Section 13 – A Synthesis Topic

As always, our discussions here will be greatly simplified.

DISCUSSION 13-1

> You're quite a ways into the course now, so let me ask you a question. Suppose you have two theories. Each is capable of making predictions that can be tested against measurements in the real world. Theory A's predictions are off by a bit, but Theory B's are off by twice as much. Which would you accept as the 'correct' theory?

Congratulations! You just decided that the sun orbits the earth! The notion that the earth orbits the sun can be traced back to at least the second century B.C. It was discounted for a number of reasons, but one argument was the lack of an observable *stellar parallax* that argued against the earth's motion. As we know now, the parallax is too small to have been measured with the instruments of the time. However, it was known that the motions of the other planets were not circular, which eventually led to the geo-centric Ptolemaic model, which you may remember required each planet to move along an epicycle while moving around the sun. Copernicus's model asserted that the planets, including the earth, orbit the sun in circles, but the error of this model in predicting the positions of the planets was somewhat greater than that of the Ptolemaic model. It was left to Kepler to suggest that the planets orbit the sun in elliptical orbits.[1]

We're going to go through these concepts in an ahistorical order. We will assume that the planets each orbit the sun in a circular orbit, unless specified otherwise. Mercury and Pluto have very high eccentricities, for different reasons; of the other seven, the worst is Mars at just under 0.1.

What do We Know about the Earth?

DISCUSSION 13-2

> What shape is the earth? Can you think of three pieces of evidence to support your claim?[2]

DISCUSSION 13-3

> Does the earth rotate, or does the sun move around the earth? Does the earth itself translate through space? Can you think of three pieces of evidence to support your claim?

DISCUSSION 13-4

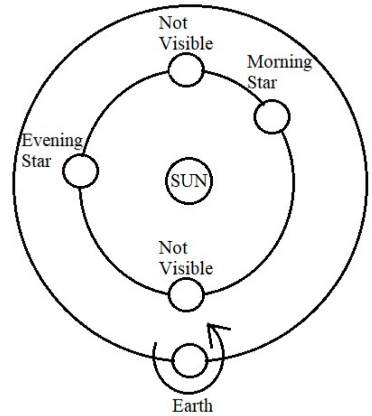
> How big is the earth? Can you think of three ways you might measure it?

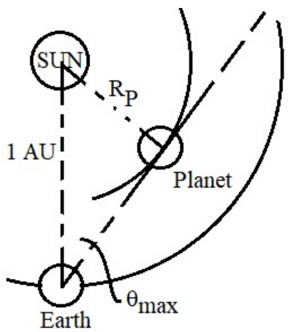
How Big is the Solar System?

[1] If you want to know more, read an Astronomy book.

[2] Clark, Daniel J., director: 2018. *Behind the Curve*. Netflix. If you have time, take a look at this film that follows a group of flat-earthers who design perfectly good experiments to test whether the earth rotates or not.

For some of our discussion, we're going to need to know the distances to the planets in our solar system. Finding the relative distances is fairly easy, although finding the values in meters is quite a bit more difficult. For convenience, let's define the radius of the earth's orbit as 1 *Astronomical Unit* (A.U.). Now let's consider an inferior planet (Mercury or Venus). Because they orbit inside the orbit of the earth, they are never seen too far away from the sun. As they and the earth move, we see them rise just before sunrise (a *morning star*), later and later until they disappear into the glare of the sun, only to reappear on the other side as an *evening star* that sets soon after the sun. The process then reverses. When looking at this diagram, remember that the earth is also orbiting. What we need to know is the maximum *elongation* of the inferior planet, that is, the maximum angular separation between the planet and the sun as seen from earth. At that time, our line of sight to the planet is necessarily tangent to the planet's orbit and perpendicular to the planet's position vector from the sun. Trigonometry tells us quickly that





$$\sin(\theta_{max}) = \frac{R_P}{1\ \mathrm{AU}} \quad \rightarrow \quad R_P = \sin(\theta_{max})\ \text{ in A.U.}$$

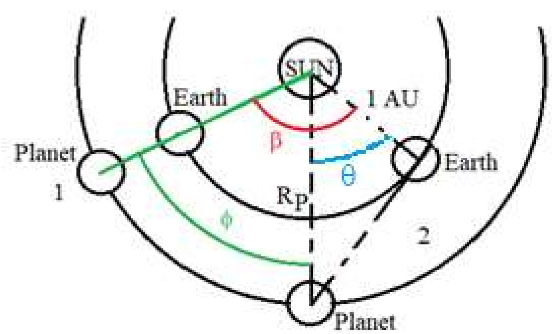
EXAMPLE 13-1

Mercury's maximum orbital elongation varies between 18° and 28°. This is because its orbit is decidedly not circular. The distance from the sun to Mercury then varies from about sin(18°) = 0.31 AU to sin(28°) = 0.47 AU

HOMEWORK 13-1

The elongation of the orbit of Venus as seen from the earth is more constant at about 46°. What is the distance from the sun to Venus in AU?

Finding the radius of the orbit of a *superior planet* (Mars, Jupiter, Saturn, Neptune, and Uranus) requires knowing the period of the earth's orbit as well as that of the planet's. We can start in Position 1 when the planets are lined up with the sun (the planet is at its highest in the sky at midnight). As time passes, both planets travel around the sun, but at different rates. We wait until the planet sets at midnight as at Position 2, at which time we know that the line from the earth to the sun and the line from the earth to the planet are perpendicular. Angle theta is the difference between beta and phi, each of which are determined by calculating the fraction of each planet's period. The radius of the planet's orbit is then



$$\cos(\theta) = \frac{1\text{ AU}}{R_P} \quad \rightarrow \quad R_P = \frac{1}{\cos(\theta)}\text{ in AU.}$$

EXAMPLE 13-2

The time between Position 1 and Position 2 for Jupiter is 89 days (we're rounding off a bit here). The orbital period of Jupiter is 4331 earth days, and for the earth of course 365 days. Then,

$$\varphi = \frac{89}{4331}360^o = 7.5^o \quad \beta = \frac{89}{365}360^o = 87.8^o \quad \theta = \beta - \varphi = 87.8^o - 7.5^o = 80.3^o$$

$$R_P = \frac{1}{\cos(80.3^o)} = 5.94\ AU\ .$$

Not too far off. The currently accepted value is about 5.2 AU.

