Section 12 - Waves and Sound

We covered single oscillators in Section 12. For the specific example of a mass/spring system, we saw that there is a natural frequency at which the system would 'like' to oscillate, given by $\omega_o = [k/m]^{1/2}$. Now, let's consider a chain of such oscillators, in this case identical masses connected by identical springs. If we apply a disturbing impulse to the left-end mass, it will move to the right, applying its own force on the next mass, *et c*. The speed of this disturbance as it moves down the chain of oscillators should depend on the masses and on the spring constants. For example, if the springs are stiff, the first mass will not need to move very far in order to apply a fairly large force on the next mass, while if the spring is flexible, the first mass would need to move a good distance to apply a sizable force to the next mass. In the same way, for a given force, smaller masses will respond more quickly and larger masses less quickly. We might surmise that the speed of the disturbance should be related to the frequency of oscillation of each mass, and based on dimensional analysis, we might suspect the relationship to be

$$v \sim \sqrt{\frac{k}{m}}\ .$$

Let's see if we can derive this result.

DERIVATION 11-1

Let x be the equilibrium position of each mass (the values are non-continuous). The masses are separated at equilibrium by distance D. Let y(x, t) be the displacement of each mass from its proper position.

Consider Newton's Second Law for the mass at position x + D:

$$F_{x+D} = ma_{x+D} = m\left.\frac{\partial^2 y}{\partial t^2}\right|_{x+D}$$

The force is due to the springs on either side of the mass (Hooke's Law), but the amount each spring is stretched or compressed depends not only on the location of the mass at x+D, but on the locations of the neighboring masses as well.

$$F_{x+D} = k[y(x+2D,t) - y(x+D,t)] - k[y(x+D,t) - y(x,t)]$$

Combine these expressions, divide each side by m, factor the k, and multiply the right side by D^2/D^2:

$$\left.\frac{\partial^2 y}{\partial t^2}\right|_{x+D} = \frac{D^2 k}{m}\left\{\frac{\left[\frac{y(x+2D,t)-y(x+D,t)}{D}\right]-\left[\frac{y(x+D,t)-y(x,t)}{D}\right]}{D}\right\} .$$

The quantities k, m, and D are 'microscopic' quantities in that they depend on the detailed structure of the chain of oscillators. Let's rewrite them in terms of 'macroscopic' quantities, L, M, and K, the total length and mass of the chain, as well as the spring constant of the chain. If there are N masses in a long chain, then[1]

$$L = ND; \;\; M = Nm, \text{and } K = \frac{k}{N} .$$

$$\left.\frac{\partial^2 y}{\partial t^2}\right|_{x+D} = \frac{L^2 K}{M}\left\{\frac{\left[\frac{y(x+2D,t)-y(x+D,t)}{D}\right]-\left[\frac{y(x+D,t)-y(x,t)}{D}\right]}{D}\right\} .$$

Now, let's smooth the system out by taking a limit by letting N go to infinity while D and m go to zero:

$$\left.\frac{\partial^2 y}{\partial t^2}\right|_{x+D} = \frac{L^2 K}{M}\lim_{D\to 0}\left\{\frac{\left[\frac{y(x+2D,t)-y(x+D,t)}{D}\right]-\left[\frac{y(x+D,t)-y(x,t)}{D}\right]}{D}\right\} .$$

We'll do this in two stages. First,

$$\left.\frac{\partial^2 y}{\partial t^2}\right|_{x+D} = \frac{L^2 K}{M}\left\{\frac{\lim\limits_{D\to 0}\frac{y(x+2D,t)-y(x+D,t)}{D}-\lim\limits_{D\to 0}\frac{y(x+D,t)-y(x,t)}{D}}{D}\right\} .$$

Remembering that

$$\lim_{\Delta x\to 0}\frac{F(x+\Delta x)}{\Delta x} = \frac{dF}{dx} ,$$

We see that the first limit is the x-derivative of y evaluated at x + D, and the second is the derivative evaluated at x:

[1] Actually, L = (N-1)D, but close enough. As for K, the effective constant of springs attached end-to-end was discussed in Section 10.

$$\lim_{D\to 0}\left.\frac{\partial^2 y}{\partial t^2}\right|_{x+D} = \frac{L^2 K}{M}\lim_{D\to 0}\left\{\frac{\left.\frac{\partial y}{\partial x}\right|_{x+D} - \left.\frac{\partial y}{\partial x}\right|_{x}}{D}\right\},$$

which we recognize as the derivative of the derivative, or

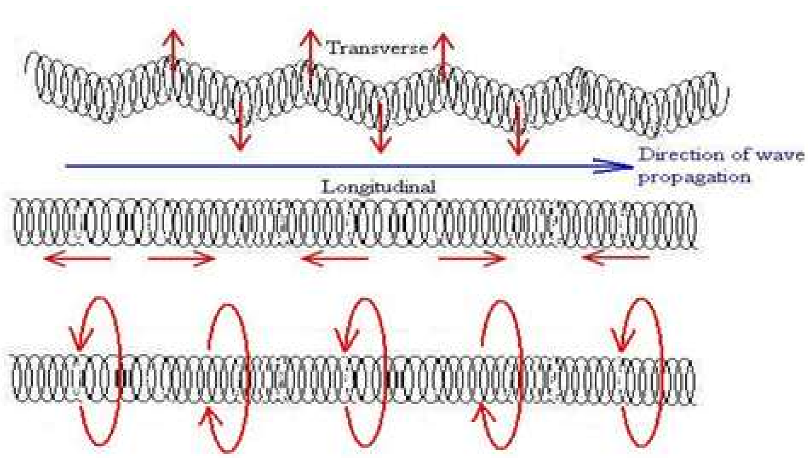
$$\left.\frac{\partial^2 y}{\partial t^2}\right|_{x} = \frac{L^2 K}{M}\left.\frac{\partial^2 y}{\partial x^2}\right|_{x}.$$

This then results in a one-dimensional version of the *wave equation*:

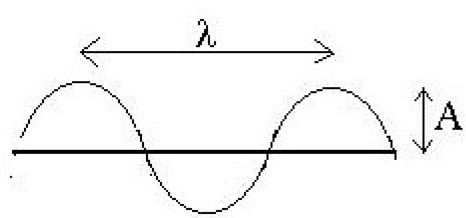
$$\frac{\partial^2 y}{\partial t^2} = \frac{L^2 K}{M}\frac{\partial^2 y}{\partial x^2}.$$

We'll come back to this result in a while.

Properties of a Wave

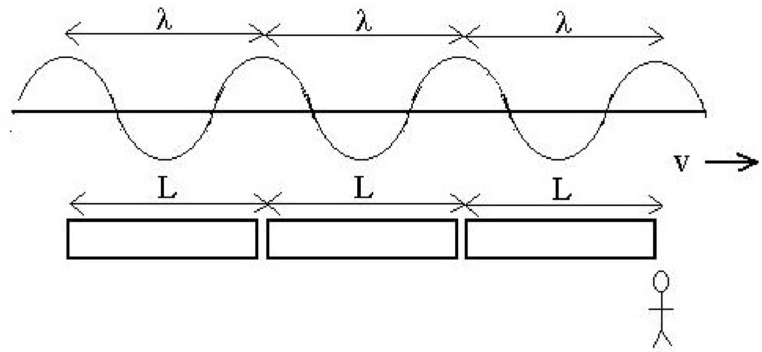


Let's consider a disturbance that repeats itself at regular intervals. Instead of individual pulses, the driving force at the end of the medium in which the disturbance travels is periodic, *i.e.*, it repeats its motion in a given amount of time, P. Now, the *wave* that is produced will have the same frequency as the driving force, even though the speed of propagation will be determined by the natural frequency of the individual oscillations (yet to be proven).We will be using one particular type of wave as our archetype; things we learn about this particular wave are transferrable to most other types of waves. We will concentrate on the *transverse wave* on a string. Waves in which the individual pieces of the medium move along the same line as the direction of propagation of the wave are referred to as *longitudinal*, while waves in which each piece of material moves along a line perpendicular to the direction of propagation, such as on a taut string, are called *transverse*. For the rest of the properties we want to discuss, let's restrict ourselves to *sinusoidal waves,* when the system is driven by a sinusoidal force with



frequency f. Now, the wave which is produced will have the same frequency as the driving force, even though the speed of propagation will be determined by the natural frequency of the individual oscillators. Let's start by defining some of the characteristics of a wave.

