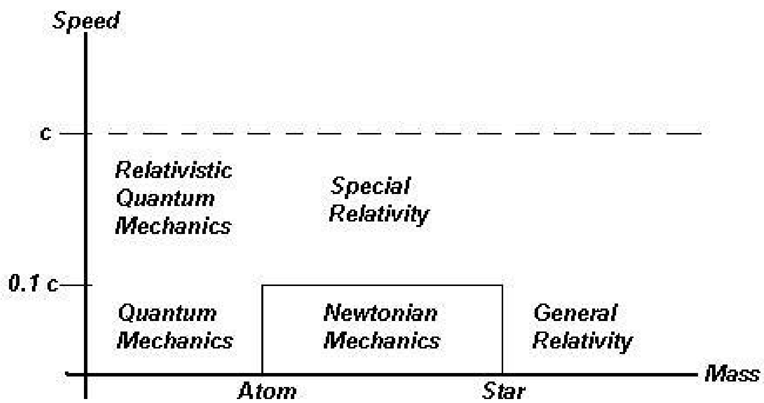
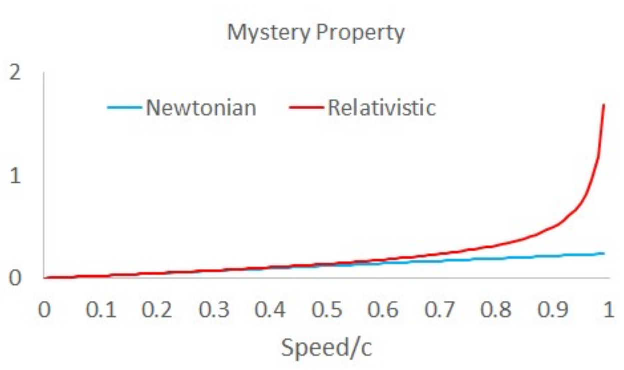
Section 1 - Background

Introduction

In this semester, we will study what is now known as *Newtonian mechanics*. The laws we shall discuss are unfortunately only approximately true, special case limits of the actual laws of the universe. However, they are sufficiently correct to agree to high precision with reality so long as certain conditions are met. The diagram below shows a rough breakdown of the approaches necessary to a given situation.



Let's consider *special relativity* as an example. Special relativity concerns 'ordinary sized' objects moving at high speeds. The relationships developed there are generally more complex than those in Newtonian physics but match experimental observations much more closely. We require that these relativistic relationships should always agree with Newtonian physics when velocities tend toward zero. This last notion is called the *correspondence principle*, and we will require it to hold for quantum mechanics in Semester Three, as well. As an example, here is a graph of a particular property of a bowling ball calculated with the Newtonian physics approximation and with the correct special relativity relationship as a function of the ball's speed. The two tend toward agreement as the speed of the bowling ball goes to low values. We have somewhat arbitrarily decided that Newtonian results are sufficiently valid for this course so long as the speed is under 10% of the speed of light, in which case the difference between the two models is under 0.5%.



So long as the speed of an object is less than about 10% of the speed of light, and the object is larger than an atom but smaller than a star, we will probably be alright.

Dimensional Analysis

Since, unlike most fields of academic inquiry, the conclusions of physics must agree with objective reality, we must be prepared to make measurements of various physical properties. Modern physicists have determined that any physical quantity can be constructed from some combination of only seven basic, fundamental quantities or *dimensions*, the choice of which is somewhat arbitrary but currently standardized:

[Length]
[Mass]
[Time]
[Electrical Current]
[Number of Particles]
[Thermodynamic Temperature]
and
[Light Intensity].

When we talk about the dimension of a quantity, we don't mean dimension in the sense of the width, length, and height of a box. Each of these specific measurements is a [Length]. The dimension of the volume of a box is $[\text{Length}]^3$. Although we haven't defined it yet, you probably have an idea of the meaning of speed; the dimension of speed is [Length]/[Time].

So, for example, next semester you will encounter the *electric potential,* which has the dimensions of $[\text{Mass}][\text{Length}]^2/[\text{Current}][\text{Time}]^3$. Note that this construct is independent of the actual units used. For example, this quantity is often called the *voltage*, since the *volt* is the standard unit for electric potential, but of course other units could just as easily be used instead. The unit might change, but the dimension will remain the same.

DISCUSSION 1-1

Dimensional analysis can be a useful tool for gaining insight into the relationships among quantities that determine the behavior of a system. For example, can we make a prediction for the dependence of the *period* (P, the time to complete one cycle) of a simple pendulum without knowing much physics? On what parameters of the system could this depend? What are the dimensions of these quantities?

EXAMPLE 1-1

A list of such quantities would perhaps include the length ℓ of the string, the mass m of the bob, the amplitude of oscillation (θ_A, the angle through which the bob swings), and perhaps the earth's gravity g, whatever that is.

period T = [Time]
mass m = [Mass]
string length ℓ = [Length]

amplitude θ_A = [1] (dimensionless, the radian is the ratio of two distances)
gravitational field strength g = [Length]/[Time]^2 (O.K., I had to give you this one.)

Since we're looking for an expression for the period, whatever combination of parameters we decide on must have dimension of [Time]. Let's suppose that

$$P \sim m^a g^b l^c \theta_A^d ,$$

where a, b, c, and d are powers of their respective variables and are to be determined. Then, looking at the dimensions,

$$[T]^1 = [M]^a \left(\frac{[L]}{[T]^2}\right)^b [L]^c (1)^d = [M]^a [T]^{-2b} [L]^{b+c} (1)^d \quad .$$

If we're going to have an equation, clearly both sides of the equation must have the same dimension. We see that there is no [Mass] on the left side, so a = 0. Continuing,

a = 0;
1 = -2b $\rightarrow$ b = -1/2;
0 = b + c $\rightarrow$ b = -c = +1/2;
d can not be determined.