DISCUSSION 13-5

Can you think of some reasons for the discrepancy?

HOMEWORK 13-2

If you were on one of the moons of Jupiter looking inward at the earth, what would be the earth's greatest elongation angle? Use 5.2 AU as the distance between Jupiter and the sun.

These methods, pursued more carefully, can give us some acceptable values for the relative spacings of the planets in our solar system. How can we know the actual values? The question of the distances to the sun and moon has been a hot topic for millennia. Many methods have been tried, such as observing the earth's shadow on the moon during lunar eclipses, parallax measurements, and transits of inferior planets across the sun's disc. The most straightforward seeming method to us today was performed in the early sixties by Gordon Pettengill,[3] who bounced a radar signal off the surface of Venus and measured its *time of flight*. Knowing the speed of radio waves in space allows for calculation of the distance and provides an accurate value for the AU of 1.496×10^8 meters.

Law of Universal Gravitation

As is common in these notes, we're going to go out of historical order. We will also assume that orbits are essentially circular, except where noted.

We have discussed in previous sections that there is a force of attraction between the earth and any object near its surface, which we call the object's weight. We also observe that the moon orbits the earth and realize that such motion requires a centripetally directed force. We may then jump to the conclusion that it is gravity that maintains the moon in its orbit. Of course, today there are

[3] Pettengill, G.H., *et al*. 1962. "A Radar Investigation of Venus." *The Astronomical Journal* 67, no. 4 (May): 181–190.

tens of thousands of objects orbiting the earth. Galileo's discovery of the four large moons orbiting Jupiter and the acceptance of a Copernican *heliocentric* model for the solar system suggest that the earth is not special, and that perhaps all objects exert this attractive gravitational force on all other objects. Newton's analysis of planetary motions led him to hypothesize that the magnitude of this force is proportional to the product of the two masses and inversely proportional to the separation between them. This result is expected to be true for two point masses, and approximately true for objects small compared to their separation. This results in what we now call the *law of universal gravitation*:

$$\mathrm{F_g} = \mathrm{G}\frac{\mathrm{M_1 M_2}}{\mathrm{r^2}} \quad \text{(always attractive)} .$$

First, let's see if we can verify at least part of this law in the wild. Let's simplify by assuming that orbits are essentially circular (which is fairly true for all the planets except Mercury) and that the sun is so much more massive than the planets that it doesn't move. Objects moving in circles require a centripetal force. At this point in our discussion, we don't know the masses of the planets or the sun. but we do know the relative distance of each from the sun in AUs. So, let's consider the *specific force*[4] experienced by each planet; this quantity is easy to calculate in that it is just equal to the centripetal acceleration:

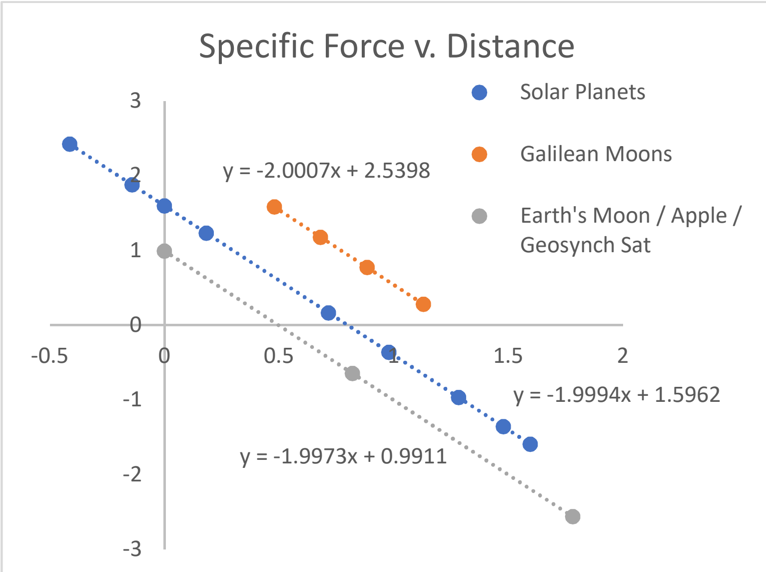
$$\frac{\mathrm{F_g}}{\mathrm{M_{PLANET}}} = \mathrm{a_C} .$$

If we plot the logs of the specific forces *v*. the logs of the planets' distances according to Newton's relationship, we should obtain a line with slope -2.

$$\log\left(\frac{\mathrm{F_g}}{\mathrm{M_{PLANET}}}\right) = \log(\mathrm{GM_{SUN}}) + (-2)\log \mathrm{r} .$$

[4] Specific, in this context, means *per* unit mass. As another example, mechanical engineers often refer to the specific volume of a material instead of the density, its reciprocal.

The Galilean moons of Jupiter give us a second system to study, although clearly the value of the central mass is different. If we push it a bit, we can add in the earth's moon, Newton's falling apple, and maybe a typical satellite as well. Note that in the graph, different, and sometimes very odd, undisclosed units are used for each curve. Regardless of the units used, we see that the specific force goes as the inverse second power of the objects' separations in each of the three cases. We've also shown that the force is proportional to the orbiting object's mass. What we have not determined is the value of G. We could however determine the products GM_{SUN}, GM_{EARTH}, and $GM_{JUPITER}$ from the curves' intercepts.



