply the same force when it is stretched a distance $X_{EFF} = X_1 + X_2$.

$$X_{EFF} = X_1 + X_2$$

$$\frac{F_{EFF}}{k_{EFF}} = \frac{F_1}{k_1} + \frac{F_2}{k_2}$$

All of the forces here are the same magnitude, so

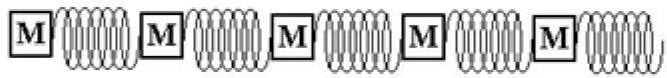
$$\frac{1}{k_{EFF}} = \frac{1}{k_1} + \frac{1}{k_2} \quad \rightarrow \quad k_{EFF} = \frac{k_1 k_2}{k_1 + k_2} = \frac{(300)(900)}{300 + 900} = 225\ \mathrm{N/m}\,.$$

Then,

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{225}{7}} = 0.90\ \mathrm{Hz}\,.$$

## Section 12 - Waves and Sound

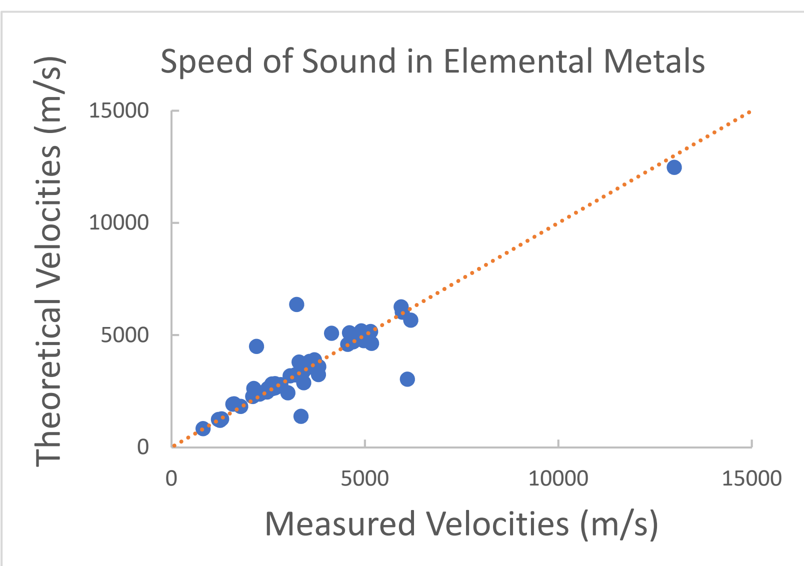
We covered single oscillators in Section 12. For the specific example of a mass/spring system, we saw that there is a natural frequency at which the system would 'like' to oscillate, given by $f_o = [k/m]^{1/2}/2\pi$. Now, let's consider a chain of such oscillators, in this case identical masses connected by identical springs. If we apply a disturbing impulse to the left-end mass, it will move to the right, applying its own force on the next mass, *et c*. The speed of this disturbance as it moves down the chain of oscillators should depend on the masses and on the spring constants. For example, if the springs are stiff, the first mass will not need to move very far in order to apply a fairly large force on the next mass, while if the spring is flexible, the first mass would need to move a good distance to apply a sizable force to the next mass. In the same way, for a given force, smaller masses will respond more quickly and larger masses less quickly. We might surmise that the speed of the disturbance should be related to the frequency of oscillation of each mass, and based on dimensional analysis, we might suspect the relationship to be




$$v \sim \sqrt{\frac{k}{m}}\ .$$

While this discussion was based on a chain of masses, we can guess that the speed of a compression wave in a material should be proportional to the square root of the ratio of an elastic property to an inertial property:

$$v \sim \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}\ .$$



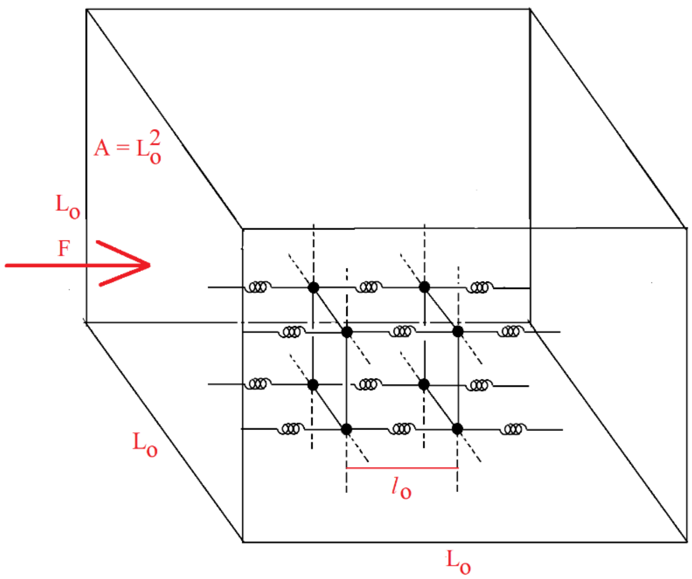

For example, the speed of a sound pulse in a metal can be shown experimentally to be

$$v = \sqrt{\frac{Y}{D}}\ ,$$

where Y is the *Young's modulus*, a measure of the springiness of the material, and D is the mass density of the material (inertial property). In the figure, the dotted line (which is not a best fit line) shows where the measured value and the theoretical values would be equal.

JUSTIFICATION 12-1*

The density and the Young's modulus are both macroscopic quantities. Let's see if we can correlate this result to microscopic quantities. From chemistry, we think that the material can be modelled by small balls of mass m (representing the atoms) connected by bonds represented by springs with stiffness k, and relaxed length $l_o$. To make things a little easier, we'll assume that the metal's atoms lie on the corners of cubes, as shown in the figure.[1] I omitted the 'springs' in two directions since the oscillations will occur in the left-right direction and we will assume that the plane of atoms will move as a single unit. Each pair of adjacent planes of atoms are connected by a large number of 'bond' springs.




For simplicity, let's say that the object is a cube of edge $L_o$ and mass M. The cross-sectional area of the left end will be $A = L_o^2$. Let's apply forces F along the length of the block, as shown. As a result, the block will contract along that axis:

$$\frac{F}{A} = Y\,\frac{\Delta L}{L_o}.$$

Here, F/A is the *stress* and $\Delta L/L_o$ is the *strain* (cause and effect).

If the cube has mass M and the mass of each atom is m, then there are N = M/m atoms. There will be $N^{1/3}$ atoms along each edge each separated from its nearest neighbor by distance $l_o$, so

$$L_o = N^{1/3} l_o \quad .$$

We might also assert that the compression of the bond length is in the same proportion to the compression of the entire length of the cube:

$$\frac{\Delta l}{l_o} = \frac{\Delta L}{L_o} \quad .$$

Similarly, the area of the left end, and therefor of each of our planes, is

$$A = L_0^2 = \left(N^{1/3} l_o\right)^2 .$$

[1] Such an arrangement is called a *simple cubic structure*. There is unfortunately only one elemental metal that has this structure, polonium, and that is highly radioactive and doesn't stick around very long before decaying into other elements.