The angle, being measured in dimensionless radians,[1] can't be determined. But, if we try a little experiment, we find that θ_A in fact has no effect on the period, so d = 0. Our final result is that we expect the period of a simple pendulum to go as

$$P \sim g^{-\frac{1}{2}} l^{\frac{1}{2}} = \sqrt{\frac{l}{g}} \quad .$$

The correct answer, as we'll see at the end of the course after much toil is

$$P = 2\pi\sqrt{\frac{l}{g}} \quad .$$

Since 2π is a dimensionless quantity, this method could not detect it. Even so, we got a good idea of how the period depends on the parameters of the system with relatively little effort.

EXERCISE 1-1

[1] The radian is defined as the ratio of two distances.

If we drop a marble from a height H above a table, it takes a certain amount of time to fall through distance H to the table. Work out roughly the relationship between the time t and the height H.

Units

DISCUSSION 1-2

Which weighs more, a pound of rocks or a pound of feathers? Which weighs more, an ounce of gold or an ounce of potatos? Which weighs more, a pound of gold or a pound of potatos?

Making measurements requires that we develop units for the measurements, and standards for these units, so that we may all understand what the measurements mean. In the example above, an ounce of gold actually weighs more than an ounce of potatos, because gold, being a precious metal, is measured in troy ounces, which are larger than the avoirdupois ounces used for food. On the other hand, a pound of potatos weighs more than a pound of gold, because there are 16 avoirdupois ounces in an avoirdupois pound but only 12 troy ounces in a troy pound. So, not only do we need to define units, we need to define which particular system of units they are associated with.

In this class, we shall use the *système international*, also known as the *MKSA system* (for meter, kilogram, second, ampère).[2,3] You are probably much more familiar with the *U.S. Customary Units System*, which is a patchwork of bizarre quantities and units. Only three nations in the world have avoided an official change to the SI; in the U.S., the conversion was to have been accomplished by 1970. Metric road signs are in use on some federal highways in Ohio, Kentucky, Tennessee, Arizona, Vermont, New Hampshire, Maine, and New York (some New York signs are also in French!), and exits are numbered by km on Rte 1 in Delaware. Here is a partial list of units used to measure distance in the United States:

inch;
foot; 1 foot = 12 inches
yard; 1 yard = 3 feet
fathom; 1 fathom = 2 yards
rod; 1 rod = 16 2/3 ft
ell; 1 ell = 2 ft
mil; 1000 mils = 1 inch
furlong; 1 furlong = 220 yards
chain; 1 chain = 66 feet
link; 100 links = 1 chain
mile; 1 mile = 5280 feet = 1760 yards = 8 furlongs
league; 3 miles = 1 league
hand; 1 hand = 4 inches

[2] There is more than one metric system, so we need to be specific.
[3] The metric system survives as one of the innovations of the First Republic (the calendar was not so lucky, but then how would we know when not to eat oysters?).

span; 1 span = 9 inches
palm; 1 palm = 3 inches
finger; 1 finger = 7/8 inch
digit; 1 digit = 1/16 foot
shaftment; 1 shaftment = 6 inches

Do you know any others?

When we describe the distance from one point to another, we usually like to use units for which the number is of a reasonable size. What I mean is, if I describe the distance between my stapler and my computer, I would say, $2^1/_2$ feet, not 4.7×10^{-4} miles. The distance between Catonsville and D.C is 39 miles, not 3120 chains. However, the conversion factors between units are quite unwieldy. The structure of the SI makes conversion between large and small units much more convenient. There is a small number of basic units, and all other units with the same dimension are some power of ten larger or smaller, usually specified with a Latin or Greek prefix:

giga = 10^9
mega = 10^6
kilo = 10^3
milli = 10^{-3}
micro = 10^{-6}
nano = 10^{-9}
et c.

For example, the *meter* is the basic unit for length, and other units include the kilometer (1000 m), the milllimeter (1/1000 m), *et c.* So, I would express the distance from Catonsville to D.C. as 62 kilometers, not as 62,000 meters.

The definitions of each unit are also well specified, although many of the definitions have evolved. For example, the meter was initially defined in the 1790s as 1/10,000,000 of the distance from the equator to the North Pole along the meridian passing through Paris.[4] Since this is not an easy standard to use, it was redefined in 1889 as the distance between two scratches on a platinum-iridium bar, kept just outside Paris. Since taking a long trip to compare measurements with the bar is inconvenient, a number of other nations were provided with their own bars (ours is in Gaithersburg). As the necessity of making more precise measurements increased, the definition of the meter was changed so that anyone with the proper equipment could reproduce the standard; in 1960, the definition was changed to the distance covered by a 1,650,763.73 wavelengths of a particular orange emission line generated by ^{86}Kr. Finally, the definition of the meter was changed again in 1983 to be the distance traveled by light in $^1/_{299,792,458}$ of a second.

Although it seems as if the progressive definitions of the metre are making it more difficult to compare our measurements to the standard, it is actually the reverse; by liberating the standard from a particular piece of matter and basing it more on the laws of the nature, which are universal,

[4] The current 2°20′14.03″ meridian.

anyone with the appropriate equipment can reproduce the standard in the comfort of his own laboratory.

Students often find converting units difficult. The *factor label method* is useful and straightforward; one only need multiply by one, albeit in a particular form.

EXAMPLE 1-2

Suppose that we wish to find out how many seconds X there are in 3 years:

$$\text{X seconds} = 3 \text{ years} \quad .$$

Note that the units are different on each side, but that the dimensions are the same, [Time]. We'll multiply the right hand side of the equation by a quantity equal to one; we do that because multiplying a number by one does not change its value. The quantity we choose to multiply by is (12 months/1 yr). Since the numerator equals the denominator and since both have dimensions of [Time], the quotient equals one, and the right hand side is still equal to three years. We cancel the units and see that :

$$\text{X seconds} = 3 \text{ years} \left(\frac{12 \text{ months}}{1 \text{ year}}\right) = 36 \text{ months} \quad .$$

Continuing, a complete calculation would look like this:

$$\text{X seconds} = 3 \text{ years} \left(\frac{12 \text{ months}}{1 \text{ year}}\right)\left(\frac{30 \text{ days}}{1 \text{ month}}\right)\left(\frac{24 \text{ hours}}{1 \text{ day}}\right)\left(\frac{60 \text{ minutes}}{1 \text{ hour}}\right)\left(\frac{60 \text{ seconds}}{1 \text{ hour}}\right)$$
$$= 9.33 \times 10^7 \text{seconds}.$$

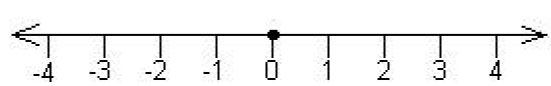
EXERCISE 1-2

A meter is 100 centimeters. Find the volume in cubic centimeters of a box with a volume of one cubic meter.

