Section 11 - Oscillations

Let's define a couple of terms. An *oscillator* is an object that moves repetitively through a given path in a given time period. In a sense, you are an oscillator, moving from home to school to work to home every day. A *simple harmonic oscillator* (SHO) is a very special case when an object moves through a cycle along a line (perhaps the x-axis) around a central point (we'll say at x = 0) such that its position x is given by

$$x(t) = A\cos(2\pi f t + \varphi) .$$

Here, t is time, A is the *amplitude*, the maximum excursion from the central point, and phi is the *phase angle* that allows us to change the cosine into a sine by shifting the curve in time. For this course, phi will always be set to zero. The symbol *f* is the *frequency*, the number of cycles completed *per* unit of time; one cycle *per* second is called one hertz (Hz). Note that we will often replace $2\pi f$ with the *angular frequency*, ω (omega). We will also define the *period* of oscillation, P, as the time to complete one cycle, and so necessarily, $P = 1/f$.

Since the object is not moving with constant velocity, there must be some force acting on it. More specifically, the force acts to return the object back towards the central point. Such a force is described as a *restoring force*.

DISCUSSION 11-1

> Can you remember a force we discussed that is opposite in direction and proportional to the displacement of an object from its equilibrium point? Is that necessarily the only such force that meets those conditions?

One such force is that exerted by a spring, F = (-) kX. Let's attached a mass m to the spring such that the end of the spring and the location of the mass are the same, *i.e*. X = x, and set it in motion. Following Newton's second law,

$$F_{\text{Spring}} = -kx = ma = m\frac{d^2x}{dt^2} .$$

So, we're looking for a function x(t) that is proportional to its own second derivative. Two such functions come to mind: the exponential function and the cosine function:

$$x(t) = Ae^{\alpha t+\varphi} \text{ and } x(t) = A\cos(\omega t + \varphi) .$$

The presence of the phase angle φ allows for combinations of sine and cosine functions. Let's be a little bit more discerning; the second derivative is to be proportional to the negative of the original function, which disqualifies the exponential. So, we have that

$$x(t) = A\cos(\omega t + \varphi) .$$

Then,

$$\frac{dx}{dt} = -\omega A \sin(\omega t + \varphi) \quad \text{and} \quad \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \varphi) \quad .$$

Let's substitute into each side of the NII equation:

$$-kA\cos(\omega t + \varphi) = m\left(-\omega^2 A \cos(\omega t + \varphi)\right)$$
$$k = m\omega^2 \ .$$

We see that our guess is a solution, but only if

$$\omega = \sqrt{\frac{k}{m}} \ .$$

The physical interpretation of this is that the frequency of oscillation is determined by these two parameters of the mass-spring system. This is called the *natural frequency* of the system. Occasionally, the frequency is expressed in cycles *per* second rather than in radians *per* second; in that case, the conventional unit is the Hertz (Hz)

CHEESEY EXPERIMENT 11-1

Oscillations.mp4

DISCUSSION 11-2

To test this result, it is more convenient to measure the natural period of oscillation, $P_o = 1/f_o$:

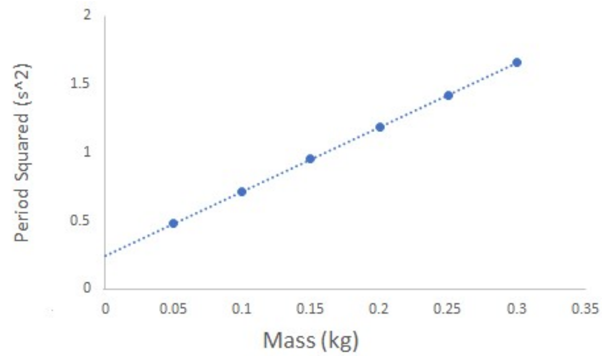
$$P_o = 2\pi\sqrt{\frac{m}{k}} \ .$$

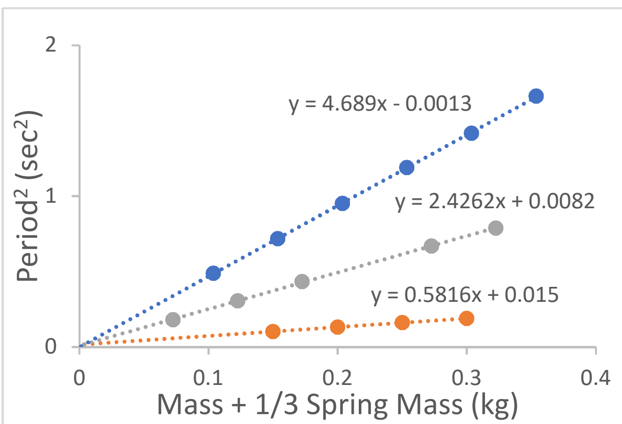
Would we expect that the period proportional to the mass and inversely proportional to the spring constant? How should the data be plotted to obtain a line? If we were to place 100 grams of mass at the end of a <u>real</u> spring, how much mass would be oscillating?

EXPERIMENT 11-1

Let's square both sides of the equation above with the intention of plotting the mass as the independent variable:

$$P_o^2 = \left(\frac{4\pi^2}{k}\right) m \ .$$





In theory, to obtain a line, the dependent variable plotted should be the square of the period. For the data shown in the graph, there is a serious problem. The theoretical relationship predicts that the intercept of the line should be zero. We asked a moment ago how much mass is actually oscillating. Since this is a real spring and not our abstract massless spring, we should take into account the parts of the spring that are also moving. We won't go into it here, but under these conditions, we should count one third of the mass of the spring.[1] Replotting and adding more springs result in lines with intercepts very close to zero. We see then that the square of the period for each is proportional to the mass plus 1/3 the mass of the spring. Now, what's different about these springs? These are the same springs we used for Hooke's relationship in Section 6. According to the relationship above, we can determine each spring constant from the slope of the associated line, since

$$\mathrm{y = (slope)x + (intercept)} \quad \leftrightarrow \quad \mathrm{P_o^2} = \left(\frac{4\pi^2}{\mathrm{k}}\right)\mathrm{m} + 0$$

$$\mathrm{slope} = \frac{4\pi^2}{\mathrm{k}} \quad \rightarrow \quad \mathrm{k} = \frac{4\pi^2}{\mathrm{slope}} \ .$$

Comparing results from this experiment and from those of section 6,

Spring	Hooke's relationship experiment	Oscillation experiment
Nr 1	8.25 N/m	8.42 N/m
Nr 2	16.13 N/m	16.27 N/m
Nr 3	64.10 N/m	67.88 N/m

gives us a bit more confidence that this relationship is correct.

[1] J.G. Fox and J. Mahanty, "The Effective Mass of an Oscillating Spring," *Am. Jour. Phys.* 38 No 1 (January 1970): 98–100.

In the discussion above, we assumed that the only force acting on the mass was that of the spring. That might be appropriate way out in space, or if the mass were mounted horizontally on an airtrack. But often, springs are arranged vertically, and so gravity plays a part. Turns out, though, that this does not affect the results for the frequency of oscillation; the mass will simply hang at a lower equilibrium point as more mass is added.

JUSTIFICATION 11-1*

When the spring is horizontal, its motion is governed by the second law,

$$-kX = ma \ .$$

When a massless spring is hung vertically, its lower end sits at the equilibrium point, X = 0. If we add some mass to the end and allow the system to come to rest, the new equilibrium point will be obtained from the second law:

$$-kX_{EQ} - gm = 0 \quad \rightarrow \quad X_{EQ} = \frac{-gm}{k} \ .$$

Note that this is negative, because the new equilibrium point will be lower than the original one. Now, let the mass oscillate about this new equilibrium point. The second law equation will be

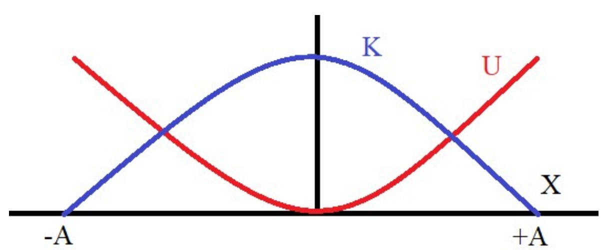
$$-kX - gm = ma \ .$$

Substitute:

$$-kX - kX_{EQ} = -k(X - X_{EQ)} = -kX' = ma \ .$$

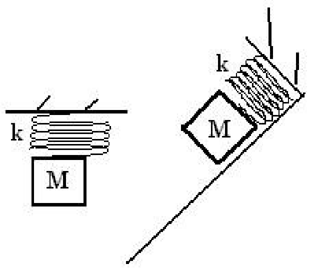
Here, X' is the displacement of the object from the <u>new</u> equilibrium point. This is the same force equation (and therefor the same motion) for the case of no gravity, except that the equilibrium point will be at X_{EQ}.