- the *frequency* (f) counts how many oscillations each piece of material experiences each second, or alternatively, how many peaks pass by an observer each second. The period P is as before the reciprocal of the frequency.
- the amplitude (A) describes the maximum deviation from equilibrium. This is easy to visualize in the example above where A refers to the maximum displacement rom equilibrium, but it can also refer to the maximum excess of pressure over average atmospheric as in sound, or the maximum electric field strength as in light, *et c*.
- the speed (discussed above).
- the wavelength (λ) measures the physical distance between corresponding points on adjacent waves (*e.g.*, from peak to peak).



There is a relationship among f, λ, and v, which we can deduce from the 'railcar analogy.'

Suppose that a train with cars of length L passes you at speed v. You count N cars in time t. The distance traveled by the train in that time is d = NL. The speed of the train is v = d/t = (NL)/t = (N/t)L. We recognize (N/t) as the frequency f (the number of peaks that pass *per* unit of time) and L as the analog of λ, so $v = f\lambda$.

Although we won't prove it, sinusoidal waves moving along the x-axis are described mathematically by the expression:

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

where φ is a *phase angle* that allows us to change the function to cosine or to some combination of sine and cosine. As in the last section, we'll let the phase angle be zero for simplicity.

Since we didn't derive this relationship, we'll at least show that it is correct and does what we need it to do. Let's examine each term one by one.

1) The shape of the curve is obviously sinusoidal.
2) The sine function varies between +1 and -1, and this function y(x, t) varies between +A and -A.
3) We've already addressed phi.
4) Suppose we freeze the wave at t = 0. Since we can assign the value of zero to any particular time we choose, this works for any instant. The equation reduces to

$$y(x,0) = A\ \sin\left(\frac{2\pi}{\lambda}x\right)\ .$$

Let's consider the location x = 0, then follow the curve though one cycle or one wavelength. The function then goes from

$$y = A\ \sin(0)\ \text{to}\ A\ \sin\left(\frac{2\pi}{\lambda}\lambda\right) = A\ \sin(2\pi)\ \ ,$$

which corresponds to one mathematical cycle, as desired.

5) Now, let's examine what happens at the particular location x = 0. Again, we can set the origin to be anywhere for our convenience. The equation reduces to

$$y(0,t) = A\ \sin(\mp 2\pi f t)\ .$$

We'll follow that bit of medium through one cycle or one period, $P = 1/f$. The function then goes from

$$y = A\ \sin(0)\ \text{to}\ y = A \sin(\mp 2\pi f P) = \ A \sin(\mp 2\pi)\ ,$$

which corresponds to one mathematical cycle, forward or backward as desired.

6) Lastly, we'll confirm that the negative sign between the terms corresponds to the wave moving to the +x-direction, and the positive sign to movement in the – x-direction. According to the equation,

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x - 2\pi f t\right)\ \ ,$$

there will be a peak in the curve when the argument of the sine function is equivalent to 90°. As time increases, the second term will become more positive, but there is negative sign which makes the argument more negative. In order to keep the argument at 90°, the x value must increase. That is, the position of the peak will move toward positive x. For the other case,

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x + 2\pi f t\right)\ \ ,$$

as time increases, the argument will become more positive, so to keep it constant at 90°, the x value must become more negative, that is, the position of the peak will move toward negative x.

So, it appears that this formula does everything we would expect of it. Note from Item 6 that it is the interplay between the spatial and temporal terms that accounts for the wave's motion.

Lets' see if this expression solves the wave equation developed above:

$$y(x,t) = A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

$$\frac{dy}{dx} = \frac{2\pi}{\lambda}A\ \cos\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

$$\frac{d^2y}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

$$\frac{dy}{dt} = -(\mp 2\pi f)A\ \cos\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

$$\frac{d^2y}{dt^2} = -(2\pi f)^2 A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)\ .$$

Then,

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = \frac{-(2\pi f)^2 A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)}{-\left(\frac{2\pi}{\lambda}\right)^2 A\ \sin\left(\frac{2\pi}{\lambda}x \mp 2\pi f t + \varphi\right)} = f^2\lambda^2 = v^2$$

Or, more usually,

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}\ .$$

Comparison of this equation with the result from the chain of oscillators tells us that the speed of a disturbance on the chain will be

$$v = \sqrt{\frac{L^2K}{M}} = D\sqrt{\frac{k}{m}}\ ,$$

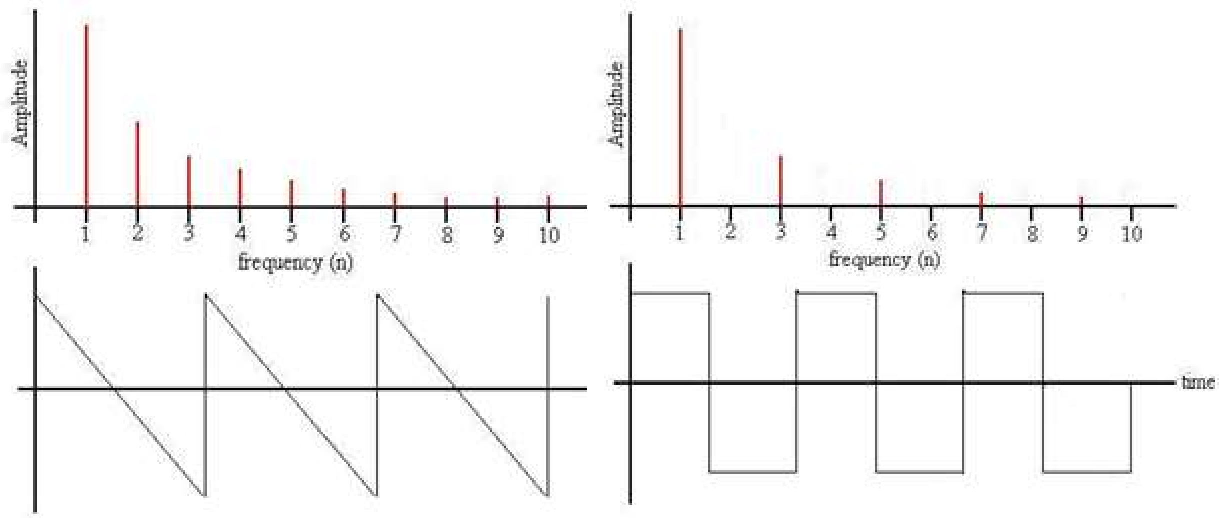
as guessed.

Non-Sinusoidal Waves*

We've been concentrating on sinusoidal waves. What about waves that are not sinusoidal? Waves that are repetitive in time can be approximated by adding up different amplitudes of many sinusoidal waves of different frequencies (*Fourier decomposition*):

$$y(t) = \sum_n A_n \sin(n2\pi f t) \quad n = 1, 2, 3, \ldots \ .$$

We can represent these amplitudes A_n graphically with a figure like this one, similar to what was shown on the screen of the *spectrum analyzer* above. For example, a 'ramp' wave is composed of waves with the form n = 1, 2, 3, … with A_n decreasing as 1/n.