EXAMPLE 13-3

Let's see if we can get an estimate for G. Consider the moon orbiting the earth. The moon is on average 3.825×10^{8} m from the center of the earth and completes one orbit every 27.3 days (angular speed is then 2.66×10^{-6} rad/sec). We'll need an estimate for the mass of the earth. We have known since antiquity that the earth is a sphere of radius 6.4×10^{6} m, and we can estimate from rocks we find on the surface that the density D is 2900 kg/m^3. We estimate the mass of the earth to be

$$M_E = D \times \mathrm{Vol} = D\,\frac{4\pi}{3}R_E^3 = 2900\,\frac{4\pi}{3}\,(6.4\times10^6)^3 = 3.18\times10^{24}\ \mathrm{kg}\ .$$

Then, by NII,

$$F_{\text{on Moon}} = M_{\text{Moon}}\, a_{\text{C Moon}}$$

$$G\,\frac{M_{EARTH}M_{MOON}}{r^2} = M_{MOON}\,\omega^2 r$$

$$G = \frac{\omega^2 r^3}{M_{EARTH}} = \frac{(2.66\times10^{-6})^2(3.825\times10^8)^3}{3.18\times10^{24}} = 1.245\times10^{-10}\ \mathrm{Nm^2/kg^2}\ .$$

In 1798, Cavendish[5] presented the results of his experiments to determine the density of the earth. This is equivalent to determining the value of G. The results from Cavendish's experiments

[5] Cavendish, Henry. 1798. "Experiments to Determine the Density of the Earth." *Philosophical Transactions of the Royal Society of London* 88 (June): 469-526.

averaged over twenty-nine trials is 6.578×10^{-11} Nm^2/kg^2. A more recent value is 6.67×10^{-11} Nm^2/kg^2.

EXPERIMENT 13-1*

We can make use of Cavendish's measurements to determine a value for the proportionality constant G. However, unlike the types of experiments we have been looking at throughout these notes, Cavendish made many extremely careful measurements of what is essentially one data point.

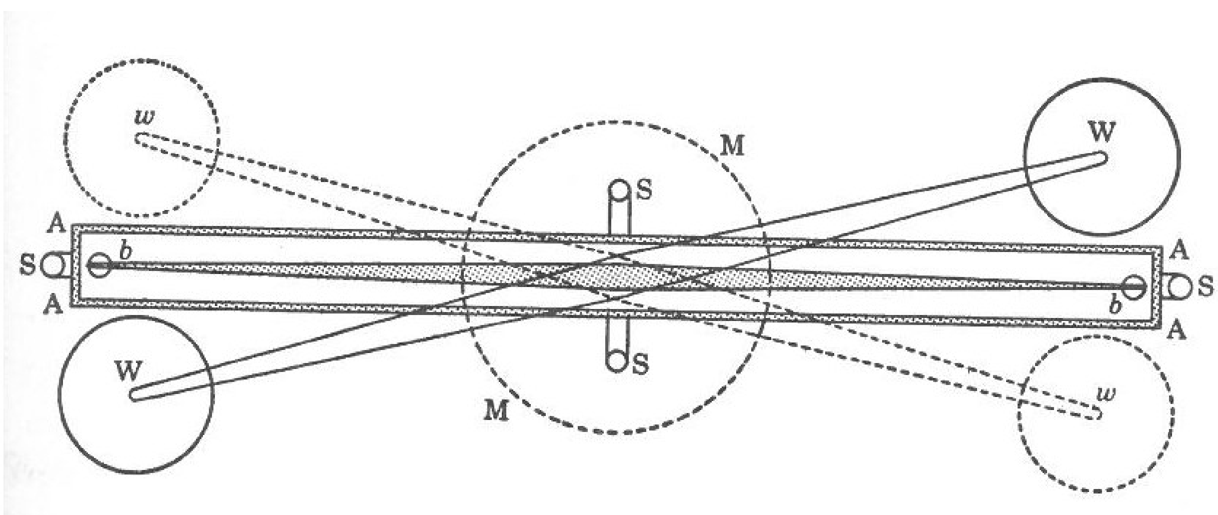
First, a measurement of the local value of g was made by adjusting the length of a pendulum so that its period[6] was two seconds; $l_{PEND} = 0.9942$ meters. Then,

$$P = 2\pi\sqrt{\frac{l_{PEND}}{g}} \quad\rightarrow\quad g = \frac{4\pi^2 l_{PEND}}{P^2} = \frac{4\pi^2(0.9942)}{2^2} = 9.81195\ \mathrm{N/kg}\,.$$

You may remember that this result comes from N II, where the restoring tangential force on the pendulum bob is the tangential component of the weight, given by

$$F = (-)gm\,\theta$$

in the small angle approximation.



The apparatus itself was, in essence, a light rod of length L = 1.862 (R = 0.931 m) meters suspended from a wire and having a 0.73 kg mass (m) at each end (labelled 'b'). The figure shows the apparatus as seen looking down from above. This forms a *torsional pendulum*, where the object rotates due to the restoring torque from the twisted suspension wire (coming up out of the page). We assume that the torque is proportional to the angular displacement of the rod:

[6] Cavendish's 'vibration' is half of what we call a period.

$$\vec{\tau} = -\kappa\, \Delta\vec{\theta} \ .$$

The period of oscillation of such a structure should be given by

$$P_{\mathrm{TORSIONAL}} = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{2mR^2}{\kappa}} \ .$$