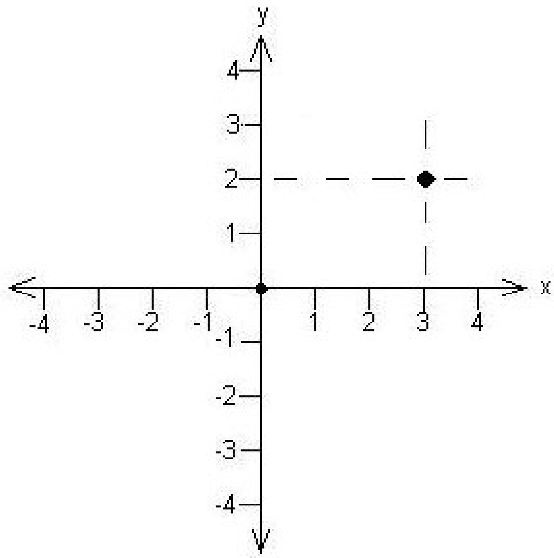
HOMEWORK 1-1

The interior of a typical ranch-style home may measure 50 ft x 24 ft x 8 ft. What is the volume of this home in cubic ft? Convert this result to cubic inches and to cubic centimeters.

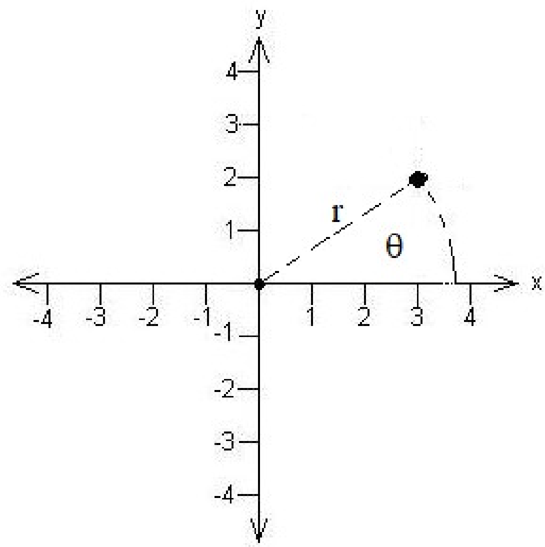
Coördinate systems



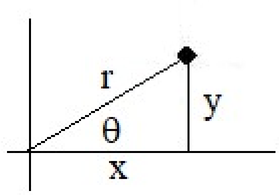
As we shall soon see, we'll need a way of keeping track of the positions of objects, as well as other quantities. In one dimension, that's fairly easy; we use the equivalent of the 'number line' we learned back in third grade, with some arbitrary point chosen as the *origin* (and usually chosen to maximize our convenience).



When we go to two dimensions, there are quite a number of systems, but the two most useful are the *rectilinear* or *Cartesian* system and the *polar* system. In the first, two 'number lines' are set up at right angles with the origins at the same spot and with equal unit spacing. We must however realize that these are not necessarily the x and y axes, but for now, let's say that they are.



The location of an object in two dimensions can be specified uniquely by reporting two numbers in an ordered pair in the form (a, b). The meaning is to start at the origin, move 'a' units in the x-direction and 'b' units in the y-direction; in this example, the location is (3, 2). The position can also be specified as a direction (usually reported as the angle measured counter-clockwise from the x-axis) and the distance from the origin, (r, θ). A negative angle is interpreted as being measured CW from the x axis. Conversion between these systems is possible through the use of the trig functions and the Pythagorean theorem:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \rightarrow y = r\sin\theta$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \rightarrow x = r\cos\theta$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \rightarrow \theta = \arctan\left(\frac{y}{x}\right)^{*}$$

$$\text{hypotenuse}^2 = \text{opposite}^2 + \text{adjacent}^2 \rightarrow r = +\sqrt{x^2 + y^2}$$

Note that r is never negative.

EXERCISE 1-3

Find r and theta for the point (3, 2) as shown in the figure above.

Now, we usually think of the lengths of the sides of a triangle as being positive numbers, which is why I introduced these relationships in the first quadrant. I assert, however, that with one small warning, these are valid in all four quadrants.

DISCUSSION 1-3

> Keeping in mind that r is never negative, in which quadrants is x/r positive and where is it negative? Where is cos θ positive and where is it negative? Do these match up? What about y/r and sin θ?
>
> Now, here is why there is an asterisk next to the arctan function. Get your calculator and find the arctangent of (2/3). Which quadrant is 33.7° in? Now find the arctangent of (-2/-3). In which quadrant should the answer be?

The problem is that your calculator does the division first, then the arctangent. It doesn't know the distinction between (-2/-3) and (2/3). Your calculator will always give you an angle between -90° and +90°; it's up to you to fix this each time. Here's my suggestion. If the angle is in fact in Quadrant I or IIII where x is positive, then the angle your calculator gives you is already correct, so you do nothing. On the other hand, if the angle is in II or III where x is negative, you must add 180°. So, the easiest test is to look at x. If x is positive, you're good. If x is negative, that's bad, and you need to fix it. I require this: if no correction is necessary, you must still indicate that you checked to see if one was necessary. I'll be happy with a √ Q on your paper.

EXERCISE 1-4

> Find the polar coördinates for the cartesian location (-3, -1).
>
> Find the cartesian coördinates for the polar location (4, 120°)

HOMEWORK 1-2

> How far from the origin is a point located at (1 m, 4 m)?

Scalars and Vectors

In this course, we deal with two types of quantities, *scalars* and *vectors*. There are other types of quantities, such as *tensors*, that thankfully we will not need to worry about. A scalar is a quantity that possesses only a size or *magnitude*. A vector possesses a magnitude and a direction.