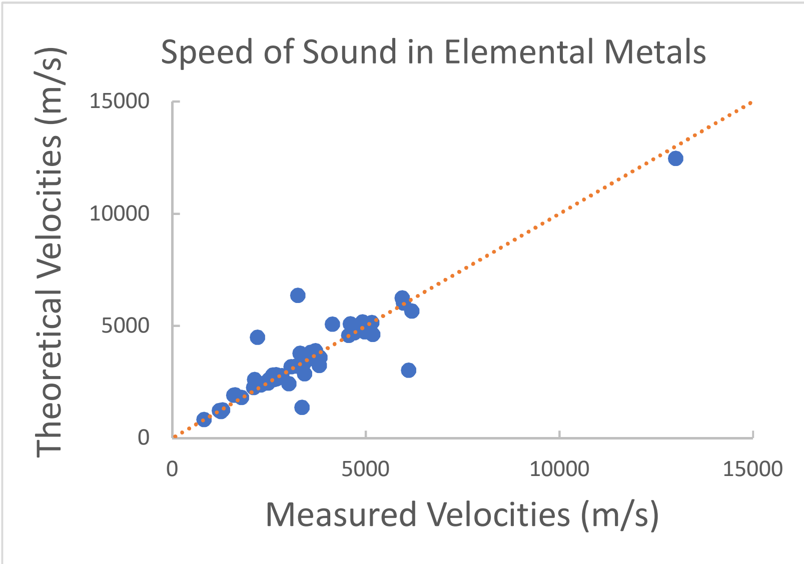


A 'square wave' is the sum of waves of the form n = 1, 3, 5, ... and the amplitude A_n decreasing as 1/n. A triangular wave is the sum of waves of the form n = 1, 3, 5, ... and the amplitude A_n decreasing as $1/n^2$.

While this discussion was based on a chain of masses, we can guess that the speed of a compression wave in a material should be proportional to the square root of the ratio of an elastic property to an inertial property:

$$v \sim \sqrt{\frac{\text{elastic property}}{\text{inertial property}}} \ .$$

EXAMPLE 11-X

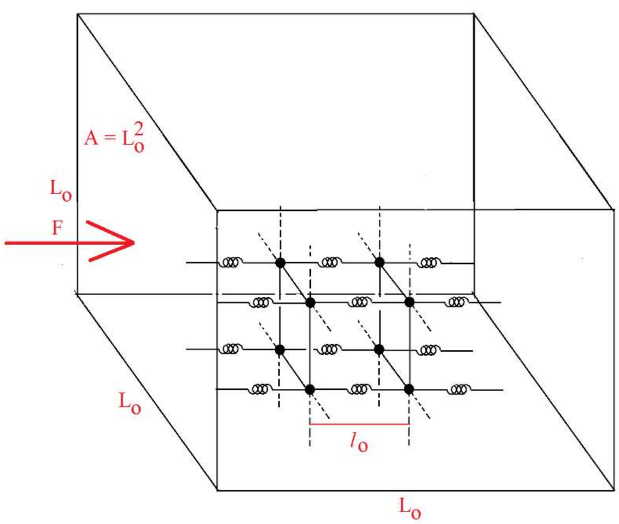


For example, the speed of a sound pulse in a metal can be shown experimentally to be

$$v = \sqrt{\frac{Y}{D}},$$

where Y is the *Young's modulus*, a measure of the springiness of the material, and D is the mass density of the material (inertial property). In the figure, the dotted line (which is not a best fit line) shows where the measured value and the theoretical values would be equal.

JUSTIFICATION 12-1*



The density and the Young's modulus are both macroscopic quantities. Let's see if we can correlate this result to microscopic quantities. From chemistry, we think that the material can be modelled by small balls of mass m (representing the atoms) connected by bonds represented by springs with stiffness k, and relaxed length l_o. To make things a little easier, we'll assume that the metal's atoms lie on the corners of cubes, as shown in the figure.[2] I omitted the 'springs' in two directions since the oscillations will occur in the left-right direction and we will assume that the plane of atoms will move as a single unit. Each pair of adjacent planes of atoms are connected by a large number of 'bond' springs.

For simplicity, let's say that the object is a cube of edge L_o and mass M. The cross-sectional area of the left end will be $A = L_o^2$. Let's apply forces F along the length of the block, as shown. As a result, the block will contract along that axis:

[2] Such an arrangement is called a *simple cubic structure*. There is unfortunately only one elemental metal that has this structure, polonium, and that is highly radioactive and doesn't stick around very long before decaying into other elements.

$$\frac{F}{A} = Y\,\frac{\Delta L}{L_o}.$$

Here, F/A is the *stress* and $\Delta L/L_o$ is the *strain* (cause and effect).

If the cube has mass M and the mass of each atom is m, then there are N = M/m atoms. There will be $N^{1/3}$ atoms along each edge each separated from its nearest neighbor by distance l_o, so

$$L_o = N^{1/3} l_o \ .$$

We might also assert that the compression of the bond length is in the same proportion to the compression of the entire length of the cube:

$$\frac{\Delta l}{l_o} = \frac{\Delta L}{L_o} \ .$$

Similarly, the area of the left end, and therefor of each of our planes, is

$$A = L_0^2 = \left(N^{1/3} l_o\right)^2 .$$

The number of atoms in such a plane is $(N^{1/3})^2$, and so the mass of such a plane will be

$$M_{PLANE} = N^{2/3} m \ .$$

Each such plane has $N^{2/3}$ springs connecting it to the adjacent plane. We saw in a Section 12 that the effective spring constant of 'parallel' springs is the sum of the spring constants, k:

$$k_{PLANE} = N^{2/3} k \ .$$

So, as we push on the left end with force F, Hooke's relationship says that we should see that

$$F = k_{PLANE} \Delta l \ .$$

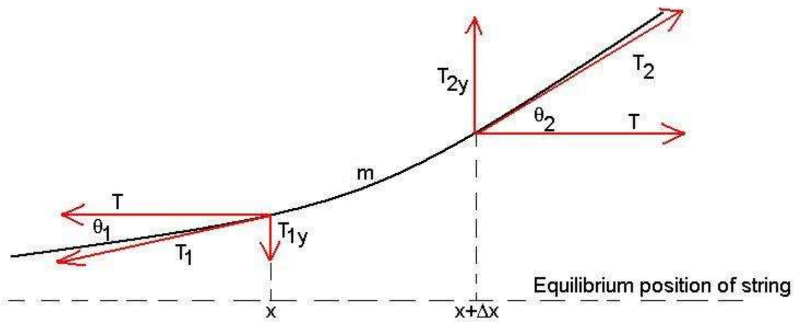
Finally, of course, the density D is the mass divided by the volume, $D = M/L_o^3 = m/l_o^3$. So, here we go.

$$v = \sqrt{\frac{Y}{D}} = \sqrt{\frac{\left(\frac{F/A}{\Delta L/L_o}\right)}{\left(\frac{M}{L_o^3}\right)}} = \sqrt{\frac{\left(\frac{F/A}{\Delta l/l_o}\right)}{\left(\frac{m}{l_o^3}\right)}} = \sqrt{\frac{F}{\Delta l}\frac{1}{N^{2/3}}\frac{l_o^2}{m}} = \sqrt{\frac{k_{PLANE}}{N^{2/3}}\frac{l_o^2}{m}} = l_o\sqrt{\frac{k}{m}} \ .$$

The speed of sound in a fluid (like air) also follows this form: $v = [B/D]^{1/2}$, where B is the *bulk modulus*, a measure of the elastic properties of a fluid.