The number of atoms in such a plane is $(N^{1/3})^2$, and so the mass of such a plane will be

$$M_{PLANE} = N^{2/3}m \ .$$

Each such plane has $N^{2/3}$ springs connecting it to the adjacent plane. We saw in a Section 12 that the effective spring constant of 'parallel' springs is the sum of the spring constants, k:

$$k_{PLANE} = N^{2/3}k \ .$$

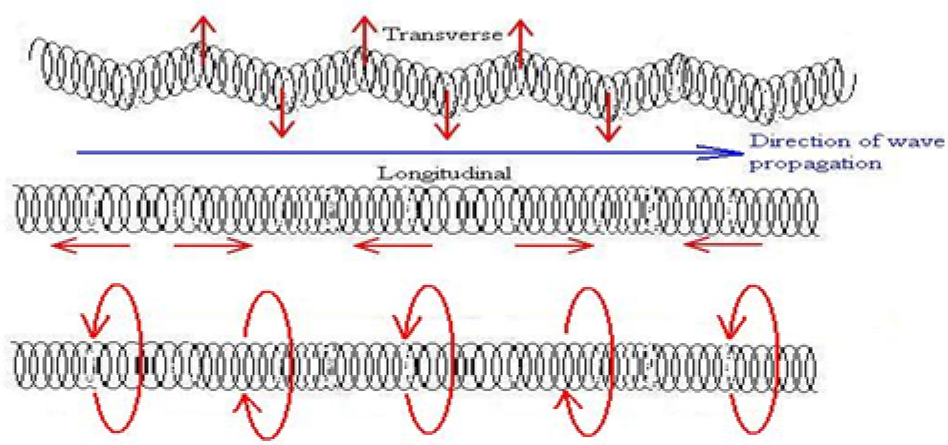
So, as we push on the left end with force F, Hooke's relationship says that we should see that

$$F = k_{PLANE}\Delta l \ .$$

Finally, of course, the density D is the mass divided by the volume, $D = M/L_o^3 = m/l_o^3$. So, here we go.

$$v = \sqrt{\frac{Y}{D}} = \sqrt{\frac{\left(\frac{F/A}{\Delta L/L_o}\right)}{\left(\frac{M}{L_o^3}\right)}} = \sqrt{\frac{\left(\frac{F/A}{\Delta l/l_o}\right)}{\left(\frac{m}{l_o^3}\right)}} = \sqrt{\frac{F}{\Delta l}\frac{1}{N^{2/3}}\frac{l_o^2}{m}} = \sqrt{\frac{k_{PLANE}}{N^{2/3}}\frac{l_o^2}{m}} = l_o\sqrt{\frac{k}{m}} \ .$$

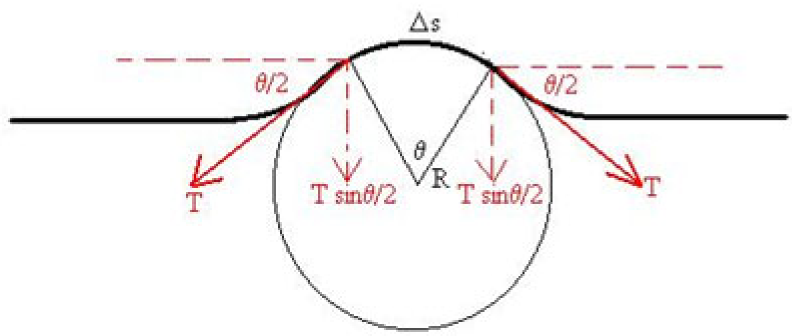
The speed of sound in a fluid (like air) also follows this form: $v = [B/D]^{1/2}$, where B is the *bulk modulus*, a measure of the elastic properties of a fluid.




We will be using one particular type of wave as our archtype; things we learn about this particular wave are transferrable to most other types of waves. We will concentrate on the *transverse wave* on a string. Waves in which the individual pieces of the medium move along the same line as the direction of propagation of the wave are referred to as *longitudinal*, while waves in which each piece of material moves along a line perpendicular to the direction of propagation, such as on a taut string, are called *transverse*. Here is a nice non-calculus derivation for the speed of a disturbance on such a string.

DERIVATION 12-1

Consider a pulse traveling along the string to the right at some speed v, and further assume that the central part of the pulse has the shape of a circular arc with radius R. The actual value of R doesn't matter, it just needs to be chosen so that part of the string lines up with the arc. If the size of the disturbance is fairly small, then the tension T is approximately constant along the string, even where the disturbance is. Now, to make this work, we will shift to a reference frame that is moving along with the pulse; in other words the pulse appears stationary and the string is moving at speed v to the left, making a small detour around the circular arc. In order to move in that circle, a small piece of the rope of length Δs (and mass Δm) would need to experience a centripetal force, $F_C$, which is provided by the two components of the tension acting toward the center of the circle:




$$F_C = 2T\sin\left(\frac{\theta}{2}\right) = a_C = \Delta m\,\frac{v^2}{R}.$$

Making use of the small angle approximation for the sine function,

$$2T\left(\frac{\theta}{2}\right) = T\theta = \Delta m\,\frac{v^2}{R}\,,$$

$$v = \sqrt{\frac{T\,\theta R}{\Delta m}} = \sqrt{\frac{T\,\Delta s}{\Delta m}}\,.$$

While we presumably know the tension T in the string, we made no specific choices about the sizes of Δs or Δm, which should depend on theta and R. But luckily, it doesn't matter; the ratio Δm/Δs depends only on the string itself. We call this ratio the *linear mass density*, or just the mass *per* unit length, μ. Then,

$$v = \sqrt{\frac{T}{\mu}}\,.$$

Once again, the speed is given by the square root of the ratio of an elastic property (how' snappy' the string is) to an inertial property.

Here is an interesting example: whips are constructed so that the linear mass density decreases nearer the tip (that is, they get thinner near the end). A pulse sent down a whip will therefor travel more and more quickly, and it is possible that some of the pieces near the end will move more quickly than the speed of sound, causing a mini *sonic boom* (see below).

## Reflections

We next looked at reflections of pulses on a string. We noted that a pulse traveling down the string is reflected with the same orientation if the end of the string is free to move, and reflected with an inverted orientation if the end of the string is fixed. Although this could be proven mathematically, we based our assertion on experiment. We can visualize what's happening, however, by imagining that the string continues beyond its actual end, and that an imaginary wave is traveling back down the string towards the end from the imaginary side. We invoke the *principle of superposition*, the notion that the total displacement of the medium is the sum of the individual displacements due to each pulse; in this way, we know that the reflected pulse for a fixed end must be inverted, since this is the only way the total displacement at the end of the string can always be zero, and then clearly, the wave and its reflection must add together in the case of a free end. Here is an animation illustrating the two types of reflections/ INSERT

Now, instead of considering the two extreme cases (completely fixed or completely free ends), think about what would happen if the end of a string were tied end to end to another string. In all cases, we would expect that some of the wave would continue down the second string with the same orientation (and frequency) as the original wave; this is called the *transmitted wave*. The argument we can give is that the end of the first string does the same thing to the second string as the hand or other agent did to the first string at the other end. We also notice that there is a *reflected wave*, the orientation (and size) of which depends on whether (and by how much) the second string is 'heavier' or 'lighter' than the first. Watch this demonstration video. INSERT

DISCUSSION 12-1

> What do you notice about the size of the pulses that are transmitted and reflected? Why does this happen?

The quantity used to measure the difficulty of a wave in passing through some medium is called the *impedance*, Z. If $Z_2 > Z_1$, the reflected wave is inverted; if $Z_2 < Z_1$, the reflected wave is upright. This is a general result, even though the exact values of the impedances are calculated in different ways for different media. In the specific example of transverse waves on a string, we have that (asserted without proof)

$$Z = \sqrt{T\mu} \ .$$

In that case, we see that our original examples correspond to $Z_2 = 0$ (end of string loose, so $\mu_2 = 0$) and $Z_2 =$ infinity (end of string tied to wall, so $\mu_2 =$ infinity). The impedances also tell us how much energy is reflected and how much is reflected.

DISCUSSION 12-2

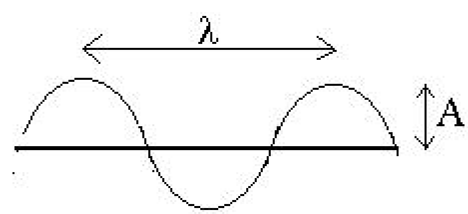
> What would happen if the two media had the same impedance? Are there occasions when we would want the incoming wave to be totally reflected? Are there occasions when we would want the incoming wave to be completely transmitted? Suppose you are walking across campus and see a friend on the other side of the quad. How would you get his attention (do

not say ‘text him’)? If he didn’t hear you, what might you do with your hands? Why does this work?