Let's have a quick discussion about the energy of a SHO. The kinetic energy is K= $^1/_2 mv^2$ and the potential energy is U = $^1/_2\, kx^2$, and the total is the sum of these two.



$$\begin{aligned} E_{TOTAL} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kX^2 \\ = \tfrac{1}{2}m(-2\pi f\ A\sin(2\pi f t))^2 \\ + \tfrac{1}{2}k(A\cos(2\pi f t))^2 \\ = \tfrac{1}{2}m\frac{k}{m}A^2\sin^2(2\pi f t) \\ + \tfrac{1}{2}kA^2\cos^2(2\pi f t) = \tfrac{1}{2}kA^2 \ . \end{aligned}$$

So, if we pull back the mass to x = A and release it, the energy will convert from potential to kinetic, then back to potential, *et c*.



HOMEWORK 11-1

An oscillator with a 0.23 second period is made from a mass M suspended from a spring of constant k. The mass is then placed on a frictionless surface which makes a 45° angle with the horizontal, and the spring is attached at the top of the incline as shown. What is the new period of the oscillation?

EXAMPLE 11-2

A 0.8 kg air-track car is attached to the end of a horizontal spring of constant k = 20 N/m. The car is displaced 12 cm from its equilibrium point and released. What is the car's maximum speed? What is the car's maximum acceleration? What is the frequency f_o of the car's oscillation?

The frequency of oscillation is given by

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20}{0.8}} = 0.80\ \text{Hz}\,.$$

The velocity of the object[2] is given by

$$v(t) = -2\pi f_o A\sin(2\pi f_o t) = -2\pi(0.80)(0.12)\sin(2\pi f_o t) = -0.60\sin(2\pi f_o t)\ .$$

The speed is a maximum whenever the sine term equals ± 1. Maximum speed is 0.60 m/s.

The acceleration is

$$a(t) = -\left(4\pi^2 f_o^2\right)A\cos(2\pi f t) = -4(9.87)(0.08^2)(0.12)\cos(2\pi f t) = -0.03\cos(2\pi f_o t).$$

The maximum magnitude acceleration is when the cosine term is ± 1, so $|a_{MAX}|$ = 0.03 m/s^2.

HOMEWORK 11-3

A mass (0.3 kg) and spring (k = 350 N/m) system oscillates with an amplitude of 6 cm. What is the total mechanical energy of the system? What is the maximum speed of the mass? What is the maximum acceleration of the mass?

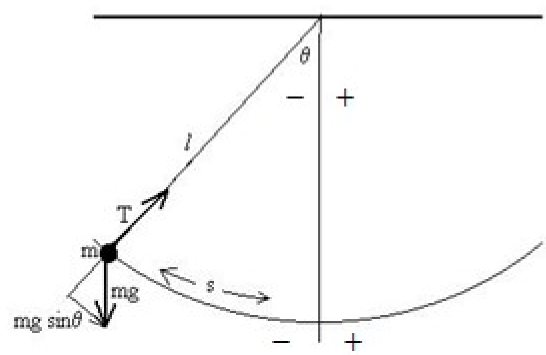
[2] Remember that we're always assuming that the object is pulled in the positive direction to X = A and released at time = zero.

The Simple Pendulum

There are many other systems which exhibit simple harmonic motion (SHM), and even more that are close enough that we can make use of the results above for a reasonably correct approximate solution. One such system is the *simple pendulum*, which is a point mass m (the bob) at the end of a massless string or stick of length *l*. Let's look at the free body diagram for such an object.

DERIVATION 11-1

We are interested in the motion along the circular arc. Let us describe the bob's position with s (= $l\ \theta$), the displacement along the arc which we shall make positive to the right and negative to the left. Theta will follow the same convention. Break the forces into tangential and radial components. We aren't really interested in the radial components, but tangentially we have



$$-\mathrm{gm}\ \sin\theta = \mathrm{ma_T}\ .$$

The negative sign is necessary to get the direction of the force correct. When the bob is on the left side of the figure where s and theta are negative, we want the force to be in the positive direction. Similarly, when the bob is on the right side of the figure where s and theta are positive, the force must be to the negative direction.