EXAMPLE 11-X

Find the speed of a transverse wave on a string of mass M and length L.



Consider a small piece of the string with length Δx and mass m. The forces acting on m are shown in the figure above. Each end of m is pulled by the adjacent piece of string along a line tangent to the string at the end (T_1 and T_2). Each of these forces can be broken up into components; since we do not expect any motion in the horizontal direction, those components (T) should cancel, and individually should be equal to the tension in the string. The vertical components will act to accelerate m. Let's find them:

$$\frac{T_{1y}}{T} = \tan\theta_1 \qquad \frac{T_{2y}}{T} = \tan\theta_2 \ .$$

Let's remember that the tangents of these angles are the slopes of the curves at those points, so

$$T_{1y} = T\left.\frac{\partial y}{\partial x}\right|_x \qquad T_{2y} = T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} \ .$$

Use NII:

$$T_{2y} - T_{1y} = ma$$

$$T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T\left.\frac{\partial y}{\partial x}\right|_x = m\frac{d^2y}{dt^2}$$

$$\frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x}{\Delta x} = \frac{m}{T\,\Delta x}\frac{\partial^2 y}{\partial t^2} \ .$$

Note that m/Δx is the linear mass density, λ. To avoid confusion in this section, however, we will use μ = m/Δx.

$$\frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x}{\Delta x} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} \ .$$

Take the limit of Δx→0. We've already seen an example of this; the left side becomes the second derivative of y w.r.t. x.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} \quad \rightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} \quad .$$

Comparison with the generic wave equation developed above then indicates that the speed of this wave along the string is $v = [T/\mu]^{1/2}$.

Reflections

We next looked at reflections of pulses on a string. We noted that a pulse traveling down the string is reflected with the same orientation if the end of the string is free to move, and reflected with an inverted orientation if the end of the string is fixed. Although this could be proven mathematically, we based our assertion on experiment. We can visualize what's happening, however, by imagining that the string continues beyond its actual end, and that an imaginary wave is traveling back down the string towards the end from the imaginary side. We invoke the *principle of superposition*, the notion that the total displacement of the medium is the sum of the individual displacements due to each pulse; in this way, we know that the reflected pulse for a fixed end must be inverted, since this is the only way the total displacement at the end of the string can always be zero, and then clearly, the wave and its reflection must add together in the case of a free end. Here is an animation illustrating the two types of reflections/ INSERT

Now, instead of considering the two extreme cases (completely fixed or completely free ends), think about what would happen if the end of a string were tied end to end to another string. In all cases, we would expect that some of the wave would continue down the second string with the same orientation (and frequency) as the original wave; this is called the *transmitted wave*. The argument we can give is that the end of the first string does the same thing to the second string as the hand or other agent did to the first string at the other end. We also notice that there is a *reflected wave*, the orientation (and size) of which depends on whether (and by how much) the second string is 'heavier' or 'lighter' than the first. Watch this demonstration video. INSERT

DISCUSSION 12-1

> What do you notice about the size of the pulses that are transmitted and reflected? Why does this happen?

The quantity used to measure the difficulty of a wave in passing through some medium is called the *impedance*, Z. If $Z_2 > Z_1$, the reflected wave is inverted; if $Z_2 < Z_1$, the reflected wave is upright. This is a general result, even though the exact values of the impedances are calculated in different ways for different media. In the specific example of transverse waves on a string, we have that (asserted without proof)

$$Z = \sqrt{T\mu} \ .$$

In that case, we see that our original examples correspond to $Z_2 = 0$ (end of string loose, so $\mu_2 = 0$) and Z_2 = infinity (end of string tied to wall, so μ_2 = infinity). The impedances also tell us how much energy is reflected and how much is reflected.

DISCUSSION 12-2

> What would happen if the two media had the same impedance? Are there occasions when we would want the incoming wave to be totally reflected? Are there occasions when we would want the incoming wave to be completely transmitted? Suppose you are walking across campus and see a friend on the other side of the quad. How would you get his attention (do not say 'text him')? If he didn't hear you, what might you do with your hands? Why does this work?

Abrupt changes in impedances between two media results in strong reflection. To minimize reflections, the transition from one material to the other should be made as gradual as possible. Cupping your hands around your mouth makes the transition from a tube of several centimeters diameter to the wide-open atmosphere smoother, resulting in more transmission of sound. Coxswains of racing shells often use megaphones, a hollow truncated cone used to amplify the voice. Some of you may remember the days of rabbit ear antennas for your televisions. The impedance of the ribbon cable is 300 *ohms*,[3] which matches that of the screw terminals on the back of the TV. When cable came along, people with older TVs had to buy matching transformers to convert the cable's 75 ohm impedance to the TV's 300 ohms. Failure so to do resulted in multiple reflections, appearing as 'ghosts' on the screen.

Standing Waves

The *principle of superposition* states that, if more than one wave is passing through a given point, the total displacement is the sum of the displacements due to each individual wave. Suppose that we set up two sinusoidal waves in a (one dimensional) medium which are identical in every way except their directions. A specific example would be two waves moving in opposite directions along a very long string, each end being jiggled at the same amplitude and frequency. What happens when these waves meet? We use the principle of superposition to find the result by adding the two individual waves. We saw that the resulting wave did not appear to travel at all; this type of wave is called a *standing wave*, although *stationary wave* might be more apt. Let's examine this more mathematically.

DERIVATION 12-2

$$y_1 = A\sin\left(\frac{2\pi}{\lambda}x - 2\pi f t\right) \quad \text{(moves to the right)}, \quad \text{and}$$

$$y_2 = A\sin\left(\frac{2\pi}{\lambda}x + 2\pi f t\right) \quad \text{(moves to the left)} \ .$$

We'll expand these out using the trig identity

$$\sin(\alpha \pm \beta) = \sin\alpha \, \cos\beta \, \pm \, \cos\alpha \, \sin\beta \ .$$

[3] An ohm is a unit for electrical impedance.

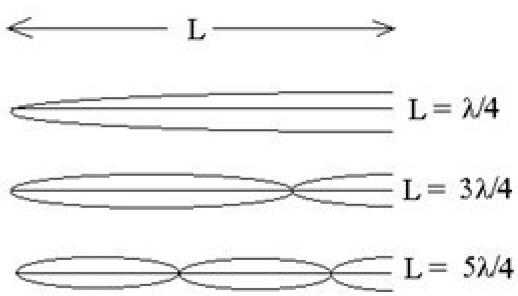
$$
\begin{aligned}
y_{\text{TOTAL}} = y_1 + y_2 \\
&= A\left[\sin\left(\tfrac{2\pi}{\lambda}x\right)\cos(2\pi f t) - \cos\left(\tfrac{2\pi}{\lambda}x\right)\sin(2\pi f t)\right] \\
&+ A\left[\sin\left(\tfrac{2\pi}{\lambda}x\right)\cos(2\pi f t) + \cos\left(\tfrac{2\pi}{\lambda}x\right)\sin(2\pi f t)\right] = 2A\sin\left(\tfrac{2\pi}{\lambda}x\right)\cos(2\pi f t) \ .
\end{aligned}
$$

Here, we see that, in contrast to the original waves which had mixed spatial and temporal components, this wave has separate spatial and temporal components. Each piece of the medium undergoes SHM, but with an amplitude that depends on its x-position.

We can see that there are spots that never oscillate (*nodes*) and spots that have maximum oscillation (*anti-nodes*). Each type of location is separated from its adjacent neighbor by one-half of the wavelength.