Abrupt changes in impedances between two media results in strong reflection. To minimize reflections, the transition from one material to the other should be made as gradual as possible. Cupping your hands around your mouth makes the transition from a tube of several centimeters diameter to the wide-open atmosphere smoother, resulting in more transmission of sound. Coxswains of racing shells often use megaphones, a hollow truncated cone used to amplify the voice. Some of you may remember the days of rabbit ear antennas for your televisions. The impedance of the ribbon cable is 300 *ohms*,[2] which matches that of the screw terminals on the back of the TV. When cable came along, people with older TVs had to buy matching transformers to convert the cable’s 75 ohm impedance to the TV’s 300 ohms. Failure so to do resulted in multiple reflections, appearing as ‘ghosts’ on the screen.

## Sinusoidal Waves

Instead of pulses, let's discuss a more commonly studied type of wave, one in which the system is driven by a sinusoidal force with frequency *f*. Now, the wave which is produced will have the same frequency as the driving force, even though the speed of propagation will be determined by the natural frequency of the individual oscillators. Let’s start by defining some of the characteristics of a wave.

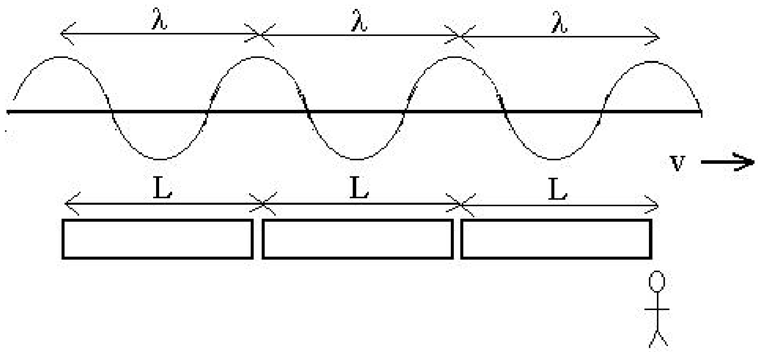



- the *frequency* (*f*) counts how many oscillations each piece of material experiences each second, or alternatively, how many peaks pass by an observer each second. The period P is as before the reciprocal of the frequency.
- the amplitude (A) describes the maximum deviation from equilibrium. This is easy to visualize in the example above where A refers to the maximum displacement rom equilibrium, but it can also refer to the maximum excess of pressure over average atmospheric as in sound, or the maximum electric field strength as in light, *et c*.
- the speed (discussed above).
- the wavelength (λ) measures the physical distance between corresponding points on adjacent waves (*e.g.*, from peak to peak).

There is a relationship among *f*, λ, and v, which we can deduce from the 'railcar analogy.'

Suppose that a train with cars of length L passes you at speed v. You count N cars in time t. The distance traveled by the train in that time is d = NL. The speed of the train is v = d/t = (NL)/t =

[2] An ohm is a unit for electrical impedance.




(N/t)L. We recognize (N/t) as the frequency $f$ (the number of peaks that pass *per* unit of time) and L as the analog of λ, so $v = f\lambda$.

Although we won't prove it, sinusoidal waves moving along the x-axis are described mathematically by the expression:

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

where φ is a *phase angle* that allows us to change the function to cosine or to some combination of sine and cosine. As in the last section, we'll let the phase angle be zero for simplicity.

Since we didn't derive this relationship, we'll at least show that it is correct and does what we need it to do. Let's examine each term one by one.

1) The shape of the curve is obviously sinusoidal.
2) The sine function varies between +1 and -1, and this function y(x, t) varies between +A and -A.
3) We've already addressed phi.
4) Suppose we freeze the wave at t = 0. Since we can assign the value of zero to any particular time we choose, this works for any instant. The equation reduces to

$$y(x,0) = A\ \sin\left(\frac{2\pi}{\lambda}x\right)\ .$$

   Let's consider the location x = 0, then follow the curve though one cycle or one wavelength. The function then goes from

$$y = A\ \sin(0) \text{ to } A\ \sin\left(\frac{2\pi}{\lambda}\lambda\right) = A\ \sin(2\pi)\ \ ,$$

   which corresponds to one mathematical cycle, as desired.

5) Now, let's examine what happens at the particular location x = 0. Again, we can set the origin to be anywhere for our convenience. The equation reduces to

$$y(0,t) = A\ \sin(\mp 2\pi f t)\ .$$

   We'll follow that bit of medium through one cycle or one period, $P = 1/f$. The function then goes from

$$y = A\ \sin(0) \text{ to } y = A\sin(\mp 2\pi f P) = A\sin(\mp 2\pi)\ ,$$

which corresponds to one mathematical cycle, forward or backward as desired.

6) Lastly, we'll confirm that the negative sign between the terms corresponds to the wave moving to the +x-direction, and the positive sign to movement in the – x-direction. According to the equation,

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x - 2\pi f t\right)\ ,$$

there will be a peak in the curve when the argument of the sine function is equivalent to 90°. As time increases, the second term will become more positive, but there is negative sign which makes the argument more negative. In order to keep the argument at 90°, the x value must increase. That is, the position of the peak will move toward positive x. For the other case,

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x + 2\pi f t\right)\ ,$$

as time increases, the argument will become more positive, so to keep it constant at 90°, the x value must become more negative, that is, the position of the peak will move toward negative x.

So, it appears that this formula does everything we would expect of it. Note from Item 6 that it is the interplay between the spatial and temporal terms that accounts for the wave's motion.

## Standing Waves

The *principle of superposition* states that, if more than one wave is passing through a given point, the total displacement is the sum of the displacements due to each individual wave. Suppose that we set up two sinusoidal waves in a (one dimensional) medium which are identical in every way except their directions. A specific example would be two waves moving in opposite directions along a very long string, each end being jiggled at the same amplitude and frequency. What happens when these waves meet? We use the principle of superposition to find the result by adding the two individual waves. We saw that the resulting wave did not appear to travel at all; this type of wave is called a *standing wave*, although *stationary wave* might be more apt. Let's examine this more mathematically.

DERIVATION 12-2

$$y_1 = A\sin\left(\tfrac{2\pi}{\lambda}x - 2\pi f t\right)\ \ \text{(moves to the right), and}$$

$$y_2 = A\sin\left(\tfrac{2\pi}{\lambda}x + 2\pi f t\right)\ \ \text{(moves to the left)}\ .$$

We'll expand these out using the trig identity

$$\sin(\alpha \pm \beta) = \sin\alpha\ \cos\beta\ \pm\ \cos\alpha\ \sin\beta\ \ .$$

$$
\begin{aligned}
y_{\mathrm{TOTAL}} &= y_1 + y_2 \\
&= A\left[\sin\left(\frac{2\pi}{\lambda}x\right)\cos(2\pi f t) - \cos\left(\frac{2\pi}{\lambda}x\right)\sin(2\pi f t)\right] \\
&\quad + A\left[\sin\left(\frac{2\pi}{\lambda}x\right)\cos(2\pi f t) + \cos\left(\frac{2\pi}{\lambda}x\right)\sin(2\pi f t)\right] = 2A\sin\left(\frac{2\pi}{\lambda}x\right)\cos(2\pi f t)\ .
\end{aligned}
$$

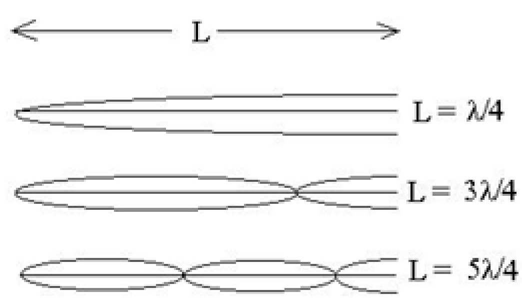
Here, we see that, in contrast to the original waves which had mixed spatial and temporal components, this wave has separate spatial and temporal components. Each piece of the medium undergoes SHM, but with an amplitude that depends on its x-position. INSERT VIDEO ANIMATION

We can see that there are spots that never oscillate (*nodes*) and spots that have maximum oscillation (*anti-nodes*). Each type of location is separated from its adjacent neighbor by one-half of the wavelength.