Cavendish's calculations are couched in proportions; let's try to place them in a more equation-oriented format. Let's suppose that there is some particular value of the torsional constant, κ_o, such that the period P_o of oscillation of a massless rod of length L_o (= $2R_o$) with a mass m on each end is exactly 2 seconds, and the restoring forces (one exerted at each end), as for a simple pendulum, are given by F = - gm $\Delta\theta$. As such, the frequency of oscillation must be the same as for a simple pendulum of the same length R_o (= 0.9942 m). Comparing,

$$-\kappa_o\, \Delta\theta = \tau = 2FR_o = -2gmR_o\, \Delta\theta \quad \rightarrow \quad \kappa_o = 2gmR_o \ .$$

Now, let's change the torsional constant to some actual value, κ, and find the force required at each end to displace the rod by an angle $\Delta\theta$ when the length of the rod is changed to 2R.

$$\begin{aligned} F = \tfrac{1}{2}\frac{\tau}{R} = -\frac{\kappa}{2R}\,\Delta\theta = -\frac{\left(\frac{4\pi^2 2mR^2}{P^2}\right)}{2R}\,\Delta\theta = -\frac{(4\pi^2 2m)R}{2P^2}\,\Delta\theta = -\frac{\left(\frac{P_o^2\kappa_o}{R_o^2}\right)R}{2P^2}\,\Delta\theta \\ = -\frac{\left(\frac{P_o^2(2gmR_o)}{R_o^2}\right)R}{2P^2}\,\Delta\theta = -\frac{P_o^2}{P^2}\frac{R}{R_o}gm\,\Delta\theta \\ = -\frac{2^2}{P^2}\frac{0.931}{0.9941}(9.81195)(.72977)\Delta\theta = -\frac{26.82384}{P^2}\,\Delta\theta\,. \end{aligned}$$

This is a very round-about way of characterizing the torsional constant. For a particular support wire, we set the rod into oscillation and measure the period, P to determine κ. What we've done is calibrate an imaginary torsional pendulum against a real simple pendulum, then calibrated a real torsional pendulum against the imaginary one.

Much of the Cavendish paper reports on testing for the effects of temperature, magnetism, air currents, and the like. There are also some corrections due to the geometry of the apparatus and the fact that the rod here is not really massless. I'm just going to throw those in without further justification. The result is then

$$F = -\frac{27.157}{P^2}\Delta\theta\,.$$

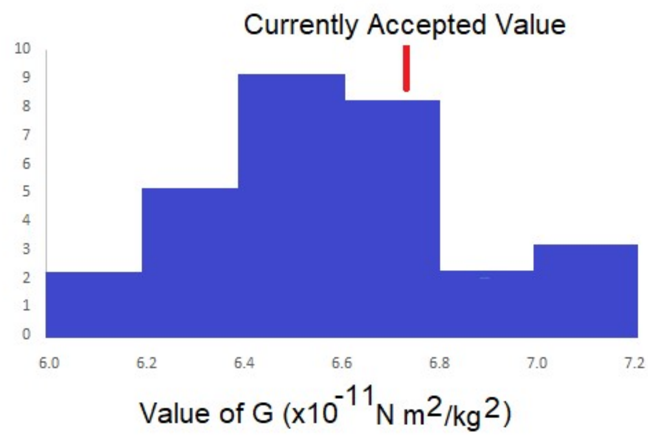
Next, the actual experiment. Masses M of 158.045 kg are placed at the locations labelled W, approximately 0.2248 meters from the small masses, m, and the amount of deflection is then measured in radians. The large masses are then switched to the locations labelled w and the measurements repeated in the other direction in an attempt to eliminate bias. The masses were never varied, and the separation was always about the same, which is why I said earlier that really only one point was measured, albeit many times.

So, if we accept that

$$F = \frac{GMm}{r^2} ,$$

then,

$$G = \frac{Fr^2}{Mm} .$$



The histogram shows the distribution of values obtained from Cavendish's data. All values are less than 8.5% from the currently accepted value. Cavendish himself estimated his uncertainty to be about 7%.

HOMEWORK 13-3*

The first measurement Cavendish made with his second wire had the following results. The period of oscillation of the torsional pendulum was 13 mins 48 seconds. The angular displacement angle was measured at the end of a 38.3 inch arm that moved laterally 2.95 20^{th}s of an inch. Find G from these data.

DISCUSSION 13-6

Why do you suppose that the estimate of the value of G in Example 11-X is so far off from the accepted value of today?

Kepler's Third Law of Planetary Motion

Johannes Kepler developed three valid laws based on the motions of the planets. The third of these (K III) is that the ratio of the square of the period of the orbit to the cube of the *semi-major axis*[7] of the orbit is the same for any object orbiting a particular central mass. If we restrict ourselves to circular orbits, this is quite easy to show. Starting from NII, and assuming that $M_{OBJECT} << M_{CENTRAL}$ so that the central mass remains at the center of the circular orbit,

$$F_g = \frac{GM_{CENTRAL}M_{OBJECT}}{r^2} = M_{OBJECT}a_C = M_{OBJECT}\omega^2 r = M_{OBJECT}\left(\frac{2\pi}{P}\right)^2 r ,$$

$$\frac{r^3}{P^2} = \frac{GM_{CENTRAL}}{4\pi^2} .$$

[7] For circular orbits, the semi-major axis is also the radius.

So, any small objects orbiting a common central mass will have the same constant. For the special case of planets orbiting the sun, we usually use AUs and Earth years, so that the constant equals 1. Plus, in general, since we now know the value of G, we can determine the mass of a central body by analyzing the motions of its satellites.

EXAMPLE 13-4

Mars is 1.524 AU from the sun. How many earth years does it take for Mars to complete one orbit about the sun?