We have two types of variables here, one tangential and the other angular. We need them to be the same type. Substitute s/*l* for theta:

$$-\mathrm{gm}\ \sin\left(\frac{s}{l}\right) = \mathrm{ma_T}\ .$$

Now, this is <u>not</u> the same as the equation for the mass/spring system, since the force F is proportional to the <u>sine</u> of the displacement, not to the displacement itself. In fact, this is a moderately difficult equation to solve. So, we will do what physicists often do, we will look at a special case, when the angle theta is 'small.' If an angle is small, the sine of the angle is approximately equal to the angle itself in radians.

MATHEMATICAL DIGRESSION

Put your calculator in radians mode. Take the sine of 0.0001 radians. How close it the result to 0.0001? Is 0.01% much of a difference? Repeat for 0.001, 0.01, and 0.1. Are the two values diverging slightly? Repeat for 0.5 radians and you will see that the result is about 4% off from the input. The art here is determine how much of a divergence is acceptable, or how small is 'small.' In a course like this one, we usually accept the approximation up to about 30°.

Continuing,

$$-\frac{\text{gm}}{l}\,\text{s} = \text{ma}_\text{T}\ .$$

Turns out, we've already solved this problem. The acceleration is negatively proportional to the position, same as for the mass on a spring. So, all of the steps we went through to solve that problem are the same steps we would go through here, except k is replaced with gm/l. Then,

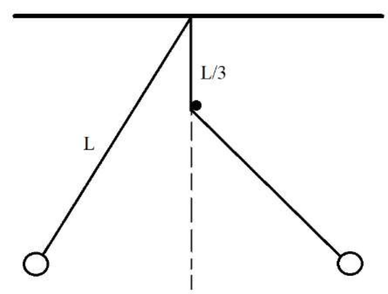
$$f_{\text{o Mass on Spring}} = \frac{1}{2\pi}\sqrt{\frac{\text{k}}{\text{m}}} \quad \rightarrow \quad f_{\text{o Simple Pendulum}} = \frac{1}{2\pi}\sqrt{\frac{\frac{\text{gm}}{l}}{\text{m}}} = \frac{1}{2\pi}\sqrt{\frac{\text{g}}{l}}\ .$$

Some of you may have verified the relationship between the frequency and the length in lab.

HOMEWORK 11-4

Mary-Kate (m = 50 kg) is swinging on a tire tied by a (light) rope (L = 3 m) to a tree limb. Her twin Ashley comes along and squeezes into the tire with her. Assume that at all times the center of mass of the person(s) riding the tire is at the end of the rope. What was the period of oscillation for Kate alone? What is the period of oscillation for the twins together?

HOMEWORK 11-5



A pendulum bob on a light string of length L is arranged as shown in the figure. There is a peg stuck into the wall a distance L/3 below the point of suspension. What is the period of small oscillations for this system?

DISCUSSION 11-3

Because we're not solving the actual equation for the motion of the simple pendulum, the correct result differs to some degree from what we've derived. Do you think that actual period of a simple pendulum at large angles is larger or smaller that for small angles?

EXAMPLE 11-X*

Let's try to solve the simple pendulum correctly. This is a difficult calculation, and one which I think is only worth setting up, not necessarily solving. Consider the time t necessary for the bob to move from its greatest displacement at θ_o to the bottom of its arc; this is then one quarter of the period, P/4. In time interval dt, the bob moves a distance ds along the arc, such that

$$d\text{t} = \frac{d s}{\text{v}}\ .$$

We can find an expression for the speed v by using conservation of mechanical energy; the tension does no work and the weight is a conservative force and is considered as a potential energy difference term.

$$\mathrm{gmy_o} = \mathrm{gmy} + \frac{1}{2}\mathrm{mv}^2 \quad \rightarrow \quad \mathrm{v} = \sqrt{2\mathrm{g}(\mathrm{y_o} - \mathrm{y})} \quad .$$