This little derivation assumes that the strings are very long. In the real world, string (or other similar systems) have a finite length. In such systems, there will be many reflections from each end which will have to be added to determine the overall behavior of the string.

We started with a string fixed at each end, and we excited waves with different frequencies. In some cases, we noted that the waves all added up to a random pattern, eventually canceling out. In other cases, we saw that the initial and reflected waved added to produce a standing wave. What conditions need to be met to do this? We could do a very mathematical derivation of this, but it is just as correct to base our investigation on observation.

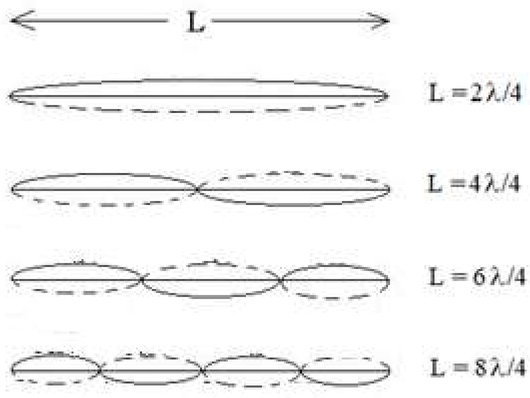


System 'fixed' at one end and 'free' at the other - We saw a series of patterns like those shown in the figure as we increased the frequency at which the system was excited. The lines indicate the limits of the oscillations of the string (the envelope). The fixed end must always correspond to a node (no motion) and the free end, where the string can move the most, corresponds to an antinode. As we increase the frequency (or shorten the wavelength), we must always add in an additional node and an additional antinode to fulfill the conditions above, thereby adding two quarter wavelengths. We notice that the length of the system must be an odd natural number multiple of a quarter wavelength, or

$$L = \frac{n\lambda}{4} \ , \qquad n = 1, 3, 5, 7, \dots \ .$$

Since we already know that $v = f\lambda$, we can solve for the allowed frequencies for these stationary waves:

$$f_n = \frac{nv}{4L} \quad n = 1, 3, 5, 7, \dots \ .$$



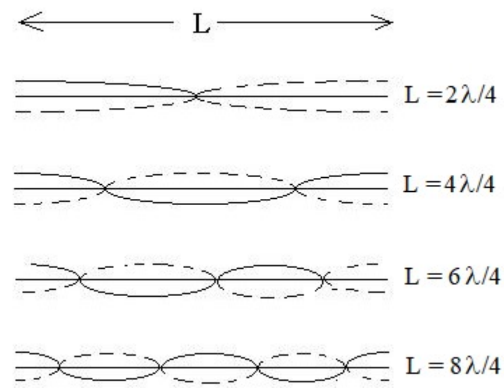
System 'fixed' at both ends - We saw a series of patterns like those shown at left as we increased the frequency at which the system was excited. Here, we see that the length is an even natural number multiple of a quarter wavelength:

$$L = \frac{n\lambda}{4}\ , \qquad n = 2, 4, 6, 8, \ldots\ ,$$

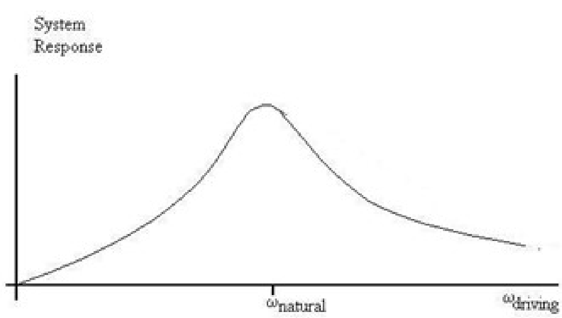
and so,

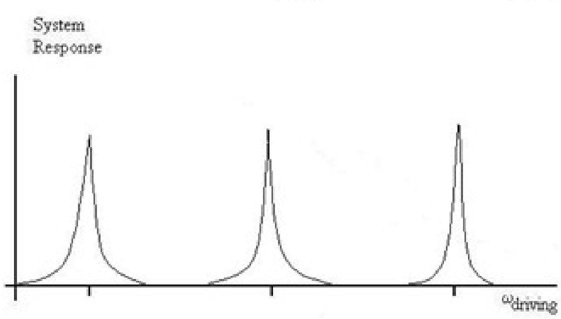
$$f_n = \frac{nv}{4L} \qquad n = 2, 4, 6, 8, \ldots\ .$$

Finally, when both ends are 'free,' we see that the result is the same as for both ends fixed but with the nodes and anti-nodes reversed.



So, now we have an easy-ish relationship and an easy way to remember which numerical values to insert: if the ends are of the same type, or 'even,' use the even values and if the ends are different, or 'odd,' use the odd values. We see that, unlike for a single oscillator with a single natural frequency, we here have a system with many natural frequencies:





Remember that, even though we derived these results for transverse waves on a string, the results are valid for other system. For example, consider a stopped organ pipe, which means that it is open at one end and closed off at the other. At the open end, air is free to vibrate, while at the closed end, no vibration is possible because of the stopper. This pipe will support standing waves of the form $f_n = nv/4L$, n = 1, 3, 5, ... How are these frequencies produced? In an organ, air is pumped into the pipe against a sharp edge, which produces all frequencies. However, those frequencies which do not correspond to the favored frequencies reflect back and forth in the pipe and, on average, cancel themselves. But the few special frequencies re-inforce one another and produce standing waves. These frequencies are often referred to as the *harmonics* of the system. On occasion, they are referred to as the *fundamental* (n = 1) and the *overtones* (n > 1).

EXERCISE 12-1

The human ear canal is about 3 cm long. If it is regarded as a tube open at one end and closed at the other, what is the fundamental frequency of a standing wave in the ear?? Let v_{sound} = 340 m/s. Since these frequencies are re-inforced in the ear canal, human hearing is just a little bit better at this frequency that at others.

HOMEWORK 12-1

A tuning fork is sounded above a (narrow) resonating tube as in lab. The first resonant situation occurs when the water level is 0.08 m from the top of the tube, and the second when the level is at 0.24 m from the top. Let v_{sound} = 340 m/s.
a) Where is the water level for the third resonant point?
b) What is the frequency of the tuning fork?