This little derivation assumes that the strings are very long. In the real world, string (or other similar systems) have a finite length. In such systems, there will be many reflections from each end which will have to be added to determine the overall behavior of the string.

VIDEO

We started with a string fixed at each end, and we excited waves with different frequencies. In some cases, we noted that the waves all added up to a random pattern, eventually canceling out. In other cases, we saw that the initial and reflected waved added to produce a standing wave. What conditions need to be met to do this? We could do a very mathematical derivation of this, but it is just as correct to base our investigation on observation.

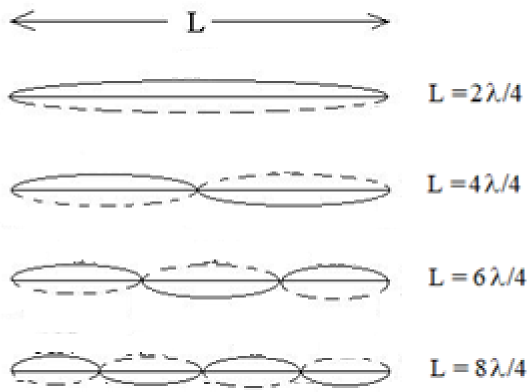



System 'fixed' at one end and 'free' at the other - We saw a series of patterns like those shown in the figure as we increased the frequency at which the system was excited. The lines indicate the limits of the oscillations of the string (the envelope). The fixed end must always correspond to a node (no motion) and the free end, where the string can move the most, corresponds to an antinode. As we increase the frequency (or shorten the wavelength), we must always add in an additional node and an additional antinode to fulfill the conditions above, thereby adding two quarter wavelengths. We notice that the length of the system must be an odd natural number multiple of a quarter wavelength, or

$$L = \frac{n\lambda}{4}\ , \qquad n = 1, 3, 5, 7, \ldots\ .$$

Since we already know that $v = f\lambda$, we can solve for the allowed frequencies for these stationary waves:

$$f_n = \frac{nv}{4L} \quad n = 1, 3, 5, 7, \dots \ .$$



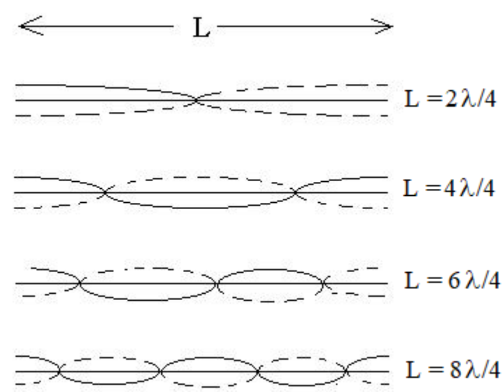

System 'fixed' at both ends - We saw a series of patterns like those shown at left as we increased the frequency at which the system was excited. Here, we see that the length is an even natural number multiple of a quarter wavelength:

$$L = \frac{n\lambda}{4} \ , \qquad n = 2, 4, 6, 8, \dots \ ,$$

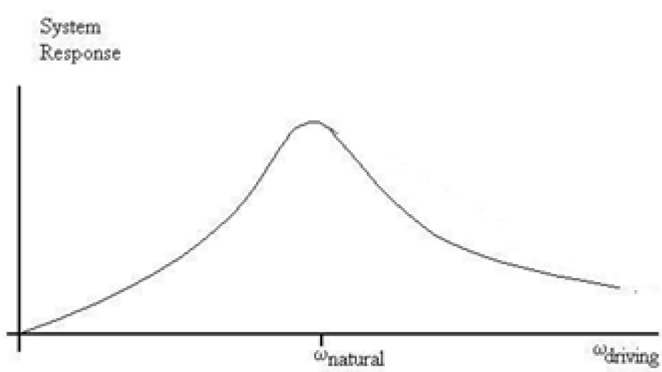
and so,

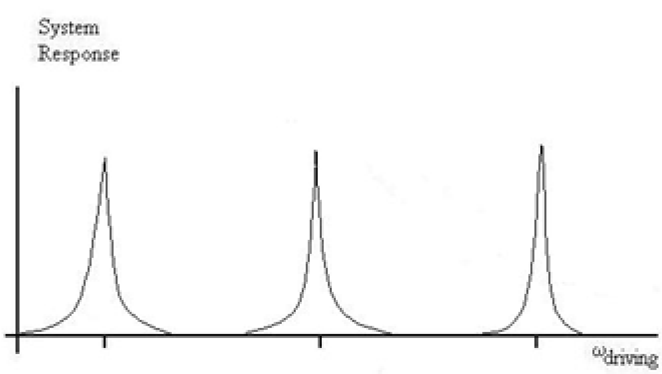
$$f_n = \frac{nv}{4L} \quad n = 2, 4, 6, 8, \dots \ .$$

Finally, when both ends are 'free,' we see that the result is the same as for both ends fixed but with the nodes and anti-nodes reversed.




So, now we have an easy-ish relationship and an easy way to remember which numerical values to insert: if the ends are of the same type, or 'even,' use the even values and if the ends are different, or 'odd,' use the odd values. We see that, unlike for a single oscillator with a single natural frequency, we here have a system with many natural frequencies:




Remember that, even though we derived these results for transverse waves on a string, the results are valid for other system. For example, consider a stopped organ pipe, which means that it is open at one end and closed off at the other. At the open end, air is free to vibrate, while at the closed end, no vibration is possible because of the stopper. This pipe will support standing waves of the form $f_n = nv/4L$, n = 1, 3, 5, .... How are these frequencies produced? In an organ, air is pumped into the pipe against a sharp edge, which produces all frequencies. However, those frequencies which do not correspond to the favored frequencies reflect back and forth in the pipe and, on average, cancel themselves. But the few special frequencies re-inforce one another and produce standing waves. These frequencies are often referred to as the *harmonics* of the system. On occasion, they are referred to as the *fundamental* (n = 1) and the *overtones* (n > 1).

EXERCISE 12-1

> The human ear canal is about 3 cm long. If it is regarded as a tube open at one end and closed at the other, what is the fundamental frequency of a standing wave in the ear?? Let $v_{sound}$ = 340 m/s. Since these frequencies are re-inforced in the ear canal, human hearing is just a little bit better at this frequency that at others.

HOMEWORK 12-1

> A tuning fork is sounded above a (narrow) resonating tube as in lab. The first resonant situation occurs when the water level is 0.08 m from the top of the tube, and the second when the level is at 0.24 m from the top. Let $v_{sound}$ = 340 m/s.
> a) Where is the water level for the third resonant point?
> b) What is the frequency of the tuning fork?