The altitudes of the bob above the lowest point of the swing can be written in terms of the angle the string makes with the vertical:

$$\mathrm{y} = \mathrm{L} - \mathrm{L}\cos\theta \quad ,$$

so that

$$\mathrm{v} = \sqrt{2\mathrm{gL}(\cos\theta - \cos\theta_o)} \quad .$$

Next, *d*s can be written in terms of the angle as well:

$$d\mathrm{s} = \mathrm{L}\, d\theta \ .$$

Put it together:

$$d\mathrm{t} = \frac{d\mathrm{s}}{\mathrm{v}} = \frac{\mathrm{L}\, d\theta}{\sqrt{2\mathrm{gL}(\cos\theta - \cos\theta_o)}} = \sqrt{\frac{L}{2g}}\, (\cos\theta - \cos\theta_o)^{-1/2}\, d\theta.$$

Finally, we integrate both sides corresponding to the travel from highest point to lowest point

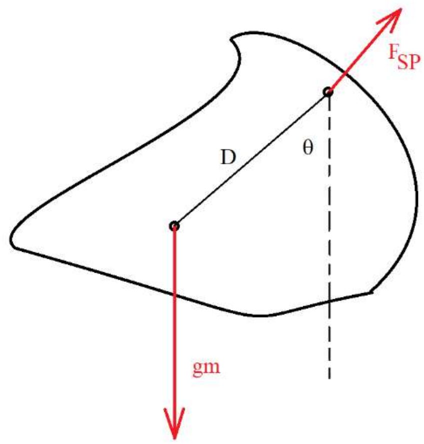
$$\int_0^{P/4} d\mathrm{t} = \sqrt{\frac{L}{2g}} \int_{\theta_o}^{0} (\cos\theta - \cos\theta_o)^{-1/2}\, d\theta$$

I'll do the left hand integral, you can do the right hand one:

$$P = 4\sqrt{\frac{L}{2g}} \int_{\theta_o}^{0} (\cos\theta - \cos\theta_o)^{-1/2}\, d\theta \quad .$$

The Physical Pendulum

DERIVATION 11-2



Consider an object of indeterminate shape hanging from an axis, as show. This is known as a *physical pendulum*. D is the distance between the point of suspension and the center of mass. The forces acting on the object comprise a force at the suspension point (the pivot) and the weight, which can be assumed to act at the center of mass. Consider the torques acting on the object when it has been displaced from equilibrium by angle theta:

$$-\mathrm{Dgm}\sin\theta + (0)\mathrm{F_{SP}}\sin(?) = \mathrm{I}\alpha \ .$$

If we once again restrict ourselves to 'small' angles,

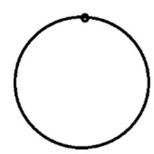
$$-\mathrm{Dgm}\,\theta = \mathrm{I}\alpha \ ,$$

we see that this problem is the same as for the mass on a spring (alpha is to theta as a is to x) and that we have already solved it. The result is found by replacing k with Dgm and m with I:

$$f_{\mathrm{o\ Physical\ Pendulum}} = \frac{1}{2\pi}\sqrt{\frac{\mathrm{Dgm}}{\mathrm{I}}} \ .$$

EXAMPLE 11-1*

Find the frequency of small oscillations for a vertically suspended disk of radius R and mass M if it is attached to an axis at its top.



We can make use of the results above, but we will need to determine D and I. D is the distance between the suspension point and the center of mass, so, D = R. The moment of inertia of a disk about its center is $^1/_2MR^2$, but we've moved the axis a distance h = R, so we'll invoke the parallel axis theorem:

$$\mathrm{I_{EDGE}} = \mathrm{I_{CM}} + \mathrm{Mh^2} = \tfrac{1}{2}\mathrm{MR^2} + \mathrm{MR^2} = \tfrac{3}{2}\mathrm{MR^2} \ .$$

$$f_{\mathrm{o\ Physical\ Pendulum}} = \frac{1}{2\pi}\sqrt{\frac{\mathrm{RgM}}{\tfrac{3}{2}\mathrm{MR^2}}} = \frac{1}{\pi}\sqrt{\frac{g}{6R}} \ .$$

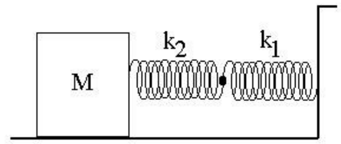
HOMEWORK 11-X

Find the natural frequency of oscillation of a thin rod of length L and mass M rotating about an axis through an end.

HOMEWORK 11-6

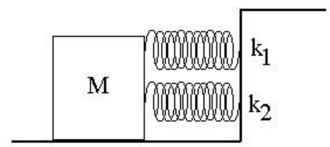
First, find the frequency of a simple pendulum with a point mass bob of mass M of a light string of length L. Then, find the frequency of a spherical bob of mass M and radius R = 0.1L at the end of that same light string. Calculate a *per cent* difference.