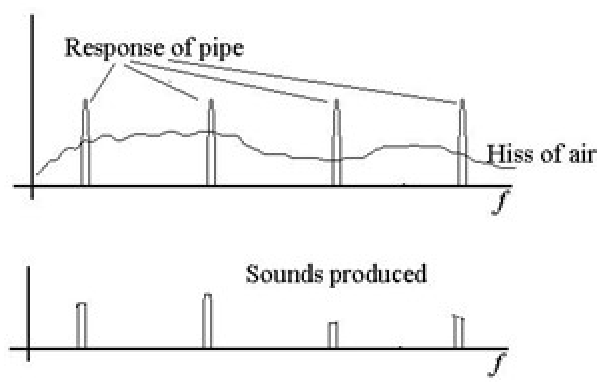
HOMEWORK 12-2

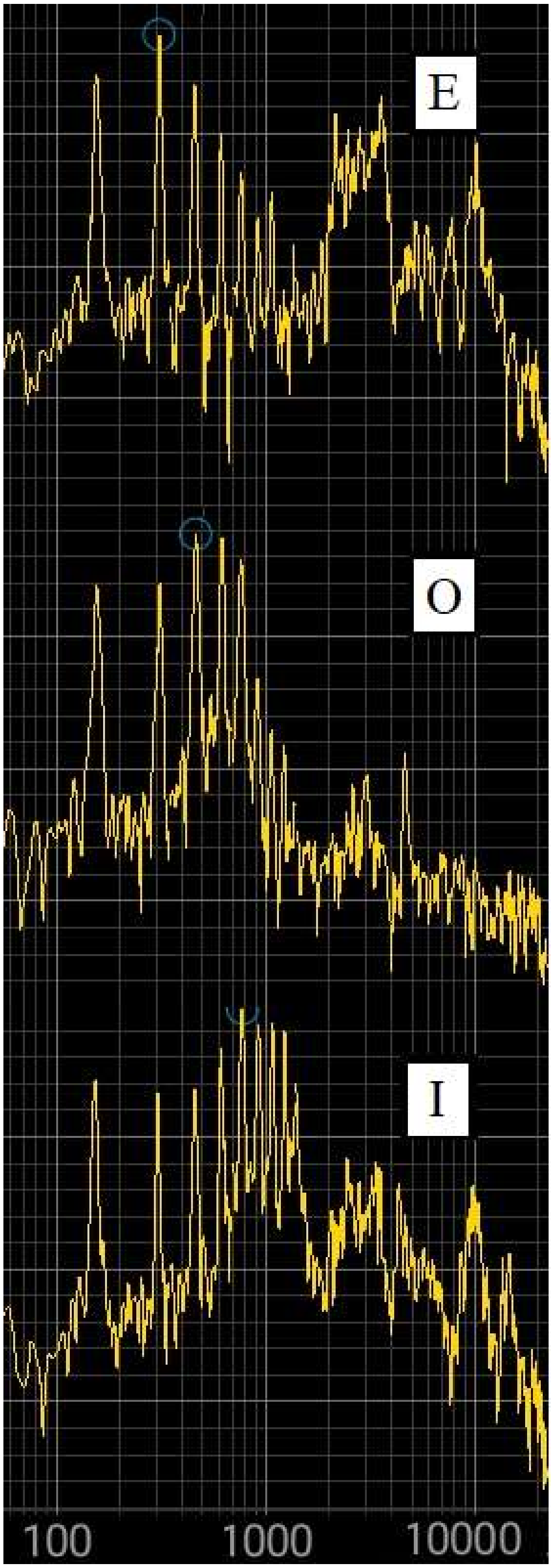
A 2 m long air column is open at both ends. The frequency of a certain harmonic is 400 Hz, and the frequency of the next higher harmonic is 480 Hz. Find the speed of sound in the air column.

DISCUSSION 12-3

Now here's a question. How can a listener distinguish different musical instruments that are playing the same note?

For example, an oboe and a clarinet are both essentially cylindrical tubes, closed at one end and open at the other, and so they produce the same sequence of harmonics, $f_n = nv/4L$. The answer is that, because of the exact shape of the bore, they each put slightly different amounts of energy into the different harmonics, and it is that distribution that your brain remembers and labels as one instrument or the other. The same goes for vowels in speech.





In the figure, you can see the *spectra* of three long vowels as pronounced by an Upstate New Yorker. The axes are logarithmic, with frequency along the horizontal axis and the strength of each frequency along the vertical axis. The actual frequencies produced are the same in each case, but the strengths of each are very different for the three sounds.

Also, it is possible to suppress certain harmonics. Those of you who play guitar know that plucking the string in different spots produced different sounds on the same fundamental. Plucking a string at its center will deliver more energy into frequencies with an anti-node there, *i.e.* n = 4, 8, 12, 16, *et c*. Touching a string lightly one third of the way from one end will suppress any frequencies with an anti-node there.

Intensity

A wave can be defined as a transfer of energy without a net movement of matter. For example, if I send a pulse down a string, the other end of the string can exert a force on some object to which it is tied, and possibly do work. For sound (and later, light), we measure the rate of energy transfer *per* unit area as the *intensity*, I, with the corresponding units of watts/m^2.

Consider a fire siren which broadcasts *isotropically* P joules of sound energy *per* second. Draw an imaginary sphere of radius R with the center at the siren; all the energy must eventually pass through that sphere, and the intensity will be

$$I = \frac{P}{4\pi R^2} \ .$$

We see that if we make the sphere larger, the energy will be distributed over a larger area, and the intensity will be reduced (that is, each square meter of area will receive less energy). This *$1/r^2$ dependence* is fairly common, and we shall see it again.

EXAMPLE 12-1

At noon, you can hear two towns' sirens going off. If Siren A has twelve times the intensity of Siren B, what is the ratio of their distances from you?

So, if we assume that the sirens are identical, the power outputs should be the same:

$$P_A = P_B$$

$$I_A 4\pi r_A^2 = I_B 4\pi r_B^2$$

$$\frac{r_A}{r_B} = \sqrt{\frac{I_B}{I_A}} = \sqrt{\frac{I_B}{12 I_B}} = \sqrt{\frac{1}{12}} = 0.29$$

EXERCISE 12-2

If Star A and Star B appear equally bright to your eye, but you know that Star B is twenty times further away from earth than is Star A, what is the ratio of the powers radiated by each?

HOMEWORK 12-3

At the fireworks show, a particularly loud explosion occurred 100m directly above Susique's head. Hank is 50 meters from Susique and Earl is 100 meters from Susique. Compare the intensities heard by each.

An alternate way of expressing intensity is in units of *decibel*s. The decibel scale is logarithmic, and thus follows more closely the actual size of the signal sent from human ear to human brain. The bel is named for Alexander Graham Bell, who was not, as one might suppose, American, but rather a Scot-born Canadian working in Boston. A reference intensity I_o is defined as 10^{-12} W/m^2, which corresponds roughly to the quietest sound a normal human can hear. The intensity to be converted is compared to this standard, and the log base ten is taken of the ratio. This gives the number of *bels*, so the number of decibels (dB) must be ten times more:

$$\beta = 10 \log_{10} \frac{I}{I_o} \ .$$

EXAMPLE 12-2

Suppose that the sound that one professor produces from the front of the classroom has an intensity of 10^{-7} W/m^2. How many dB does this correspond to?

$$\beta = 10 \log_{10} \frac{10^{-7}}{10^{-12}} = 10 \log_{10} 10^5 = 10\,(5) = 50 \text{ dB} \ .$$

EXERCISE 12-3

Now, suppose there are ten professors at the front of the room, talking incoherently.[4] How many decibels would ten professors produce? What about one hundred?

Note that this is not a linear relationship. A multiplicative factor of ten in intensity is an additive increase of ten in decibels.

Here are the intensity levels of some common situations:

Situation	Intensity	Intensity level
Threshold of Hearing	10^{-12} W/m^2	0 dB
Library Reading Room	10^{-9} W/m^2	30 dB
Conversation	10^{-6} W/m^2	60 dB
Vacuum Cleaner	10^{-3} W/m^2	90 dB
Rock Concert	10^{-1} W/m^2	110 dB
Thunder	10 W/m^2	130 dB

Of course, these values depend on the distance between source and listener. Prolonged exposure to sounds above 90 dB will cause permanent damage, and exposure to sounds over 110dB will be painful. Here are some helpful hints: ALWAYS wear ear protection in noisy situations, such as lawn mowing or vacuuming and on up. If you must wear headphones to listen to music, place them just in front of your ears, not right over them.

HOMEWORK 12-4

After Route 100 was built through Columbia, some homes experienced an average decibel level of 110 dB, caused by 200 cars passing *per* minute. When the inspector arrived to test this, there were only 25 cars passing *per* minute. What average decibel level did the inspector measure?

HOMEWORK 12-5

Five identical machines operating in a factory produce an average sound intensity level of 87 dB. If three additional machines are put online in the same location, what will the new average sound intensity level be?

Beats

DERIVATION 12-3

[4] No jokes, please. Here, incoherent means that there are no particular relationships among the sounds produced by N profs, in which case the intensity is simply N times the intensity die to one prof. If they were all chanting or reciting together, the sounds would be coherent and the result more complicated.