Since Mars and the earth both orbit the sun, we can write

$$\frac{r_E^3}{P_E^2} = \frac{r_M^3}{P_M^2} \quad \rightarrow \quad P_M = P_E \sqrt{\frac{r_M^3}{r_E^3}} = 1 \sqrt{\frac{1.524^3}{1^3}} = \text{1.88 earth years} \; .$$

EXAMPLE 13-5

Calculate the mass of the earth based on the movement of the moon.

The moon's motion about the earth is not very circular, but we'll give it a try. Distance to the moon averages out to 3.844×10^8 m with an orbital period of 27.3 earth days. We worked out earlier that

$$\frac{r^3}{P^2} = \frac{GM_{CENTRAL}}{4\pi^2} \quad \rightarrow \quad M_{CENTRAL} = 4\pi^2 \frac{r^3}{G\,P^2} = 4\pi^2 \frac{(3.844 \times 10^8)^3}{6.67 \times 10^{-1}\;(2.358 \times 10^6)^2}$$
$$= 6.06 \times 10^{24}\ \text{kg} \; .$$

HOMEWORK 13-4

The moon Miranda orbits Uranus at a distance of 129,400 km with a period of 1.41 earth days. Find the mass of Uranus.

Geo-stationary satellites

DISCUSSION 13-6

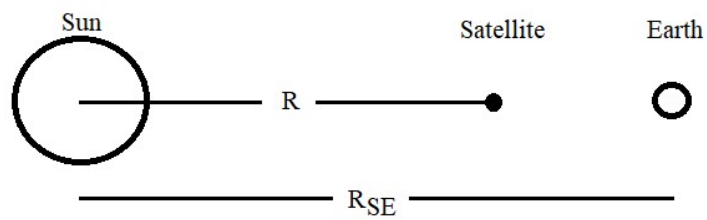
Do you have satellite tv? You may have noticed that your receiving dish always points to the same spot in the sky where, one presumes, your provider's satellite is lurking. But, doesn't the earth rotate? How is it that you don't need to continuously repoint your dish?

Satellites orbit the earth, in a sense, constantly falling toward the earth, but also moving horizontally so that they keep missing it. A truly stationary satellite would simply fall straight down to the earth. The idea here is to place the satellite into a special orbit so that, as seen from the earth, it appears stationary. This places three basic constraints on the orbit. First, it must be a circle centered on the center of the earth. Second, it must run above the earth's equator. Third, the distance from earth to the satellite should be such that it completes an orbit in 23 hours, 56 minutes.[8]

HOMEWORK 13-5

> Using the result presented in the last section, find the radius of the orbit of a truly geo-stationary satellite.

Lagrange Points*



One might wonder if there is an analog to synchronous satellites in a two body system such as the earth and sun. For example, is there a spot between the sun and earth where one could place a satellite such that it remains on the line connecting the two bodies, even as it orbits the larger? Such a spot is called the *Lagrange 1 Point* (L1). It is not just a question of finding a location where the gravitational forces from earth and sun cancel; there must be <u>some</u> net force toward the sun to keep the satellite in a circular orbit. Nor can we use the third law of planetary motion, since there is more than just the one body acting on the satellite. Let's make toward the sun be a positive force and away negative (*i.e.*, centripetal force is positive, centrifugal is negative, as always). We want to find R, the distance from the center of the sun to the L1 point. From NII,

$$\sum_n \vec{F}_{C\,n} = m\vec{a}_C$$

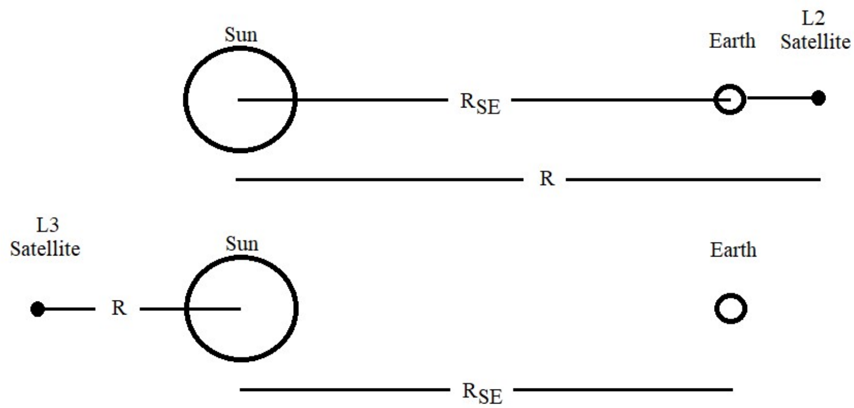
$$+G\frac{M_{SUN}M_{SAT}}{R^2} - G\frac{M_{EARTH}M_{SAT}}{(R_{SE}-R)^2} = M_{SAT}a_C = M_{SAT}\omega^2 R\ .$$

The angular speed ω for the satellite should be the same as for the earth, 2π radians/year. Let's eliminate M_{SAT} and re-arrange:

$$(\omega^2)R^5 - (2\omega^2 R_{SE})R^4 + (\omega^2 R_{SE}^2)R^3 - \big(G(M_S - M_E)\big)R^2 + (2GM_S R_{SE})R - (GM_S R_{SE}^2) = 0$$

[8] If one can tolerate some small apparent motion in the sky, some of these restrictions can be relaxed.