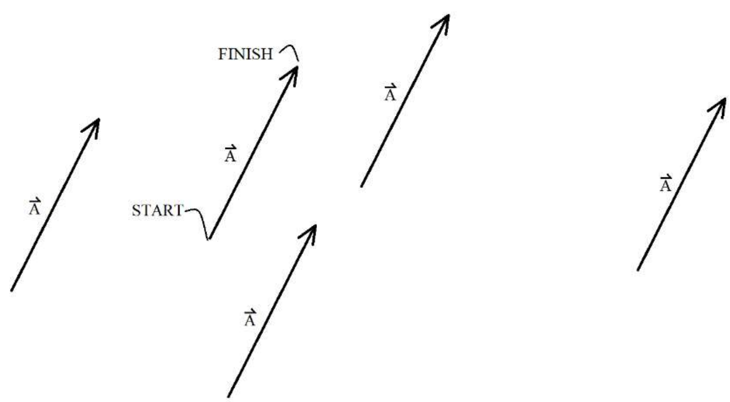
DISCUSSION 1-4

> Consider the evening weather report. Which quantities are vectors and which are scalars?

The notation for vectors is to use bold type or to place a half arrow above the symbol: **A** or $\vec{A}$. The magnitude only is written as A or less ambiguously as $|\vec{A}|$. During this course, we will sometimes drop the arrow and rely on your sense of context to know which quantities are vectors.

We often represent vectors with arrows drawn on for example a paper sheet. Arrows also have two properties we can make use of: they have direction and they have length. We can make the directions be the same, and make the length of the arrow be proportional to the magnitude of our vector.

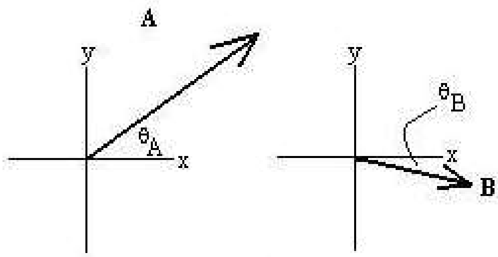
We want to investigate some properties of vectors. To do so, let's jump the gun a bit and introduce the vector *displacement*. The displacement represents the movement of an object. We can think of it as pointing from the starting position to the final position. This makes the visualization a bit easier at the start. Later, vectors will represent much more abstract quantities, such a momentum, magnetic fields, or nuclear spin. We do need to be careful; once a vector is defined, it has only two properties, magnitude and direction. We can move the vector around as much as we wish so long as those two properties remain constant. For example, in the figure, vector $\vec{A}$ was constructed to represent the displacement from the START to the FINISH, but all of the other vectors drawn are just as validly vector $\vec{A}$.

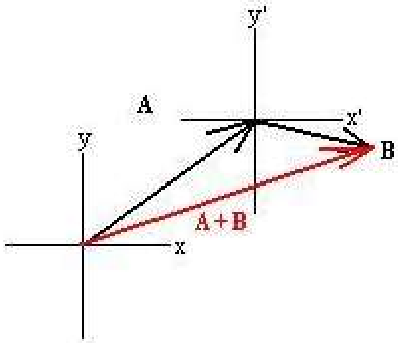


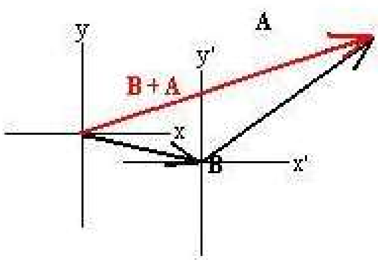
We can visualize adding vectors in terms of displacements: $\vec{A} + \vec{B}$ says that we should start at our origin and travel A meters in a direction given by θ_A, then from that intermediate destination, travel B meters in the direction given by θ_B. Conceptually, this is known as the *tail-to-tip method of addition*.[5] The red vector is the sum, or *resultant*, of $\vec{A} + \vec{B}$. Now, look at the bottom diagram. If we were to perform the motion described by $\vec{B}$ first, then perform $\vec{A}$, we would wind up in the same place. That means that vector addition is *commutative*. The order of addition doesn't matter:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \ .$$

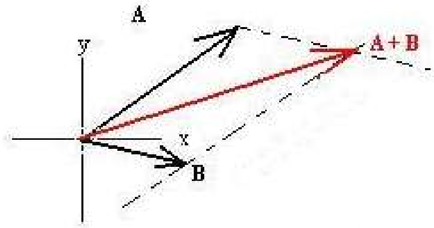
[5] Well, O.K. it's actually called the tip-to-tail method, but that makes no sense. Let's make it a thing.







An alternate, but equivalent, method of addition is the parallelogram method. This helps explain the contention of commutativity; the two long sides are each A and the two short sides are each B. The resultant will be the diagonal of the parallelogram.



When more than two vectors are added graphically, we must do one at a time, so

$$\vec{A}+\vec{B}+\vec{C}+\vec{D}=\left(\left(\vec{A}+\vec{B}\right)+\vec{C}\right)+\vec{D}$$

Vector addition is also *associative*:

$$\vec{A}+\left(\vec{B}+\vec{C}\right)=\left(\vec{A}+\vec{B}\right)+\vec{C}$$

EXERCISE 1-5

Make an argument that vector addition is associative. Try a graphical solution with three vectors.

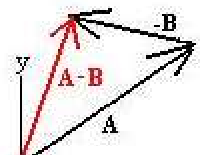
HOMEWORK 1-3

Anne walks a certain distance due north, then turns due east and walks twice as far. At the end of her trip, she is 450 meters from her starting point, as the crow flies. What is the length of each leg of the trip? What is the direction of her displacement relative to north?