Suppose that we have two nearly identical waves passing through a spot in space (call it the origin so x = 0), so that the time dependences (we'll ignore the spatial dependence) are given by

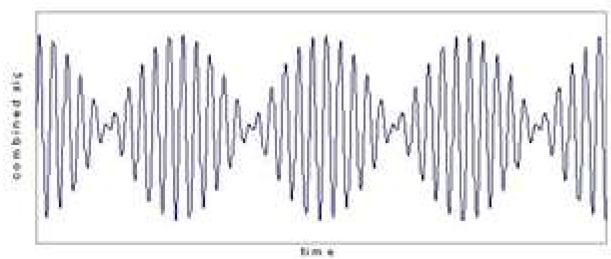
$$y_1 = A\sin(2\pi f_1 t) \quad \text{and} \quad y_2 = A\sin(2\pi f_2 t) \ .$$

Using the principle of superposition, we get the total 'displacement' from equilibrium from the sum of these two expressions:

$$y_{TOTAL} = A\sin(2\pi f_1 t) \ + \ A\sin(2\pi f_2 t) \ .$$

Now, we'll make use of a trig identity, sin a + sin b = 2 sin[(a+b)/2] cos[(a-b)/2], so that

$$y_{TOTAL} = 2A\cos\left(2\pi\left(\frac{f_1 - f_2}{2}\right)t\right)\ \sin\left(2\pi\left(\frac{f_1 + f_2}{2}\right)t\right) \ .$$



From this we see that the frequency of oscillation is the average of the two original frequencies, but also that the amplitude of the oscillation is modulated by an envelope with a frequency equal to the difference of the original frequencies, $|f_1 - f_2|$. The formula says that the frequency of the envelop is half the difference in the frequencies, but there are two pulses, or beats, *per* cycle.

This can be (and often is) used as a method for tuning pianos and other such instruments. Once a 'C' string is tuned to the correct pitch, it and the 'G' a fifth above it above are struck simultaneously. The third harmonic of the C and the second harmonic of the G are the same note, and so the G string's tension is adjusted until no beats are heard between those two harmonics (or in other tuning schemes, a certain number of beats *per* second should be heard, but that's a whole 'nother story...).

Beating is also used in some radar guns. An example will follow in the next section.

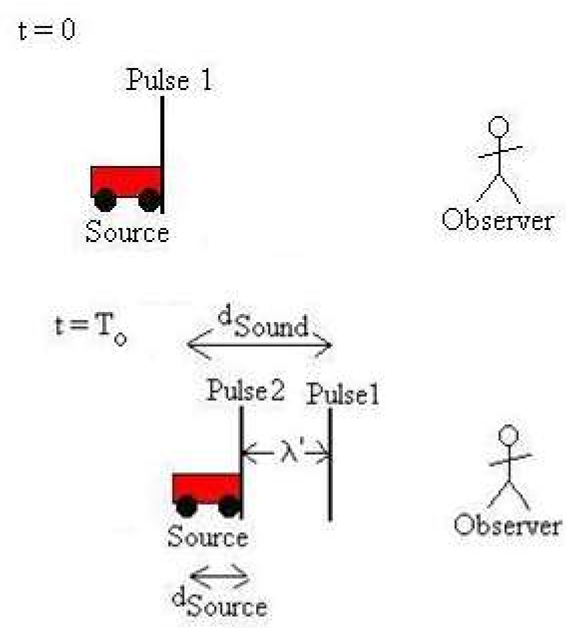
HOMEWORK 12-6

> Consider two identical strings under the same tension but with different lengths. The n = 2 harmonic of the first string (400 Hz) beats at 6 Hz with the n = 2 harmonic of the longer string. What is the difference in the lengths of the strings if the speed of transverse waves on these strings is 120 m/s?

The Doppler Effect

You're probably familiar with this effect: a car or train passes you while blowing its horn, so that the pitch of the sound rises while the vehicle is moving toward you, but sounds lower when the vehicle is moving away from you. We shall look at a couple of special cases, and then integrate the results for all such cases into a single relationship. Note, however, that the results will only be true if there is no wind, that is, the medium (usually air or water) is stationary. Also, our derivations will be done for a one dimensional universe.

DERIVATION 12-4



Consider a source moving at speed v_{Source} toward (*approaching*) a stationary observer (or listener, if you insist). Instead of having the source emit a sinusoidal wave, let's assume that it emits pulses; we can later correlate these pulses to the peaks of a sinusoidal wave, if necessary. Let the frequency of the pulse emitted by the source be f_o, and the time between the emission of pulses be $T_o = 1/f_o$. Here at t = 0, the source releases a pulse, which then travels to the right at speed v_{Sound}. Now, let's look at the locations of everything a time t equal to one period later. Pulse 1 has traveled a distance $d_{Sound} = v_{Sound} T_o$, while the source has traveled a distance $d_{Source} = v_{Source} T_o$, at which point it emits Pulse 2. Now, the wavelength that the observer will measure is the distance between the two pulses:

$$\lambda' = d_{Sound} - d_{Source} = v_{Sound}T_o - v_{Source}T_o \quad .$$

Now, T is the period between pulses as 'heard' by the source, and so we remember that $f = 1/T$ and that $f\lambda = v_{wave}$, and so this last expression can be re-written as

$$\frac{v_{Sound}}{f'} = \frac{v_{Sound}}{f_o} - \frac{v_{Source}}{f_o} \quad ,$$

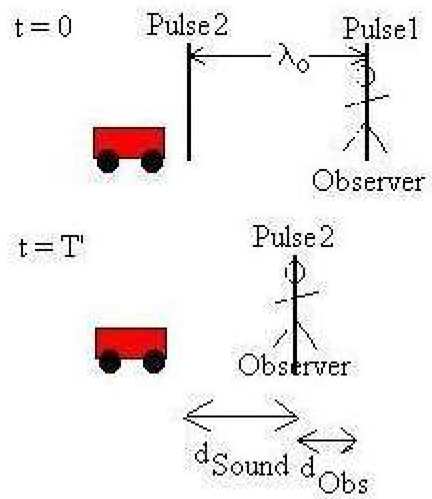
$$f' = f_o \frac{v_{Sound}}{v_{Sound} - v_{Source}} \quad .$$

Now, if the source had instead been moving the other way (*receding*), those two distances would have had to have been added, changing the minus sign to a plus sign:

$$f' = f_o \frac{v_{Sound}}{v_{Sound} + v_{Source}} \quad .$$

Now, suppose instead that it were the observer moving toward (approaching) the stationary source at speed $v_{Observer}$. Once again, let the source emit pulses at an interval of T_o. Let t = 0 when Pulse

1 arrives at the observer. The second pulse arrives at the observer at time T', during which interval the pulse has traveled distance (to the right) $d_{Sound} = v_{Sound}\,T'$ and the observer has traveled distance (to the left) $d_{Observer} = v_{Observer}\,T'$. T' is now the time between pulses, as heard by the observer.



The sum of these distances is the original wavelength as produced by the source, λ_o:

$$\lambda_o = d_{Sound} + d_{Observer} = v_{Sound}T' + v_{Source}T' \quad .$$

Remembering that $f' = 1/P'$ and that $f'\lambda' = v_{wave}$, we substitute to obtain

$$\frac{v_{Sound}}{f_o} = \frac{v_{Sound}}{f'} + \frac{v_{Observer}}{f'} \quad ,$$

$$f' = f_o\,\frac{v_{Sound} + v_{Observer}}{v_{Sound}} \quad .$$

Once again, if the observer had been receding from the source, there would have been a sign reversal to

$$f' = f_o\,\frac{v_{Sound} - v_{Observer}}{v_{Sound}} \quad .$$

Now, we can combine all these relationships, if we're careful. First, we need to define better the terms 'approach' and 'recede.' 'Approach' is to head in the direction of the other object, <u>regardless</u> of whether the distance between the objects is becoming smaller or not, and 'recede' means to head in the opposite direction of the other object, whether the distance between is increasing or not. Then,

$$f' = f_o\,\frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} \quad ,$$ [5]

where the upper sign is used if that object is approaching and the lower sign if that object is receding.