EXERCISE 11-1*



A mass is attached to two massless springs as shown in the figure. What is the natural frequency of oscillation f_o if M = 7 kg, k_1 = 300 N/m, and k_2 = 900 N/m? Assume no friction. HINT: Find the effective spring constant of the two springs is they were replaced by a single spring that does the same job.

HOMEWORK 11-7



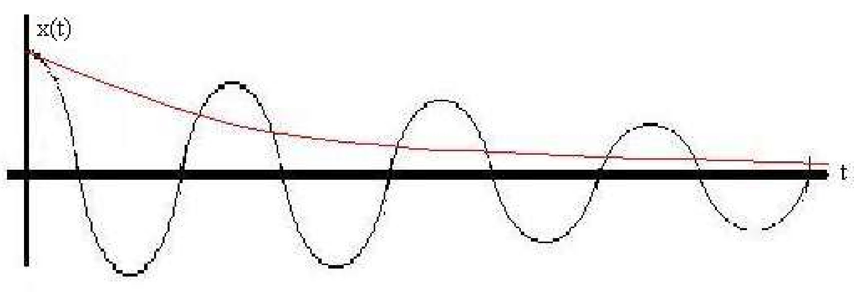
A mass is attached to two massless springs as shown in the figure. What is the natural frequency of oscillation f_o if M = 7 kg, k_1 = 300 N/m, and k_2 = 900 N/m? Assume no friction.

Damped Oscillations

We spoke briefly about *damped oscillations*. The discussion above suggest that, if one sets the mass/spring system into oscillation, the total energy of the system remains constant and the mass will vibrate forever with the same amplitude. In fact, we know that the mass will slow a bit on each pass due to friction with the air (usually assumed to be a *drag force* of the form F_{Drag}= -bv) or the table; energy is removed as friction performs negative work on the mass. A typical NII equation for laminar flow of a fluid around a moving object is

$$-\mathrm{kx} - \mathrm{b}\frac{d\mathrm{x}}{d\mathrm{t}} = \mathrm{m}\frac{d^2\mathrm{x}}{d\mathrm{t}^2} \ .$$

The figure shows a *lightly damped* system (black curve) and an *overdamped* system (red line), which loses so much energy so quickly that it never oscillates even once. A good example of the overdamped system is the car shock absorber. The car (m) is supported by springs (k), so that SHM is possible. If one were to drive over a bump with faulty shocks, the car would then continue to oscillate at about 1 Hz for several seconds. Shock absorbers dampen the system so that the ride smooths out without the oscillations.



DISCUSSION 11-4

The natural frequency of a lightly damped oscillation is slightly lower than that of an undamped system. Can you give a brief, non-mathematical reason for this? Consider the very first swing of the bob. What does the retarding force do to the speed of the bob?

The frequency of oscillation for a lightly damped system is given by[3]

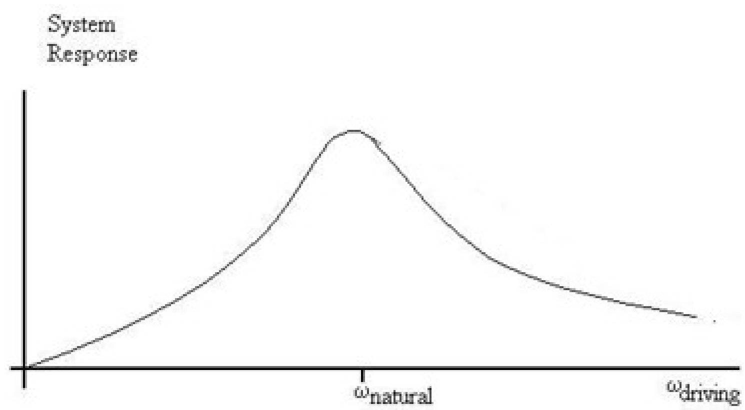
$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

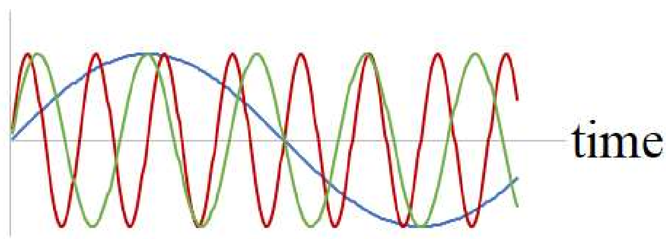
HOMEWORK 11-X

> Show that when the damping coëfficient b is high enough, oscillation is not possible. Find that critical value of b.