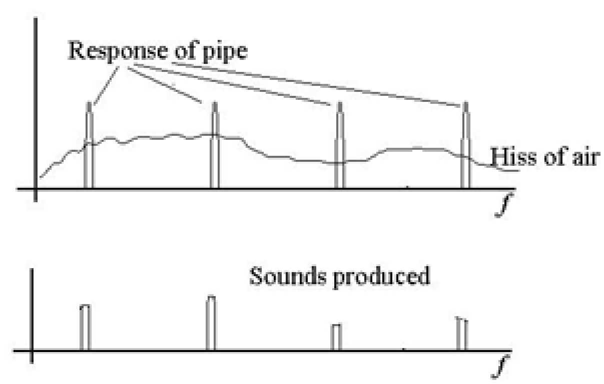
HOMEWORK 12-2

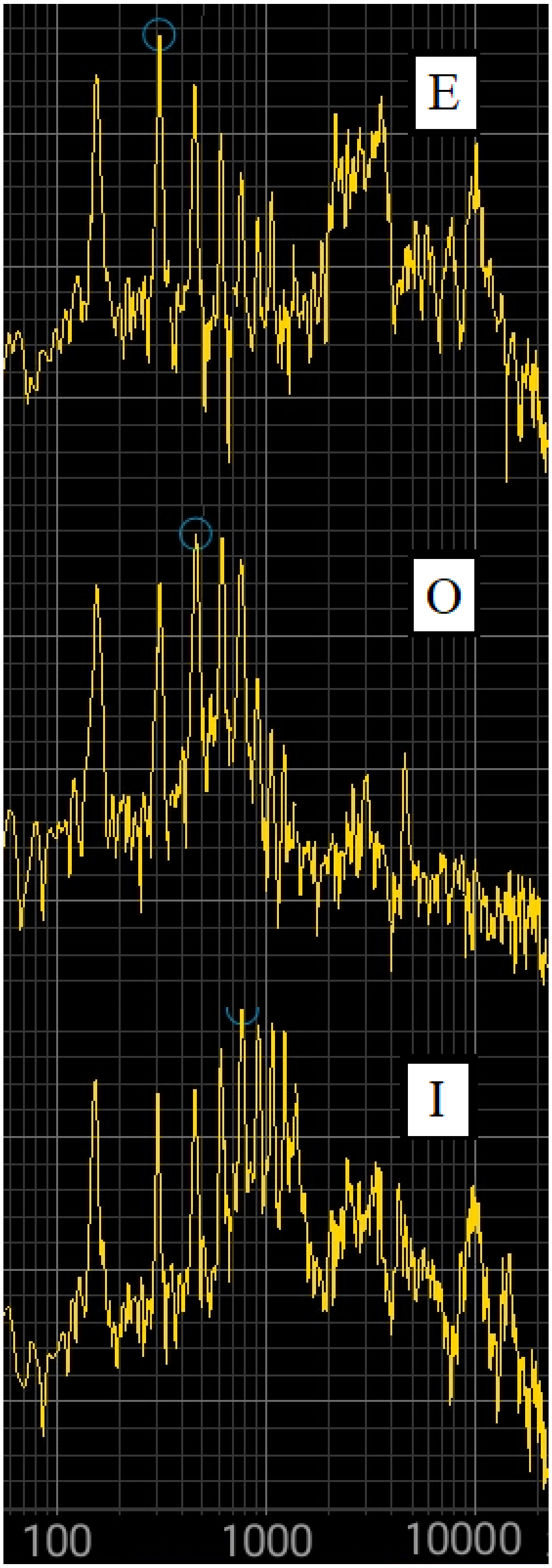
> A 2 m long air column is open at both ends. The frequency of a certain harmonic is 400 Hz, and the frequency of the next higher harmonic is 480 Hz. Find the speed of sound in the air column.

DISCUSSION 12-3

> Now here's a question. How can a listener distinguish different musical instruments that are playing the same note?

For example, an oboe and a clarinet are both essentially cylindrical tubes, closed at one end and open at the other, and so they produce the same sequence of harmonics, $f_n = nv/4L$. The answer is that, because of the exact shape of the bore, they each put slightly different amounts of energy into the different harmonics, and it is that distribution that your brain remembers and labels as one instrument or the other. The same goes for vowels in speech.






In the figure, you can see the *spectra* of three long vowels as pronounced by an Upstate New Yorker. The axes are logarithmic, with frequency along the horizontal axis and the strength of each frequency along the vertical axis. The actual frequencies produced are the same in each case, but the strengths of each are very different for the three sounds.

Also, it is possible to suppress certain harmonics. Those of you who play guitar know that plucking the string in different spots produced different sounds on the same fundamental. Plucking a string at its center will deliver more energy into frequencies with an anti-node there, *i.e.* n = 4, 8, 12, 16, *et c*. Touching a string lightly one third of the way from one end will suppress any frequencies with an anti-node there.

DEMONSTRATION 12-1

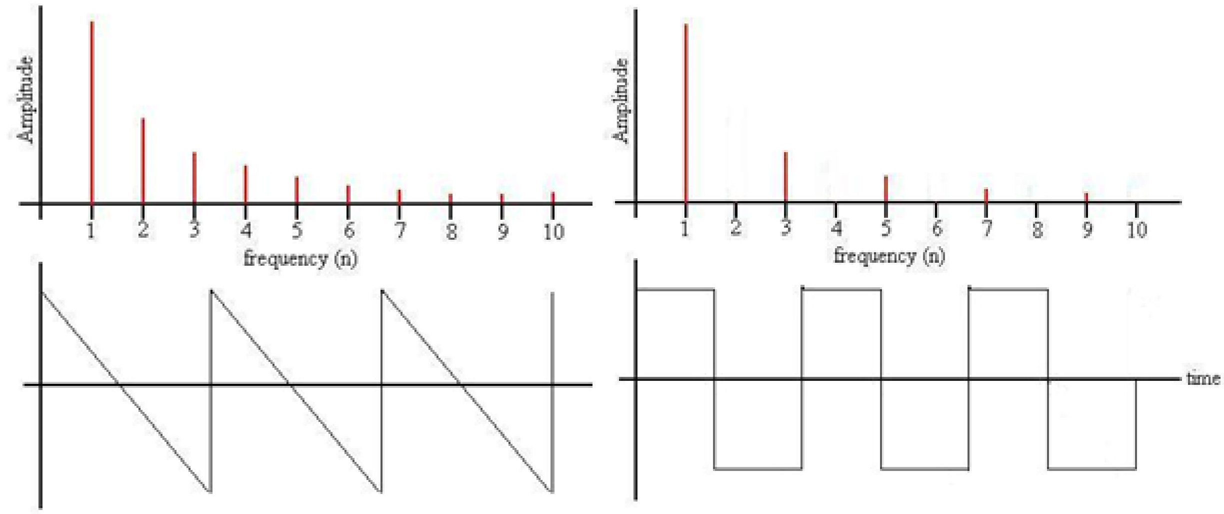
VIDEO al rod

## Non-Sinusoidal Waves*

We've been concentrating on sinusoidal waves. What about waves that are not sinusoidal? Waves that are repetitive in time can be approximated by adding up different amplitudes of many sinusoidal waves of different frequencies (*Fourier decomposition*):

$$y(t) = \sum_n A_n \sin(n2\pi f t) \quad n = 1, 2, 3, \ldots \ .$$

We can represent these amplitudes $A_n$ graphically with a figure like this one, similar to what was shown on the screen of the *spectrum analyzer* above. For example, a 'ramp' wave is composed of waves with the form n = 1, 2, 3, … with $A_n$ decreasing as 1/n.




A ‘square wave’ is the sum of waves of the form n = 1, 3, 5, ... and the amplitude $A_n$ decreasing as 1/n. A triangular wave is the sum of waves of the form n = 1, 3, 5, ... and the amplitude $A_n$ decreasing as $1/n^2$.

## Intensity

A wave can be defined as a transfer of energy without a net movement of matter. For example, if I send a pulse down a string, the other end of the string can exert a force on some object to which it is tied, and possibly do work. For sound (and later, light), we measure the rate of energy transfer *per* unit area as the *intensity*, I, with the corresponding units of watts/m$^2$.

Consider a fire siren which broadcasts *isotropically* P joules of sound energy *per* second. Draw an imaginary sphere of radius R with the center at the siren; all the energy must eventually pass through that sphere, and the intensity will be

$$I = \frac{P}{4\pi R^2} \ .$$

We see that if we make the sphere larger, the energy will be distributed over a larger area, and the intensity will be reduced (that is, each square meter of area will receive less energy). This *$1/r^2$ dependence* is fairly common, and we shall see it again.

EXAMPLE 12-1

At noon, you can hear two towns’ sirens going off. If Siren A has twelve times the intensity of Siren B, what is the ratio of their distances from you?