DISCUSSION 12-3

What exactly would one do if there were wind?

EXAMPLE 12-3

[5] For light, the result is somewhat different. See Notes for PHYS 3.

A police car moving at 150 kph is chasing a speeder traveling at 100 kph on a calm day. If the cop's siren has a frequency of 500 Hz, what frequency will the speeder hear?

First, what is the speed of sound on a calm day? We typically use 340 m/s as a default value. The cop is approaching the speeder, while the speeder is receding form the cop. Remember that these terms have nothing to do with the distance between them increasing or decreasing. Converting the speeds to meters *per* second, we get

$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} = 500 \frac{340 - 27.8}{340 - 41.7} = 523.3 \text{ Hz} .$$

AN ADMONITION

It's tempting to try to cut a corner and use relative velocities. Bad idea. Let's try it. Suppose the speeder is motionless and the cop is approaching him at 50 kph.

$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} = 500 \frac{340 - 0}{340 - 13.9} = 521.3 \text{ Hz} .$$

These values are close, but the second is wrong because we left out the fact that there is now a 100 kph wind blowing against the motion of the cop. The speed of sound is therefore not 340 m/s but rather 312.2 m/s. Let's try again.

$$f' = f_o \frac{v_{Sound} \pm v_{Observer}}{v_{Sound} \mp v_{Source}} = 500 \frac{3312.2 - 0}{312.2 - 13.9} = 523.3 \text{ Hz} .$$

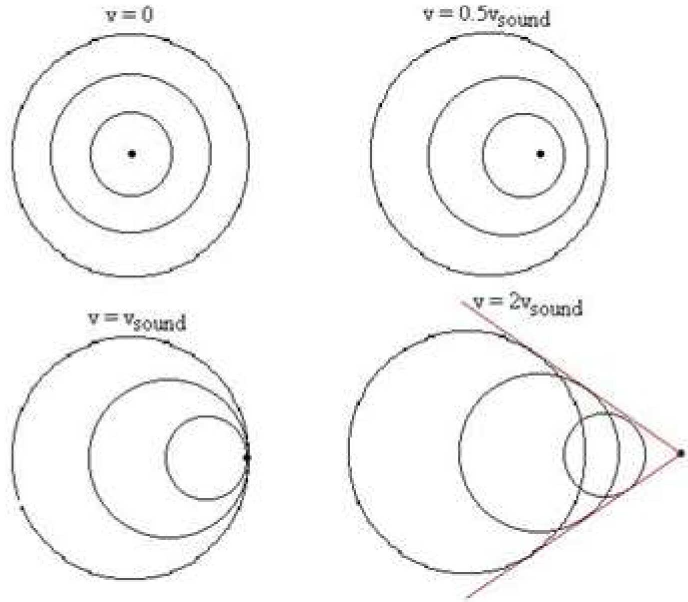
HOMEWORK 12-7

If you move at 15 m/s (relative to the ground) toward a sound source (3000 Hz) which is also moving toward you at 45 m/s (relative to the ground), what frequency will you hear from the object? Assume that v_{sound} = 340 m/s.

DISCUSSION 12-4

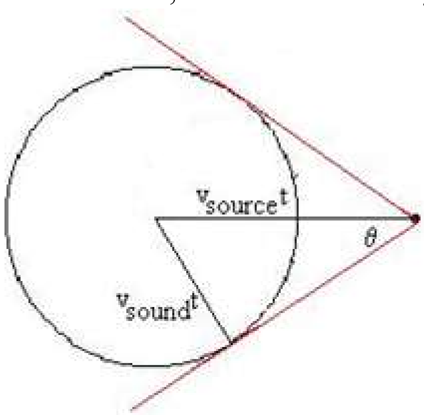
What happens if the source travels more quickly than sound? Consider the denominator of the Doppler relationship we derived.

The upper left figure shows the locations of the crests of three pulses emitted by a source such as a jet while it is stationary. The pulses move out in a spherical form centered on the spot at which they were produced. The upper right figure shows the same when the source is moving to the right at about 0.5 the speed of sound; note that the wavelengths will be shorter for listeners in the path of the source, but longer for listeners from which the source is receding. When the source reaches the speed of the wave in that medium, a *bow shock wave* is generated; this is most easily seen when generated by a boat, but recent photos of jets breaking the 'sound barrier' have caught these shock waves as they condensed water gas in the air. Once the speed of the source exceeds the speed of the wave in that medium, the crests of all waves co-incide to produce a giant shock wave (red line); for jets, this results in the familiar *sonic boom*. It can also be seen as the bow wave of a boat in water.



Some random notes:

1) The reason the Concorde was so quiet (to the passengers anyway, not to those living on the flight path) during supersonic flight is that the noise from the engines could not keep up with the cabin; one could hear only the noise transmitted through the body of the plane.



2) The apex angle (θ) of the cone formed by the shockwave depends on the speed of the source. The wave shown was generated at the instant that the source was at its center. In time interval t, the source moved to the right a distance $v_{source}t$ and the sound moved outward a distance $v_{sound}t$. Consider the right triangle formed in the diagram. We see that

$$\sin\theta = \frac{v_{Sound}t}{v_{Source}t} = \frac{v_{Sound}}{v_{Source}} \quad .$$

The inverse of this ratio is referred to as the *Mach number*. The official speed record of any jet aircraft is about Mach 9.6, set by the unmanned X43A. The Space Shuttle (when we had a set) entered earth's atmosphere at about Mach 25. Be aware though that the Mach speed is NOT just a multiple of the speed of sound at sea level (340 m/s); the speed of sound varies with altitude.

EXERCISE 12-1 Solution

Since the two ends are different, or 'odd,'

$$f_n = \frac{nv}{4L} \quad n = 1, 3, 5, 7, \ldots \ .$$

$$f_1 = \frac{1 \times 340}{4 \times 0.03} = 2833 \text{ Hz} \ .$$

Let's just keep increasing n until we exceed the limit of human hearing at 20,000 Hz:

8500 Hz, 14,167 Hz, 19, 833 Hz.

EXERCISE 12-2 Solution

The same brightness means the same intensity as seen from earth. So,

$$I = \frac{P_A}{\pi r_A^2} = \frac{P_B}{\pi r_B^2}$$

$$\frac{P_A}{P_B} = \frac{r_A^2}{r_B^2} = \frac{r_A^2}{(20r_A)^2} = 2.5 \times 10^{-3} \ .$$

Or the other way round, Star B produces 400 times more energy each second than does Star A.

EXERCISE 12-3

Assume that the sources are incoherent, ten professors would have an intensity of 10×10^{-7} W/m^2. Then,

$$\beta_{10} = 10 \ \log_{10} \frac{10 \times 10^{-7}}{10^{-12}} = 10 \ \log_{10} 10^6 = 10 \ (6) \ = 60 \text{ dB}$$

and

$$\beta_{100} = 10 \ \log_{10} \frac{100 \times 10^{-7}}{10^{-12}} = 10 \ \log_{10} 10^7 = 10 \ (7) \ = 70 \text{ dB} \ .$$

As was stated, a factor of ten in intensity I is an addition of 10 in intensity level beta.