I have no idea how to subtract vectors, but I know a trick from grade school. When I learned to add, for example 5 + 2, I started at the origin of the number line and moved five to the right, then another two to the right. To subtract, say 5 – 2, I moved five to the right, then two to the left, that is, I did 5 + (-2). Let’s try this:

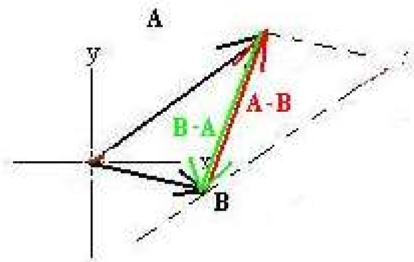
$$\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right).$$

The question is then, what is $-\vec{B}$? I think we would want to require



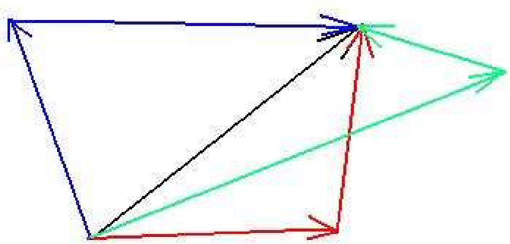
$$\vec{B} + (-\vec{B}) = \vec{0} \ ,$$ [6]

That is, $-\vec{B}$ must have the same magnitude as $\vec{B}$, but point in exactly the opposite direction

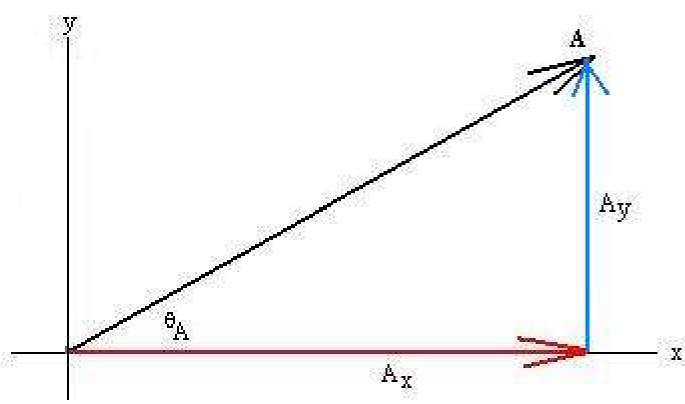


Comparison to the parallelogram method reveals that $\vec{A} - \vec{B}$ is the other diagonal of the parallelogram (as is $\vec{B} - \vec{A}$, the same diagonal but pointing in the other direction).

Once again, to add vectors graphically, one would take paper, ruler and protractor, choose a scale, and draw arrows to represent the vectors such that the length of each is proportional to the magnitude of the corresponding vector. To find the resultant, measure the length of the resultant with the ruler and back convert to find the magnitude, and use the protractor to find the direction.



Well, we really don't want you adding vectors with rulers and protractors for the rest of the semester. Let's investigate an analytic method. Now that we can add vectors, we can also see that any given vector (shown in black) can be written as the sum of two (or more) other vectors. In the diagram, you can see that the black vector is the sum of the two red vectors, but it is also the sum of the two green vectors as well as the sum of the two blue vectors. If that's true, we might as well choose two vectors that will be convenient for us. If we make the two vectors perpendicular, we might be able to use trig relationships to suss out some info.



A_x is called the *x-component* of **A** and A_y is the *y-component* of **A**, that is, how much the vector points in each direction. A_x and A_y are actually scalars, although they can be positive or negative or even zero. We convey the directional information through the use of the *unit vectors* $\hat{\imath}$ (x direction), $\hat{\jmath}$ (y direction), and $\hat{k}$ (z direction). Unit vectors have length one and are dimensionless (that information is carried in the components). Sticking with two dimensions for now, we can write that $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$. From trig, we see that $A_x = A \cos\theta_A$ and that $A_y = A \sin\theta_A$. Note that if

[6] Technically speaking, this zero is also a vector, the *null vector*.

we measure θ_A CCW from the x axis, that the signs of the trig functions correctly give the signs of the components.

EXAMPLE 1-3

Let $\vec{A}$ be 15 m at θ_A= 120°, which is in the second quadrant. We find that

$$A_x = A\ \cos\theta_A = (15\ \text{m})\cos 120^\circ = -7.5\ \text{m}$$
$$A_y = A\ \sin\theta_A = (15\ \text{m})\sin 120^\circ = +13\ \text{m}$$

So, $\vec{A} = -7.5\,\hat{\imath} + 13\,\hat{\jmath}$ meters .

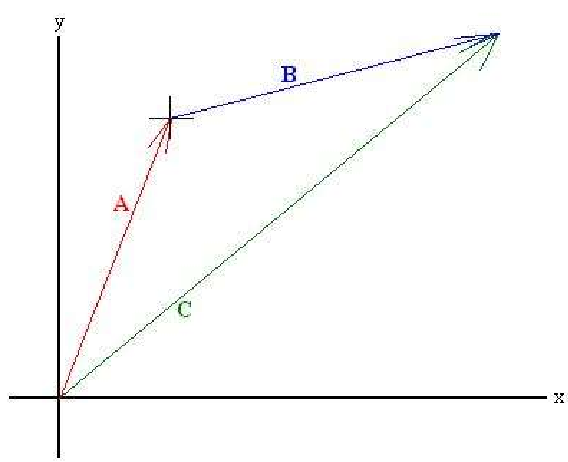
and the signs of these components match what we know about the direction of $\vec{A}$.

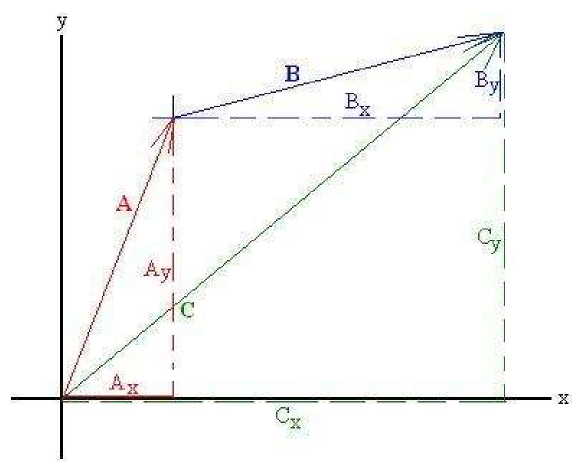
HOMEWORK 1-4

The direction of a vector is 127° measured from the x-axis, and its y-component is 12.0 units. Find the x-component of the vector and the magnitude of the vector.