So, if we assume that the sirens are identical, the power outputs should be the same:

$$P_A = P_B$$

$$I_A 4\pi r_A^2 = I_B 4\pi r_B^2$$

$$\frac{r_A}{r_B} = \sqrt{\frac{I_B}{I_A}} = \sqrt{\frac{I_B}{12 I_B}} = \sqrt{\frac{1}{12}} = 0.29$$

EXERCISE 12-2

If Star A and Star B appear equally bright to your eye, but you know that Star B is twenty times further away from earth than is Star A, what is the ratio of the powers radiated by each?

HOMEWORK 12-3

At the fireworks show, a particularly loud explosion occurred 100m directly above Susique's head. Hank is 50 meters from Susique and Earl is 100 meters from Susique. Compare the intensities heard by each.

An alternate way of expressing intensity is in units of *decibel*s. The decibel scale is logarithmic, and thus follows more closely the actual size of the signal sent from human ear to human brain. The bel is named for Alexander Graham Bell, who was not, as one might suppose, American, but rather a Scot-born Canadian working in Boston. A reference intensity $I_o$ is defined as $10^{-12}$ W/m$^2$, which corresponds roughly to the quietest sound a normal human can hear. The intensity to be converted is compared to this standard, and the log base ten is taken of the ratio. This gives the number of *bels*, so the number of decibels (dB) must be ten times more:

$$\beta = 10 \log_{10} \frac{I}{I_o} \quad .$$

EXAMPLE 12-2

Suppose that the sound that one professor produces from the front of the classroom has an intensity of $10^{-7}$ W/m$^2$. How many dB does this correspond to?

$$\beta = 10 \log_{10} \frac{10^{-7}}{10^{-12}} = 10 \log_{10} 10^5 = 10\,(5) = 50 \text{ dB} \quad .$$

EXERCISE 12-3

Now, suppose there are ten professors at the front of the room, talking incoherently.[3] How many decibels would ten professors produce? What about one hundred?

[3] No jokes, please. Here, incoherent means that there are no particular relationships among the sounds produced by N profs, in which case the intensity is simply N times the intensity die to one prof. If they were all chanting or reciting together, the sounds would be coherent and the result more complicated.

Note that this is not a linear relationship. A multiplicative factor of ten in intensity is an additive increase of ten in decibels.

Here are the intensity levels of some common situations:

| Situation | Intensity | Intensity level |
|---|---|---|
| Threshold of Hearing | $10^{-12}$ W/m$^2$ | 0 dB |
| Library Reading Room | $10^{-9}$ W/m$^2$ | 30 dB |
| Conversation | $10^{-6}$ W/m$^2$ | 60 dB |
| Vacuum Cleaner | $10^{-3}$ W/m$^2$ | 90 dB |
| Rock Concert | $10^{-1}$ W/m$^2$ | 110 dB |
| Thunder | 10 W/m$^2$ | 130 dB |

Of course, these values depend on the distance between source and listener. Prolonged exposure to sounds above 90 dB will cause permanent damage, and exposure to sounds over 110dB will be painful. Here are some helpful hints: ALWAYS wear ear protection in noisy situations, such as lawn mowing or vacuuming and on up. If you must wear headphones to listen to music, place them just in front of your ears, not right over them.

HOMEWORK 12-4

After Route 100 was built through Columbia, some homes experienced an average decibel level of 110 dB, caused by 200 cars passing *per* minute. When the inspector arrived to test this, there were only 25 cars passing *per* minute. What average decibel level did the inspector measure?

HOMEWORK 12-5

Five identical machines operating in a factory produce an average sound intensity level of 87 dB. If three additional machines are put online in the same location, what will the new average sound intensity level be?

## Beats

DERIVATION 12-3

Suppose that we have two nearly identical waves passing through a spot in space (call it the origin so x = 0), so that the time dependences (we'll ignore the spatial dependence) are given by

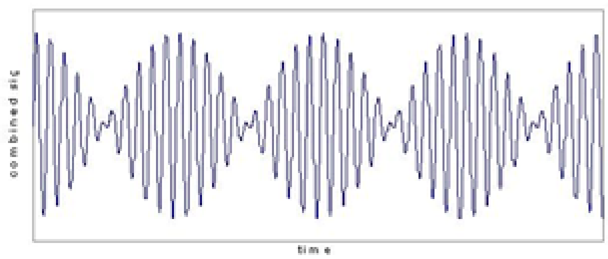
$$y_1 = A\sin(2\pi f_1 t) \quad \text{and} \quad y_2 = A\sin(2\pi f_2 t) \ .$$

Using the principle of superposition, we get the total 'displacement' from equilibrium from the sum of these two expressions:

$$y_{\mathrm{TOTAL}} = \mathrm{A}\sin(2\pi f_1 \mathrm{t}) \; + \; \mathrm{A}\sin(2\pi f_2 \mathrm{t}) \;\; .$$

Now, we'll make use of a trig identity, sin a + sin b = 2 sin[(a+b)/2] cos[(a-b)/2], so that

$$y_{\mathrm{TOTAL}} = 2\mathrm{A}\cos\left(2\pi\left(\frac{f_1 - f_2}{2}\right)\mathrm{t}\right)\;\sin\left(2\pi\left(\frac{f_1 + f_2}{2}\right)\mathrm{t}\right)\;.$$




From this we see that the frequency of oscillation is the <u>average</u> of the two original frequencies, but also that the amplitude of the oscillation is modulated by an envelope with a frequency equal to the difference of the original frequencies, $|f_1 - f_2|$. The formula says that the frequency of the envelop is half the difference in the frequencies, but there are two pulses, or beats, *per* cycle.

This can be (and often is) used as a method for tuning pianos and other such instruments. Once a 'C' string is tuned to the correct pitch, it and the 'G' a fifth above it above are struck simultaneously. The third harmonic of the C and the second harmonic of the G are the same note, and so the G string's tension is adjusted until no beats are heard between those two harmonics (or in other tuning schemes, a certain number of beats *per* second should be heard, but that's a whole 'nother story...).

Beating is also used in some radar guns. An example will follow in the next section.

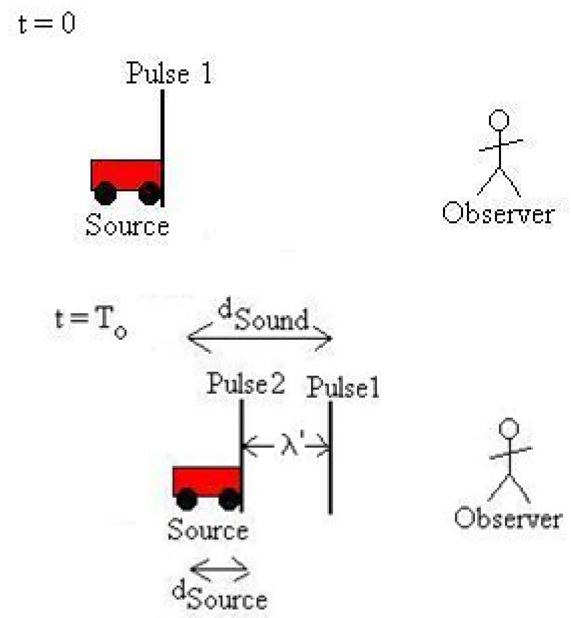
HOMEWORK 12-6

> Consider two identical strings under the same tension but with different lengths. The n = 2 harmonic of the first string (400 Hz) beats at 6 Hz with the n = 2 harmonic of the longer string. What is the difference in the lengths of the strings if the speed of transverse waves on these strings is 120 m/s?

## The Doppler Effect

You're probably familiar with this effect: a car or train passes you while blowing its horn, so that the pitch of the sound rises while the vehicle is moving toward you, but sounds lower when the vehicle is moving away from you. We shall look at a couple of special cases, and then integrate the results for all such cases into a single relationship. Note, however, that the results will only be true if there is no wind, that is, the medium (usually air or water) is stationary. Also, our derivations will be done for a one dimensional universe.