Resonance

If we were to disturb the mass/spring system in some way and step back, the system will oscillate with natural frequency $\omega_o = [k/m]^{1/2}$. If it's disturbed again in a different manner then left to itself, the system will again oscillate at that same natural frequency, until its energy is depleted. If we want the system to continue to oscillate, we must replace the energy lost to dissipative forces. Let's jiggle the other end of the spring, applying a force though a distance (*i.e.*, doing work), at some frequency *f*, which is then known as the *driving frequency*. Let us vary the driving frequency to see the effect on the system. If we jiggle the spring at a very low frequency, we see that the mass oscillates with the same frequency at which it is driven, but with a small amplitude. Changing to very high driving frequency, we see once again that the mass oscillates at the driving frequency, but with a very small amplitude. However, if we excite the system at a driving frequency very near to the natural frequency, we see that the response of the system, as demonstrated by the amplitude of oscillation, increases. If we plot this response as a function of the driving frequency, we see the curve shown in the figure. Consider this simplified situation:





The green line represents oscillation at the natural frequency. If we were to apply the force as the shown by the blue line at a frequency less than f_o, we would see that sometimes the force is acting in the direction of motion of the mass, but at other times, against the motion. On average, then, no work is done by that force. The red line indicates the force with frequency > f_o, and the argument is the same. When

[3] If you like, you can check this. Assume a solution of the form $x(t) = Ae^{i\omega' t}$, where i is the root of -1.

we vary the applied force at the same frequency as the natural frequency, we are always applying force in the direction of motion of the mass, so all work we do is positive. If the rate of doing work is greater than the rate of energy dissipation, the amplitude of the oscillation will increase. The condition when the system is driven at its natural frequency and delivers its greatest response is called *resonance*. Sometimes resonance is desirable, sometimes not. For example, if one wants to push a small child on a swing, the greatest amount of fun (or terror) is attained when one pushes the swing at its natural frequency. On the other hand, if the ground shakes at the natural frequency of a skyscraper, the building may respond with an amplitude beyond the limits of structural integrity. The Tacoma Narrows Bridge collapse occurred because the wind passing over the bridge excited one of the span's torsional oscillation modes, resulting in the collapse about three hours later. Are you surprised at the incredible elasticity of steel and concrete? Only a dog lost its life in the collapse, because the owner left it behind when he abandoned his car on the bridge (Hmm!). The bridge had exhibited strange effects for the three months it was open. There are films of the deck of the bridge oscillating in a vibrational mode much like waves in the ocean; cars could actually disappear from view behind the humps which rose and fell in the roadway. A related system is that of tall skyscrapers. Once again, if the wind were to gust at the natural frequency of the building, it might cause collapse; modern buildings often have a mechanism to 're-tune' the vibrational modes of the building away from the current driving frequency of the wind.

Exercise 11-1 Solution

Suppose that Spring 1 is stretched from its equilibrium length a distance X_1. To do this, a force of $F_1 = k_1X_1$ is required. This force is applied by Spring 2. Suppose that Spring 2 is stretched a distance X_2 from its equilibrium length. This requires a force $F_2 = k_2X_2$. By the third law, this is the same magnitude force as F_1 and the force applied to the mass. We want to replace the two springs with a single spring of constant k_{EFF} that will apply the same force when it is stretched a distance $X_{EFF} = X_1 + X_2$.

$$X_{EFF} = X_1 + X_2$$

$$\frac{F_{EFF}}{k_{EFF}} = \frac{F_1}{k_1} + \frac{F_2}{k_2}$$

All of the forces here are the same magnitude, so

$$\frac{1}{k_{EFF}} = \frac{1}{k_1} + \frac{1}{k_2} \quad \rightarrow \quad k_{EFF} = \frac{k_1 k_2}{k_1 + k_2} = \frac{(300)(900)}{300 + 900} = 225\ \mathrm{N/m}\,.$$

Then,

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{225}{7}} = 0.90\ \mathrm{Hz}\,.$$