Now, we have an alternate manner of adding vectors using the components. Let $\vec{C} = \vec{A} + \vec{B}$.

I hope it's clear that $C_x = A_x + B_x$ and $C_y = A_y + B_y$. We might say that **the components of the sum are the sums of the components**. Once we have the components of $\vec{C}$, we can convert them back to a magnitude and a direction angle.





EXAMPLE 1-4

Let $\vec{C} = \vec{A} + \vec{B}$. Find the magnitude and direction angle of $\vec{C}$.

A = 7 m $\theta_A = 35^\circ$
B = 12 m $\theta_B = 155^\circ$

First, we find the components of $\vec{A}$ and $\vec{B}$:

$$A_x = A\ \cos\theta_A = (7\text{ m})\cos 35^o = +5.73\text{ m}$$
$$A_y = A\ \sin\theta_A = (7\text{ m})\sin 35^o = +4.02\text{ m}$$
$$B_x = B\ \cos\theta_B = (12\text{ m})\cos 155^o = -10.88\text{ m}$$
$$B_y = B\ \sin\theta_B = (12\text{ m})\sin 155^o = +5.07\text{ m}\quad .$$

Then we do with the components what we're asked to do with the vectors:

$$C_x = A_x + B_x = 5.73 + (\text{-}10.88) = \text{-}5.15\text{ m}$$
$$C_y = A_y + B_y = 4.02 + 5.07 = 9.09\text{ m}\quad .$$

Then, we reconstitute the components of $\vec{C}$ back into a magnitude and direction:

$$C = +\sqrt{C_x^2 + C_y^2} = \sqrt{(\text{-}5.15)^2 + 9.09^2} = 10.45\text{ m}\ ,$$
$$\theta_C = \arctan\left(\frac{C_y}{C_x}\right)^* = \arctan\left(\frac{9.09}{\text{-}5.15}\right) = \arctan(\text{-}1.76) = \text{-}60.47^o\ .$$

Are we done? No, we need to check the quadrant of the angle to see if the calculator's answer is correct. In this case, it is not. Because $C_x<0$, we need to add 180° to the result. So

$$\theta_C = -60.47 + 180 = 119.53^o\quad .$$

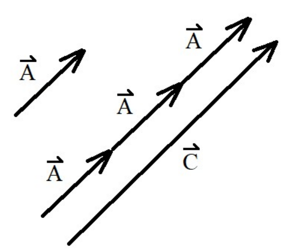
HOMEWORK 1-5

Vector $\vec{A}$ has magnitude 8.0 units at an angle of 60° from the x-axis. Vector $\vec{B}$ has magnitude 6.0 at an angle of -30° from the x-axis. Find the magnitude and direction of vector $\vec{C} = \vec{A} + \vec{B}$.

Vector Multiplication

There are a number of ways vectors can be multiplied; we'll deal with three.
The first type of multiplication is perhaps familiar from grade school. Let's multiply $\vec{A}$ by a scalar, 3, and call that $\vec{C}$:

$$\vec{C} = 3\,\vec{A}\ .$$



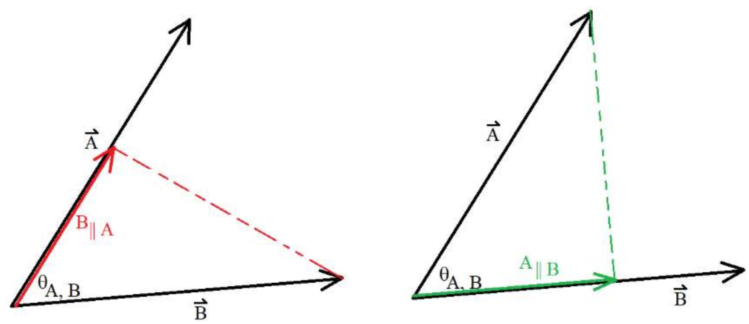
This type of multiplication is repeated addition.

$$\vec{C} = \vec{A} + \vec{A} + \vec{A}\,.$$

We can see that $\vec{C}$ is in the same direction of $\vec{A}$ but with three times the magnitude. It should be easy to see that we could expand this notion to non-integer multiples as well. It's a little more complicated when the scalar is not a dimensionless number, but the notion is the same; the value and dimension of the magnitude will change, but the direction will remain the same.[7]

Next, we will define the *scalar product* (also called the *inner product* or the *dot product*) of two vectors $\vec{A}$ and $\vec{B}$ to be:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta_{A,B} \ ,$$



that is, the magnitude of $\vec{A}$ times the magnitude of $\vec{B}$ times the cosine of the angle between them if they were placed tail to tail. The dot product is defined to be a scalar. One interpretation of this definition is that we are multiplying the magnitude of $\vec{A}$ by the component, or *projection*,[8] of $\vec{B}$ that lies in the direction of $\vec{A}$:

$$\vec{A} \cdot \vec{B} = A\, B_{||} = A\,(B \cos\theta_{A,B}) = |\vec{A}||\vec{B}| \cos\theta_{A,B} \ ,$$

as shown in the figure on the left. Clearly, though, we could just as well think of it as the magnitude of $\vec{B}$ times the projection of $\vec{A}$ on $\vec{B}$:

$$|\vec{A}||\vec{B}| \cos\theta_{A,B} = B\,(A \cos\theta_{A,B}) = B\, A_{||} = \vec{B} \cdot \vec{A} \ .$$

The dot product is therefor commutative.

Keep in mind that there is nothing magical about the dot product. It is simply a shorthand way of writing a particular process; as the course progresses, we'll see that we are often interested in how much of one vector is in the direction of another.

DERIVATION 1-1*

Alternatively, we can write the vectors $\vec{A}$ and $\vec{B}$ in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Remember that $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$. Then,

$$\begin{aligned}\vec{A} \cdot \vec{B} = &\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\cdot\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)\\ &= A_xB_x\hat{\imath}\cdot\hat{\imath} + A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_xB_z\hat{\imath}\cdot\hat{k} + A_yB_x\hat{\imath}\cdot\hat{\jmath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\hat{\jmath}\cdot\hat{k}\\ &\quad + A_zB_x\hat{\imath}\cdot\hat{k} + A_zB_y\hat{\jmath}\cdot\hat{k} + A_zB_z\hat{k}\cdot\hat{k}\end{aligned}$$

[7] Looking way ahead, the momentum p of an object is given by the mass times the velocity v. Momentum and velocity are in the same direction, but they have very different dimensions.