DERIVATION 12-4




Consider a source moving at speed $v_{Source}$ toward (*approaching*) a stationary observer (or listener, if you insist). Instead of having the source emit a sinusoidal wave, let's assume that it emits pulses; we can later correlate these pulses to the peaks of a sinusoidal wave, if necessary. Let the frequency of the pulse emitted by the source be $f_o$, and the time between the emission of pulses be $P_o = 1/f_o$. Here at t = 0, the source releases a pulse, which then travels to the right at speed $v_{Sound}$. Now, let's look at the locations of everything a time t = T later. Pulse 1 has traveled a distance $d_{Sound} = v_{Sound}\,T$, while the source has traveled a distance $d_{Source} = v_{Source}\,T$, at which point it emits Pulse 2. Now, the wavelength that the observer will measure is the distance between the two pulses:

$$\lambda' = d_{Sound} - d_{Source} = v_{Sound}T - v_{Source}T \ .$$

Now, T is the period between pulses as 'heard' by the source, and so we remember that $f = 1/T$ and that $f\lambda = v_{wave}$, and so this last expression can be re-written as

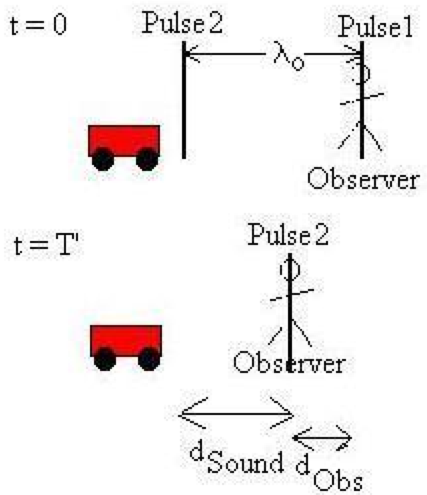
$$\frac{v_{Sound}}{f'} = \frac{v_{Sound}}{f_o} - \frac{v_{Source}}{f_o} \ ,$$

$$f' = f_o \frac{v_{Sound}}{v_{Sound} - v_{Source}} \ .$$

Now, if the source had instead been moving the other way (*receding*), those two distances would have had to have been added, changing the minus sign to a plus sign:

$$f' = f_o \frac{v_{Sound}}{v_{Sound} + v_{Source}} \ .$$

Now, suppose instead that it were the observer moving toward (approaching) the stationary source at speed $v_{Observer}$. Once again, let the source emit pulses at an interval of T. Let t = 0 when Pulse 1 arrives at the observer. The second pulse arrives at the observer at time T', during which interval the pulse has traveled distance (to the right) $d_{Sound} = v_{Sound}\,T'$ and the observer has traveled distance (to the left) $d_{Observer} = v_{Observer}\,T'$. T' is now the time between pulses, as heard by the observer.




The sum of these distances is the original wavelength as produced by the source, $\lambda_o$:

$$\lambda_o = d_{Sound} + d_{Observer} = v_{Sound}T' + v_{Source}T' \quad .$$

Remembering that $f' = 1/P'$ and that $f'\lambda' = v_{wave}$, we substitute to obtain

$$\frac{v_{Sound}}{f_o} = \frac{v_{Sound}}{f'} + \frac{v_{Observer}}{f'} \quad ,$$

$$f' = f_o \frac{v_{Sound} + v_{Observer}}{v_{Sound}} \quad .$$

Once again, if the observer had been receding from the source, there would have been a sign reversal to

$$f' = f_o \frac{v_{Sound} - v_{Observer}}{v_{Sound}} \quad .$$

Now, we can combine all these relationships, if we're careful. First, we need to define better the terms 'approach' and 'recede.' 'Approach' is to head in the direction of the other object, <u>regardless</u> of whether the distance between the objects is becoming smaller or not, and 'recede' means to head in the opposite direction of the other object, whether the distance between is increasing or not. Then,

$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} \quad ,$$[4]

where the upper sign is used if that object is approaching and the lower sign if that object is receding.

DISCUSSION 12-3

What exactly would one do if there were wind?

EXAMPLE 12-3

A police car moving at 150 kph is chasing a speeder traveling at 100 kph on a calm day. If the cop's siren has a frequency of 500 Hz, what frequency will the speeder hear?

First, what is the speed of sound on a calm day? We typically use 340 m/s as a default value. The cop is approaching the speeder, while the speeder is receding form the cop. Remember

[4] For light, the result is somewhat different.

that these terms have nothing to do with the distance between them increasing or decreasing. Converting the speeds to meters *per* second, we get

$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} = 500\frac{340 - 27.8}{340 - 41.7} = 523.3 \text{ Hz} .$$

AN ADMONITION

It's tempting to try to cut a corner and use relative velocities. Bad idea. Let's try it. Suppose the speeder is motionless and the cop is approaching him at 50 kph.

$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} = 500\frac{340 - 0}{340 - 13.9} = 521.3 \text{ Hz} .$$

These values are close, but the second is wrong because we left out the fact that there is now a 100 kph wind blowing against the motion of the cop. The speed of sound is therefore not 340 m/s but rather 312.2 m/s. Let's try again.

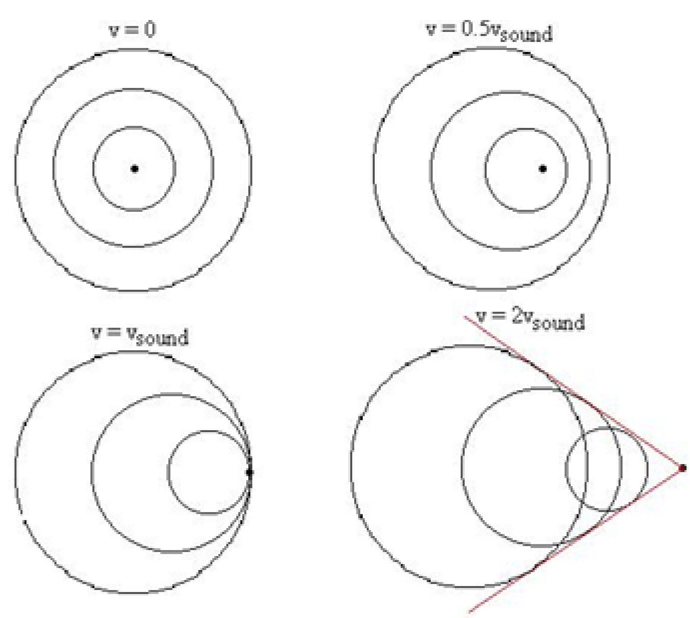
$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} = 500\frac{3312.2 - 0}{312.2 - 13.9} = 523.3 \text{ Hz} .$$

HOMEWORK 12-7

If you move at 15 m/s (relative to the ground) toward a sound source (3000 Hz) which is also moving toward you at 45 m/s (relative to the ground), what frequency will you hear from the object? Assume that $v_{sound} = 340$ m/s.

DISCUSSION 12-4

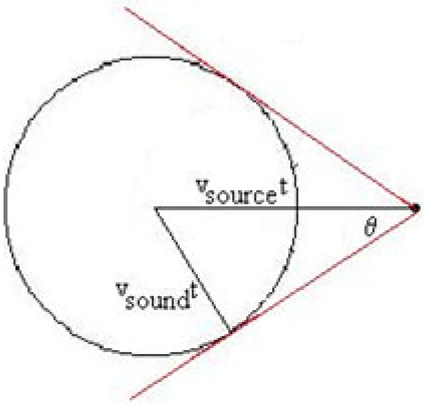
What happens if the source travels more quickly than sound? Consider the denominator of the Doppler relationship we derived.




