Hw 11-8 Soln)

First, let's show that this is correct. Assume a solution to the differential equation

$$-\mathrm{k}\mathrm{x} - \mathrm{b}\frac{d\mathrm{x}}{d\mathrm{t}} = \mathrm{m}\frac{d^2\mathrm{x}}{d\mathrm{t}^2} \ .$$

of the form

$$\mathrm{x}(\mathrm{t}) = \mathrm{A}e^{\alpha\mathrm{t}}\cos(2\pi f'\mathrm{t} + \varphi).$$

Then,

$$\frac{d\mathrm{x}}{d\mathrm{t}} = \mathrm{A}\alpha\, e^{\alpha\mathrm{t}}\cos(2\pi f'\mathrm{t} + \varphi) - \mathrm{A}\, 2\pi f' e^{\alpha\mathrm{t}}\sin(2\pi f'\mathrm{t} + \varphi)$$

and

$$\frac{d^2\mathrm{x}}{d\mathrm{t}^2} = \mathrm{A}\alpha^2\, e^{\alpha\mathrm{t}}\cos(2\pi f'\mathrm{t} + \varphi) - \mathrm{A}\alpha\, 2\pi f' e^{\alpha\mathrm{t}}\sin(2\pi f'\mathrm{t} + \varphi) \\ - \mathrm{A}\alpha\, 2\pi f' e^{\alpha\mathrm{t}}\sin(2\pi f'\mathrm{t} + \varphi) - \mathrm{A}\,(2\pi f')^2 e^{\alpha\mathrm{t}}\cos(2\pi f'\mathrm{t} + \varphi)$$

Then we have a mess:

$$-\mathrm{k}\left(\mathrm{A}e^{\alpha\mathrm{t}}\cos\left(2\pi f'\mathrm{t} + \varphi\right)\right) - \mathrm{b}\left(\mathrm{A}\alpha\, e^{\alpha\mathrm{t}}\cos\left(2\pi f'\mathrm{t} + \varphi\right) - \mathrm{A}\, 2\pi f' e^{\alpha\mathrm{t}}\sin\left(2\pi f'\mathrm{t} + \varphi\right)\right) \\ = \mathrm{m}\Big(\mathrm{A}\alpha^2\, e^{\alpha\mathrm{t}}\cos\left(2\pi f'\mathrm{t} + \varphi\right) - \mathrm{A}\alpha\, 2\pi f' e^{\alpha\mathrm{t}}\sin\left(2\pi f'\mathrm{t} + \varphi\right) \\ - \mathrm{A}\alpha\, 2\pi f' e^{\alpha\mathrm{t}}\sin\left(2\pi f'\mathrm{t} + \varphi\right) - \mathrm{A}\left(2\pi f'\right)^2 e^{\alpha\mathrm{t}}\cos\left(2\pi f'\mathrm{t} + \varphi\right)\Big) \ .$$

Now, all the As cancel as well as all of the exponential terms.

$$-\mathrm{k}\left(\cos\left(2\pi f'\mathrm{t} + \varphi\right)\right) - \mathrm{b}\left(\alpha\ \cos\left(2\pi f'\mathrm{t} + \varphi\right) - 2\pi f'\sin\left(2\pi f'\mathrm{t} + \varphi\right)\right) \\ = \mathrm{m}\Big(\alpha^2\cos\left(2\pi f'\mathrm{t} + \varphi\right) - \alpha\, 2\pi f'\sin\left(2\pi f'\mathrm{t} + \varphi\right) \\ - \alpha\, 2\pi f'\sin\left(2\pi f'\mathrm{t} + \varphi\right) - \left(2\pi f'\right)^2\cos\left(2\pi f'\mathrm{t} + \varphi\right)\Big) \ .$$

The next thing we can do is separate this into two smaller equations, one of the sine terms and the other of the cosine terms:

$$-\mathrm{k}\left(\cos\left(2\pi f'\mathrm{t} + \varphi\right)\right) - \mathrm{b}\left(\alpha\ \cos\left(2\pi f'\mathrm{t} + \varphi\right)\right) \\ = \mathrm{m}\left(\alpha^2\cos\left(2\pi f'\mathrm{t} + \varphi\right) - \left(2\pi f'\right)^2\cos\left(2\pi f'\mathrm{t} + \varphi\right)\right) \ .$$

$$-\mathrm{k} - \mathrm{b}\alpha = \mathrm{m}\,\alpha^2 - \mathrm{m}\left(2\pi f'\right)^2 \ .$$

And

$$-\mathrm{b}\left(-\,2\pi f^{'}\sin\left(2\pi f^{'}\mathrm{t}+\varphi\right)\right)=\mathrm{m}\left(-\alpha\,2\pi f^{'}\sin\left(2\pi f^{'}\mathrm{t}+\varphi\right)-\alpha\,2\pi f^{'}\sin\left(2\pi f^{'}\mathrm{t}+\varphi\right)\right)\ .$$

$$\mathrm{b}=\ -2\mathrm{m}\alpha\ .$$

So,

$$\alpha=\frac{-b}{2m}$$

and from the cosine equation, substituting for alpha,

$$-\mathrm{k}-\mathrm{b}\left(\frac{-b}{2m}\right)\ =\mathrm{m}\ \left(\frac{-b}{2m}\right)^{2}-\ \mathrm{m}\left(2\pi f^{'}\right)^{2}\ .$$

$$(2\pi f')^2=\ \frac{\mathrm{k}}{m}-\frac{\mathrm{b}}{m}\left(\frac{b}{2m}\right)+\mathrm{m}\ \left(\frac{-b}{2m}\right)^{2}.$$

$$(2\pi f')^2=\ \frac{\mathrm{k}}{m}-\frac{\mathrm{b}^2}{2m^2}+\frac{\mathrm{b}^2}{4m^2}\ .$$

$$(2\pi f')^2=\ \frac{\mathrm{k}}{m}-\frac{\mathrm{b}^2}{4m^2}\ .$$

$$f'=\frac{1}{2\pi}\ \sqrt{\frac{\mathrm{k}}{m}-\frac{\mathrm{b}^2}{4m^2}}\ .$$

O.K., now let's answer the question. As b increases, the argument of the square root decreases; at some point it will equal zero and the frequency will disappear. This happens when

$$\frac{\mathrm{k}}{m}=\frac{\mathrm{b}^2}{4m^2}$$

$$k=\frac{\mathrm{b}^2}{4m}$$

$$b=2\sqrt{km}$$