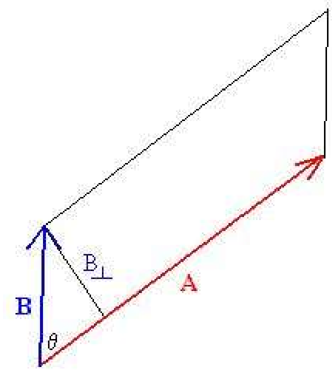
[8] You can think of a projection as analogous to a shadow, the shadow that $\vec{B}$ casts on $\vec{A}$.

$$= A_x B_x + A_y B_y + A_z B_z$$

Another type of vector multiplication is the *vector product* or the *cross product,* which we define in two parts. We define the magnitude of the cross product to be

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta_{A,B} \ .$$

that is, we're taking the magnitude of A and multiplying by the component of B that is perpendicular to A. One interpretation of the cross product's magnitude is that it is the area of the parallelogram formed by the vectors **A** and **B** when they are placed tail to tail. Using an argument like the one for the dot product, we see that



$$|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| \ .$$

However, there is a second part to the cross product, direction. We define the direction of $\vec{A} \times \vec{B}$ to be perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$. That leaves two possible directions, for example, in the diagram, into the page or out of the page. We define the direction sense using the *right-hand-rule* (RHR). Point your index finger of your right hand in the direction of $\vec{A}$ and your middle finger in the direction of $\vec{B}$; your right thumb then points in the direction of the cross product. You can then see that

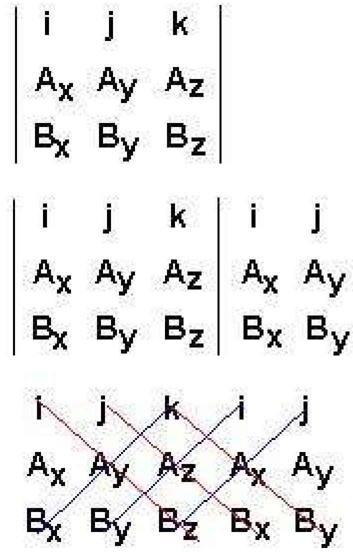
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \ .$$

DERIVATION 1-2*

Alternatively, we can write the vectors $\vec{A}$ and $\vec{B}$ in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Remember that $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$ and $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, and $\hat{k} \times \hat{\imath} = \hat{\jmath}$. Then,

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\ &= A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k} + A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} + A_yB_z\hat{\jmath} \times \hat{k} \\ &\quad + A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k} \\ &= (A_yB_z - A_zB_y)\,\hat{\imath} + (A_zB_x - A_xB_z)\,\hat{\jmath} + (A_xB_y - A_yB_x)\,\hat{k} \,.\end{aligned}$$

There is a quick way of remembering how to do this. Arrange the components into a table as seen in the top figure. Rewrite the first two columns at the right of the table, as shown in the middle figure. Lastly, multiply the quantities along each diagonal as shown. If the diagonal is to the down and to the right (red), add the product and if it's to the left (blue), subtract.



HOMEWORK 1-6*

Given that

$$\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k} \quad \text{and}\ \vec{B} = -\hat{\imath} + 3\hat{\jmath} + 2\hat{k} \quad ,$$

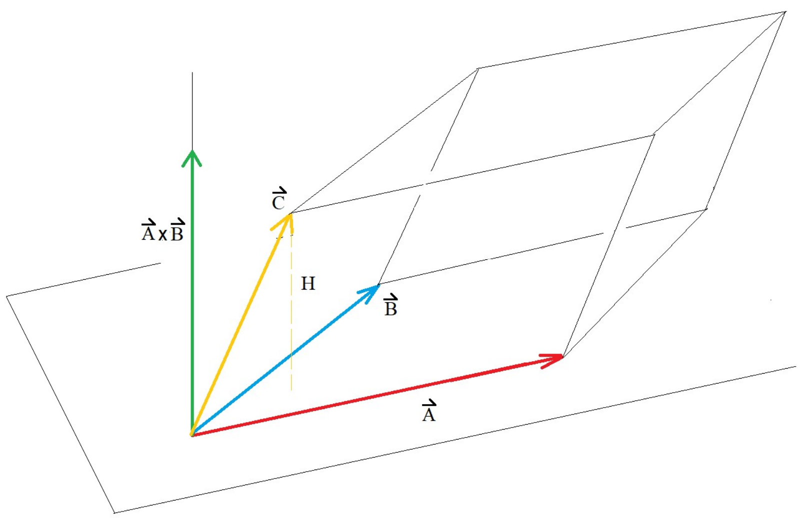
find $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$.

It is also sometimes useful to combine successive multiplications. Consider the *scalar triple product*. We'll be using this for one problem only, but this seems like the appropriate time to introduce it. Consider three vectors, not all in the same plane. The scalar triple product has an interesting useful property:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times A) = \vec{C} \cdot (\vec{A} \times \vec{B}) \ .$$

DERIVATION 1-3*

The three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, when paced tail to tail to tail, are the edges of a parallelepiped solid. As discussed above, the magnitude of the cross product of $\vec{A}$ and $\vec{B}$ gives the area of the parallelogram-shaped base of the solid. The volume V of the solid will be the base area times the height, H:



$$V = H|\vec{A} \times \vec{B}| \ .$$

The height H is the projection of $\vec{C}$ on $\vec{A} \times \vec{B}$, so

$$V = \vec{C} \cdot (\vec{A} \times \vec{B}) \ .$$

Now, we do need to be a little careful, in that $\vec{C}$ should be on the same side of the AB plane as $\vec{A} \times \vec{B}$; if not, then we get the negative of the volume instead.