Because this is a *quintic equation*, this calls for a numerical solution. We'll program this into Excel and see what value of R makes it equal zero. In fact, while we're at it, let's find L2 and L3 as well. For L2, both the earth and the sun are pulling the satellite toward the center of its orbit, and in fact also for L3, although the earth-satellite distance is quite different. The corresponding equations are then



$$+\,\mathrm{G}\frac{\mathrm{M_{SUN}M_{SAT}}}{\mathrm{R}^2}+\ \mathrm{G}\frac{\mathrm{M_{EARTH}M_{SAT}}}{(R_{SE}-R)^2}=\ M_{SAT}\omega^2R$$

$$(\omega^2)\mathrm{R}^5-(2\omega^2\mathrm{R_{SE}})\mathrm{R}^4+(\omega^2\mathrm{R_{SE}^2})\mathrm{R}^3-\big(\mathrm{G}(\mathrm{M_E}+\mathrm{M_S})\big)\mathrm{R}^2+(2\mathrm{GM_SR_{SE}})\mathrm{R}-(\mathrm{GM_SR_{SE}^2})=0$$

and

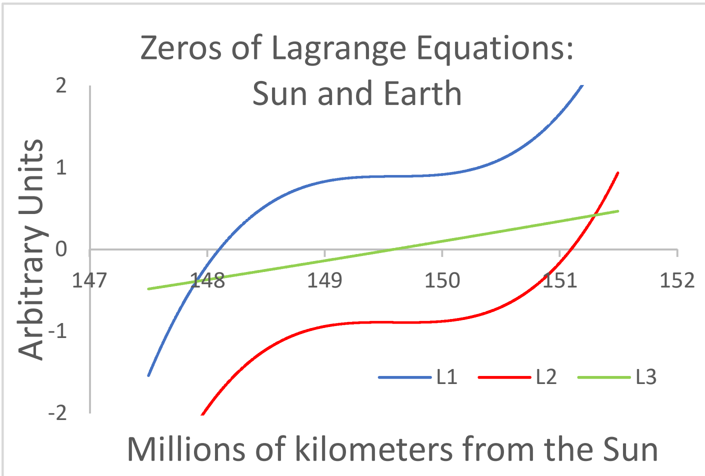
$$+\,\mathrm{G}\frac{\mathrm{M_{SUN}M_{SAT}}}{\mathrm{R}^2}+\ \mathrm{G}\frac{\mathrm{M_{EARTH}M_{SAT}}}{(R_{SE}+R)^2}=\ M_{SAT}\omega^2R$$

$$(\omega^2)\mathrm{R}^5+(2\omega^2\mathrm{R_{SE}})\mathrm{R}^4+(\omega^2\mathrm{R_{SE}^2})\mathrm{R}^3-\big(\mathrm{G}(\mathrm{M_E}+\mathrm{M_S})\big)\mathrm{R}^2-(2\mathrm{GM_SR_{SE}})\mathrm{R}-(\mathrm{GM_SR_{SE}^2})=0\ .$$

The graph plots the three functions above and is rescaled to find the zeros. The results are:

L1	148.1 Mkm[9] from the sun	1.5 Mkm from the earth towards the sun
L2	151.1 Mkm from the sun	1.5 Mkm from the earth opposite the sun
L3	149.6 Mkm from the sun	About the same distance from the sun as earth is, but on the other side of the sun from the earth

[9] We say a million kilometers, but not a gigameter!



There are two more Lagrange points, but we'll leave them for your junior level Physics class.

DISCUSSION

Can you think of an application of L1? What about L2? What kind of satellite would you place at either point? What about L3?

Gravitational Potential Energy

Back in Section 6, we discussed gravitational potential energy, but restricted ourselves to situations near the surface of the earth, where the gravitational field g was considered to be constant. In that case, we decided that the potential energy of an object was given by

$$U_g = gmy \ ,$$

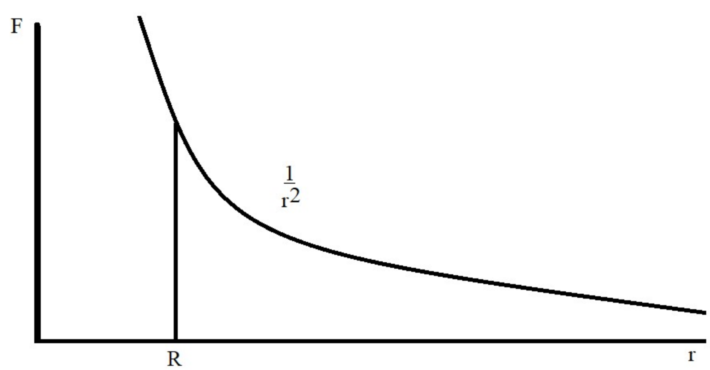
with y and therefor U often defined, arbitrarily to be sure, to be zero at the surface of the earth. In that picture, the weight of an object does the same amount of (negative) work for each meter its altitude is increased. But, if Newton's law of universal gravitation is correct, the weight is actually getting weaker every meter the object moves away from the earth, and so the work done by the weight through each meter will become less and less. Let's do this: take an object of mass m and place it a distance r = R from the earth. Then, we will very slowly push the object out towards r = infinity. If we move it very slowly, the kinetic energy K will remain about zero, and all the work we do will go into the potential energy of the mass, m.

$$W_{\mathrm{WE\ DO}} = -W_g = -(-\Delta U_g) = U_{g\,f} - U_{g\,i} = U(\infty) - U(R)$$

We remember that, in general,

$$W = F\,\Delta x \cos\theta_{F,\Delta x} \ .$$

In this case, both the force and the displacement are outward, so the cosine term gives us a +1. However, the force is not constant. So,



$$W_{\mathrm{WE\ DO}} = \int_R^\infty \frac{GM_E m}{r^2}\,dr = -GM_E m r^{-1}\big|_R^\infty = 0 - \frac{-GM_E m}{R} = U_g(\infty) - U_g(R).$$

If we set the potential energy to zero at infinity, as suggested by this formula, then

$$U_g(r) = -\frac{GM_E m}{r} \ .$$

How is this consistent with our Section 6 notion that U(y) = gmy? Suppose we start with an object of mass m right at the surface of the earth, so $r = R_E = 6.4\text{x}10^6$ m. Let's raise it a small distance $dr \ll R_E$. Then,

$$\frac{dU_g}{dr} = +\frac{GM_E m}{r^2} \ .$$

Evaluate this at the earth's surface:

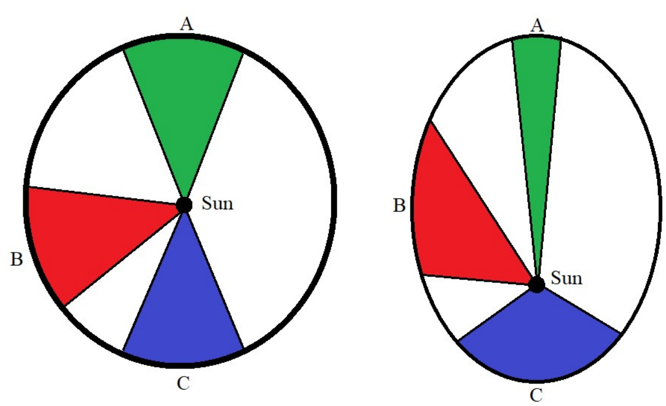
$$dU_g = \frac{GM_E m}{{R_E}^2} dr \approx \frac{(6.67 \times 10^{-11})(5.972 \times 10^{24})}{(6.37 \times 10^6)^2} m\Delta y = \left(9.82 \frac{N}{kg}\right) m\Delta y = gm\,\Delta y \ .$$

And so, we see that the two relationships are consistent at distances not too far above the surface of the earth.

HOMEWORK 13-6

> Suppose a giant asteroid started from rest a very, very large distance from the earth on a trajectory that caused it to hit the earth's surface. How quickly would it be moving as it entered the earth's atmosphere? If the mass had been 2×10^{19} kg (a typical mass for large asteroids in the solar system save for the very largest), how much energy would be released? FYI, the largest of the Soviet cold war warheads was thought to be able to release 2×10^{17} joules.

Kepler's Second Law of Planetary Motion



Now, we're going to start considering the possibility that the orbits are not circular. The first law says that the orbits of objects are actually ellipses, of which a circle is a special case. That will actually be our final topic in this section. For now, accept it and we'll work on the second law of planetary motion (KII), which states that a line connecting any orbiting object to its central mass will 'sweep out' a given area within the orbit in a given amount of time regardless of where in the orbit the object is. For a circular orbit, that should be fairly obvious. Let's start with something we know from Section 9, the torque. You may remember that the toque of a force is given by

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad \text{or } |\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta_{r,F} \ .$$

The vector $\vec{r}$ points from the Sun out to the planet, and the gravitational force on the planet is directed back along that line towards the Sun. The torque exerted on the planet is therefor zero because $\theta_{r,F} = 180^o$.

DISCUSSION 13-7

> You might remember that we have a special situation when the torque on an object is zero. What quantity is conserved in that case? How did we write that quantity back in Section 9? What can you do with that expression if we assume that the planet is small enough to be considered a point mass?

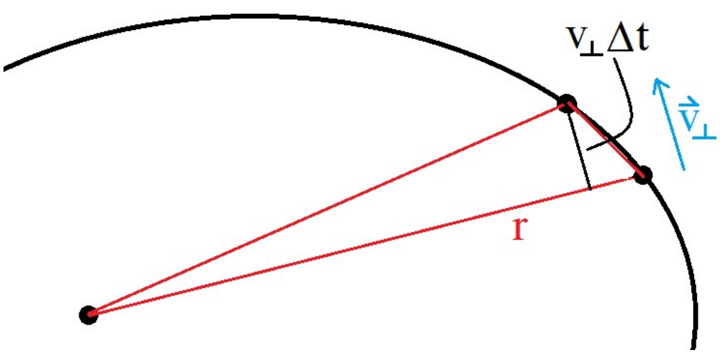
In the same way that force causes a change in momentum, torque causes a change in angular momentum. If there's no torque, then angular momentum is conserved. You may remember that one way of writing the angular momentum of a point mass about a given pivot is

$$\vec{L} = \vec{r} \times \vec{p} = m\,\vec{r} \times \vec{v} \quad \rightarrow \quad L = m\,r\,v_\perp\,,$$

where $v_\perp$ is the component of the velocity that is perpendicular to the location vector, $\vec{r}$. The angular momentum of course points out of the orbit's plane from the right-hand rule. We're setting no restriction on the shape of the orbit other than the direction of motion varies smoothly.

Now, let's consider the planet as it moves through a small distance along its orbit in a short time Δt. If the angle theta is small, we can approximate the area of the slice of the orbit as if it were a triangle with base r and height $v_\perp\,\Delta t$ r:

$$\delta A \approx \tfrac{1}{2}\, r\, (v_\perp \Delta t) = \frac{L}{2m}\,\Delta t\,.$$



The shorter we make the time interval Δt, the smaller that extra bit of area and the more accurate this relationship becomes.

$$\frac{\delta A}{\Delta t} \rightarrow \frac{dA}{dt} = \frac{L}{2m}\,.$$

We see that, since L is constant, the rate at which area is 'swept out' is the same everywhere in the orbit.

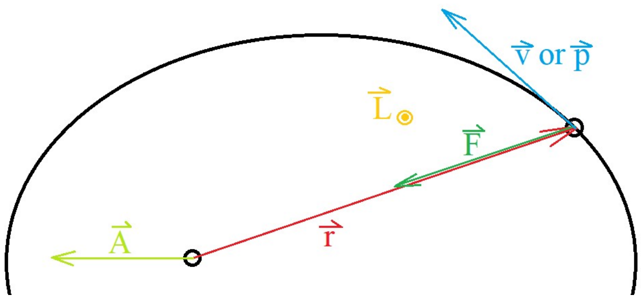
Kepler's First Law of Planetary Motion

The first law states that planets move in elliptical orbits, with the sun at one focus. We can of course generalize this to any small object orbiting a large central body.