The upper left figure shows the locations of the crests of three pulses emitted by a source such as a jet while it is stationary. The pulses move out in a spherical form centered on the spot at which they were produced. The upper right figure shows the same when the source is moving to the right at about 0.5 the speed of sound; note that the wavelengths will be shorter for listeners in the path of the source, but longer for listeners from which the source is receding. When the source reaches the speed of the wave in that medium, a *bow shock wave* is generated; this is most easily seen when generated by a boat, but recent photos of jets breaking the 'sound barrier' have caught these shock waves as they condensed water gas in the air. Once the speed of the source exceeds the speed of the wave in that medium, the crests of all waves co-incide to produce a giant shock wave (red line); for jets, this results in the familiar *sonic boom*. It can also be seen as the bow wave of a boat in water.

Some random notes:

1) The reason the Concorde was so quiet (to the passengers anyway, not to those living on the flight path) during supersonic flight is that the noise from the engines could not keep up with the cabin; one could hear only the noise transmitted through the body of the plane.




2) The apex angle (θ) of the cone formed by the shockwave depends on the speed of the source. The wave shown was generated at the instant that the source was at its center. In time interval t, the source moved to the right a distance $v_{source}t$ and the sound moved outward a distance $v_{sound}t$. Consider the right triangle formed in the diagram. We see that

$$\sin\theta = \frac{v_{Sound}t}{v_{Source}t} = \frac{v_{Sound}}{v_{Source}} \quad .$$

The inverse of this ratio is referred to as the *Mach number*. The official speed record of any jet aircraft is about Mach 9.6, set by the unmanned X43A. The Space Shuttle (when we had a set) entered earth's atmosphere at about Mach 25. Be aware though that the Mach speed is NOT just a multiple of the speed of sound at sea level (340 m/s); the speed of sound varies with altitude.

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EXERCISE 12-1 Solution

Since the two ends are different, or 'odd,'

$$f_n = \frac{nv}{4L} \qquad n = 1, 3, 5, 7, \dots \quad .$$

$$f_1 = \frac{1 \times 340}{4 \times 0.03} = 2833\text{ Hz} \quad .$$

Let's just keep increasing n until we exceed the limit of human hearing at 20,000 Hz:

8500 Hz, 14,167 Hz, 19, 833 Hz.

EXERCISE 12-2 Solution

The same brightness means the same intensity as seen from earth. So,

$$I = \frac{P_A}{\pi r_A^2} = \frac{P_B}{\pi r_B^2}$$

$$\frac{P_A}{P_B} = \frac{r_A^2}{r_B^2} = \frac{r_A^2}{(20 r_A)^2} = 2.5 \times 10^{-3} \ .$$

Or the other way round, Star B produces 400 times more energy each second than does Star A.

EXERCISE 12-3

Assume that the sources are incoherent, ten professors would have an intensity of $10{\times}10^{-7}$ W/m$^2$. Then,

$$\beta_{10} = 10 \ \log_{10} \frac{10 \times 10^{-7}}{10^{-12}} = \ 10 \ \log_{10} 10^6 = 10\ (6) \ = 60 \text{ dB}$$

and

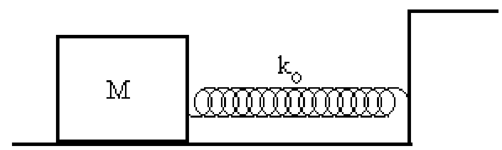
$$\beta_{100} = 10 \ \log_{10} \frac{100 \times 10^{-7}}{10^{-12}} = \ 10 \ \log_{10} 10^7 = 10\ (7) \ = 70 \text{ dB} \ .$$

As was stated, a factor of ten in intensity I is an addition of 10 in intensity level beta.

Sample Exam V

**MULTIPLE CHOICE** (4 pts each)

1) A mass M = 4 kg slides along a frictionless horizontal floor. At time = zero, it encounters a spring of constant k = 8 N/m. How long does it take the mass to come to rest?




A) 0 seconds
B) 0.2 seconds
C) 1.1 seconds
D) 35.8 seconds
E) 204.2 seconds

2) Consider a mass M oscillating at the end of a massless spring of constant k with amplitude A. At what distance(s) from the equilibrium point (x = 0) will the kinetic and potential energies be the same?

A) x = 0
B) x = ± 0.5A
C) x = ± 0.71A
D) x = ± A
E) The K and the $U_{SPRING}$ are never the same.

3) A fisherman notices that the wave crests on the water pass his anchored boat every 4 seconds. He measures the distance between crests to be 9 m. How fast are the waves traveling?

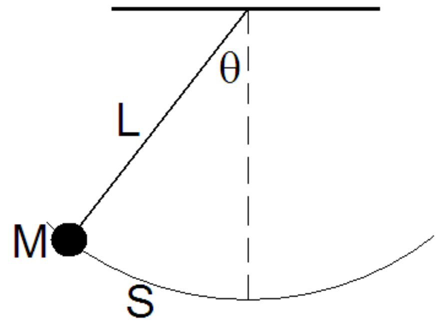
A) 0 m/s
B) 0.03 m/s
C) 0.44 m/s
D) 2.25 m/s
E) 36 m/s

4) Consider a particle oscillating with amplitude A. Through what distance does the particle move in one cycle? What is its displacement after one cycle?

A) distance = 0, displacement =2A
B) distance = 2A, displacement = 4A
C) distance = 4A, displacement = 0
D) distance = 0, displacement = 4A
E) distance = A, displacement = 0

5) The wail from an air-raid siren has an intensity of 1 Watt/$m^2$ at a distance of 1km. How far from the siren should you stand so that the intensity is only 0.01 Watt/$m^2$ ?

A) 0.01 km
B) 0.1 km
C) 1 km
D) 10 km
E) 100 km




**PROBLEM I** (20 pts)

Consider a simple pendulum of length L and mass M. Show that if the angle θ is kept small, the period of oscillation *P* is given by:

$$P = 2\pi\sqrt{\frac{L}{g}}$$

HINT: Compare the pendulum with a mass on a spring, for which the N II equation is

$$-kx = ma$$

and for which we have previously shown that

$$P = 2\pi\sqrt{\frac{m}{k}}.$$

**PROBLEM II** (20 pts)

In shallow water, the speed of a surface wave is given by the formula

$$v \approx \sqrt{gd},$$

where d is the depth of the water and g is the gravitational field strength. 'Shallow' is a relative term that means that the depth of water is much less than the wavelength of the wave, $d \ll \lambda$. This condition is met by *tsunami* waves in mid-ocean.

A) Estimate the speed of a *tsunami* in the open ocean, where the mean ocean depth is 5.6 km.

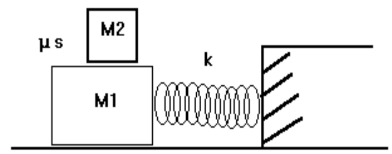
B) How long would it take such a wave to travel the approximately 9000 km from its source in, say, the mid-Pacific Ocean to, say, the California coast?

**PROBLEM III** (20 pts)

Suppose that you have a clock that operates on the motion of a mass on a spring and which keeps perfect time. Now, you take it with you when you vacation on Mars, where gravity is only one-third that on the earth. If you set your clock correctly at 12:00, what time will the clock read one hour later?

**PROBLEM IIII** (20 pts)

Consider a mass $M_2$ which sits upon a larger mass $M_1$, which in turn slides along a frictionless floor. $M_1$ is connected to a spring with constant k. The coëfficient of static friction between $M_1$ and $M_2$ is $\mu_S = 0.6$. With what maximum amplitude can $M_1$ oscillate back and forth without having $M_2$ slide off? HINT: Until the masses actually lose contact, you can treat them as if they were a single mass.




A) What type of force accelerates $M_2$ back and forth? (4 pts)

B) What is the maximum value this force can possibly have? (4 pts)

C) What is the maximum acceleration $M_2$ can experience without slipping? (4 pts)

D) Since $M_2$ hasn't slipped yet, we can still treat the two blocks as if they were one. Write Newton's second law for the motion of the combined blocks in the horizontal direction.? (4 pts)

E) Use the expression you found in Part D to find the maximum amplitude of oscillation so that $M_2$ does not slide. (4 pts)