Now, imagine that we were to roll the solid onto its BC face. The volume would be

$$V = \vec{A} \cdot \left(\vec{B} \times \vec{C}\right) \ .$$

Rolling it over again onto its AC side,

$$V = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) \ .$$

Since rolling the solid over doesn't change its volume, we have a useful relationship:

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times A\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) \ .$$

Lastly, let's consider the *vector triple product*, $\vec{A} \times \left(\vec{B} \times \vec{C}\right)$. I assert that

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C} \ .$$

DERIVATION 1-4*

The straightforward path is to write each vector in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, then perform the operations required on each side of the equation. Let's try to see if we can do it in a less tedious way.[9]

The vector $\vec{B} \times \vec{C}$ is of course perpendicular to the plane containing both $\vec{B}$ and $\vec{C}$. When we cross $\vec{A}$ with that vector, the result is perpendicular to $\vec{B} \times \vec{C}$, which means it lies back in the B-C plane. Therefore, we can write the triple product in terms of some additive combination of $\vec{B}$ and $\vec{C}$:

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \alpha \vec{B} + \beta \vec{C} \ ,$$

where alpha and beta are real numbers. The triple cross product must for this same reason also be perpendicular to $\vec{A}$, so

$$\vec{A} \cdot \left(\alpha \vec{B} + \beta \vec{C}\right) = 0 \, ,$$

$$\alpha \vec{A} \cdot \vec{B} = -\beta \vec{A} \cdot \vec{C} \ .$$

This requires that

$$\alpha = \gamma \vec{A} \cdot \vec{C} \quad \text{and} \quad \beta = -\gamma \vec{A} \cdot \vec{B} \ ,$$

[9] Ercelebi, Atilla, "A×(B×C).pdf," accessed 12/4/2020, www.fen.bilkent.edu.tr/~ercelebi/Ax(BxC).pdf.

with gamma some presently unknown number that will cancel out upon substitution back into the previous equation.[10] This relationship should be correct for <u>any</u> vectors, so let's see if we can determine gamma by applying these relationships to a specific set of vectors, î, ĵ, and k̂:

$$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \alpha\, \hat{\imath} + \beta\, \vec{\jmath} \;\;,$$

$$\hat{\imath} \times \hat{k} = (\gamma\, \hat{\imath} \cdot \hat{\jmath})\, \hat{\imath} + (-\gamma\, \hat{\imath} \cdot \hat{\imath})\, \vec{\jmath} \;\;,$$

$$-\hat{\jmath} = (0)\, \hat{\imath} + (-\gamma)\, \vec{\jmath} \;\;,$$

$$\gamma = 1 \;\;.$$

Now we have that

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \alpha\, \vec{B} + \beta\, \vec{C} = \left(\vec{A} \cdot \vec{C}\right) \vec{B} - \left(\vec{A} \cdot \vec{B}\right) \vec{C} \;\;.$$

EXERCISE 1-1 Solution

What quantities might affect the time and what are their respective dimensions? Well, we have

height H = [Length]
time t = [Time]
mass m = [Mass]
gravitational field strength g = [Length]/[Time]^2

We might guess that

$$t \sim H^a m^b g^c \;\;.$$

$$[T]^1 = [L]^a [M]^b \left(\frac{[L]}{[T]^2}\right)^c = [L]^{a+c} [M]^b [T]^{-2c} \;\;.$$

Then,

0 = a+c;
b = 0;
1 = -2c → c = -1/2;
a = -c = +1/2.

[10] In other words, α = A·C, β = A·C is not the only possible solution; α = 6.7 A·C, β = 6.7 A·B would fit as well. We need an unambiguous solution.

$$t \sim H^{1/2} g^{-1/2} = \sqrt{\frac{H}{g}} \; .$$

The correct relationship, as we will see in the next section, is

$$t = \sqrt{\frac{2H}{g}} \; .$$

EXERCISE 1-2 Solution

$$X \text{ cm}^3 = 1\text{m}^3 \left(\frac{100 \text{ cm}}{1 \text{ m}}\right)\left(\frac{100 \text{ cm}}{1 \text{ m}}\right)\left(\frac{100 \text{ cm}}{1 \text{ m}}\right) = 10^6 \text{cm}^3 \; .$$

Note that you must cancel each of the three meters in the original value.

EXERCISE 1-3 Solution

$$\theta = \arctan\left(\frac{y}{x}\right)^* = \arctan\left(\frac{2}{3}\right) = 33.7^o$$

$$r = +\sqrt{x^2 + y^2} = +\sqrt{3^2 + 2^2} = 3.61$$

EXERCISE 1-4 Solution

x = -1, y = -3 (There were no units.)

$$r = +\sqrt{x^2 + y^2} = +\sqrt{(-3)^2 + (-1)^2} = \sqrt{10} = 3.16$$

Be sure to square the negative signs!

$$\theta = \arctan\left(\frac{-3}{-1}\right)^* = \arctan(3)^* = 71.6^o$$

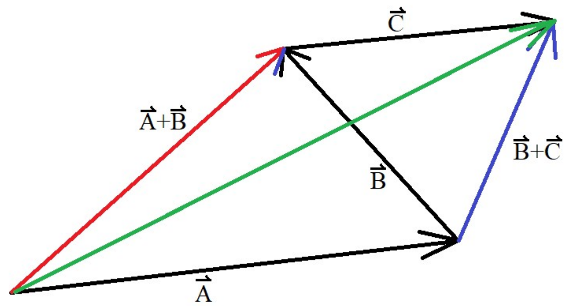
But, x is negative, so we need to add 180° to get 251.6° as the correct answer.

For the second part of the exercise, we have r = 4, θ = 120°. In this direction, there's no ambiguity.

$$x = r \cos\theta = 4 \cos 120^0 = -2$$

$$y = r \sin\theta = 4 \sin 120^o = 3.46$$

EXERCISE 1-5 Solution



This is a demonstration, not a proof:

The green vector is the sum of $\vec{A}$, $\vec{B}$, and $\vec{C}$, and can be written as $(\vec{A}+\vec{B}) + \vec{C}$, or as $\vec{A}+ (\vec{B}+\vec{C})$.