DERIVATION*

We're going to make use of a somewhat obscure quantity called the *Laplace vector*:[10]

$$\vec{A} \equiv \vec{p} \times \vec{L} - \frac{GMm^2}{r}\vec{r} \ ,$$



which is conserved for a small object m orbiting a large central body M. First, let's examine $\vec{A}\cdot\vec{L}$:

$$\vec{A}\cdot\vec{L} = \left(\vec{p} \times \vec{L}\right)\cdot\vec{L} - \frac{GMm^2}{r}\vec{r}\cdot\vec{L}$$
$$= 0\,.$$

The first term is zero because $\vec{p} \times \vec{L}$ must be perpendicular to both $\vec{p}$ and $\vec{L}$, and so the dot product with $\vec{L}$ must be zero The second term is zero because $\vec{r}$ lies in the plane of the orbit and $\vec{L}$ points out of the plane.[11] So, this means that $\vec{A}$ lies in the plane of the orbit. Next, consider $\vec{A}\cdot\vec{r}$:

$$\vec{A}\cdot\vec{r} = \vec{r}\cdot\left(\vec{p} \times \vec{L} - \frac{GMm^2}{r}\vec{r}\right) = \vec{r}\cdot\left(\vec{p} \times \vec{L}\right) - \left(\frac{GMm^2}{r}\right)\vec{r}\cdot\vec{r}\ .$$

Remember the scalar vector product from Section 1 ($\vec{r}\cdot\left(\vec{p} \times \vec{L}\right) = \vec{L}\cdot\left(\vec{r} \times \vec{p}\right)$), that $\vec{L} = \vec{r} \times \vec{p}$, and that any vector dotted with itself is the square of its magnitude. Then this becomes

$$\vec{A}\cdot\vec{r} = \vec{L}\cdot\left(\vec{r} \times \vec{p}\right) - \left(\frac{GMm^2}{r}\right)r^2 = \vec{L}\cdot\vec{L} - \left(\frac{GMm^2}{r}\right)r^2 = L^2 - GMm^2 r\,.$$

Remember that the dot product was defined in Section 1 as

$$\vec{A}\cdot\vec{r} = A\,r\cos\theta_{A,r}\ ,$$

so,

$$A\,r\cos\theta_{A,r} = L^2 - GMm^2 r \quad \rightarrow \quad \frac{1}{r} = \frac{GMm^2}{L^2}\left(1 + \frac{A}{GMm^2}\cos\theta_{A,r}\right)\,.$$

One version of the formula for a conic section is

$$\frac{1}{r} = C(1 + e\cos(\theta))$$

with e the curve's *eccentricity* and C a constant. Matching these up, we see that the eccentricity is

[10] Herbert Goldstein, *Classical Mechanics* (Reading: Addison-Wesley Publishing Company, 1980), 103-104.
[11] This assumes that the angular momentum is not itself zero, *i.e*. the object is not simply falling toward its central body.

$$e = \frac{A}{GMm^2} \ .$$

Then, if

e = 0, the orbit is circular.

0 < e <1, the orbit is elliptical.

e = 1, the orbit is parabolic.

e > 1, the orbit is hyperbolic.

MATHEMATICAL JUSTIFICATION*

Now the really hard part. We asserted that the Laplace vector $\vec{A}$ is conserved. We need to show that the instantaneous time rate of change of the Laplace vector is always zero:

$$\frac{d\vec{A}}{dt} = 0 \ .$$

Let's start by rewriting the expression for $\vec{A}$ slightly. Remembering that the momentum $\vec{p}$ = $m\vec{v}$ and that $\vec{r}/r$ can be written as $\hat{r}$, we have that

$$\vec{A} = \vec{p} \times \vec{L} - \frac{GMm^2}{r}\ \vec{r}$$

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}\left(\vec{p} \times \vec{L}\right) - GMm^2\frac{d\hat{r}}{dt} = 0$$

$$\frac{d\vec{p}}{dt} \times \vec{L} + \vec{p} \times \frac{d\vec{L}}{dt} = GMm^2\frac{d\hat{r}}{dt}.$$

We remember from impulse-momentum, that

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{and here,} \quad \vec{F} = -\frac{GMm}{r^2}\hat{r}$$

and also that the angular momentum is a constant and equal to $m\,\vec{r} \times \vec{v}$, so

$$\left(-\frac{GMm^2}{r^2}\hat{r}\right) \times (\vec{r} \times \vec{v}) = GMm^2\frac{d\hat{r}}{dt},$$

$$-\frac{1}{r^3}\ \vec{r} \times \left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{d\hat{r}}{dt}.$$

Making use of the triple cross product,

$$-\frac{1}{r^3}\left(\left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right)\vec{r} - \ (\vec{r} \cdot \vec{r})\frac{d\vec{r}}{dt}\right) = \frac{d\hat{r}}{dt}.$$

$$-\frac{1}{\mathrm{r}^3}\left(\left(\vec{\mathrm{r}}\cdot\frac{d\vec{\mathrm{r}}}{d\mathrm{t}}\right)\vec{\mathrm{r}} - \mathrm{r}^2\frac{d\vec{\mathrm{r}}}{d\mathrm{t}}\right) = \frac{d\hat{\mathrm{r}}}{d\mathrm{t}}.$$

Now, some sneaky stuff:

$$\frac{d}{d\mathrm{t}}(\vec{\mathrm{r}}\cdot\vec{\mathrm{r}}) = 2\,\vec{\mathrm{r}}\cdot\frac{d\vec{\mathrm{r}}}{d\mathrm{t}} \text{ but it is also } \frac{d}{d\mathrm{t}}(\vec{\mathrm{r}}\cdot\vec{\mathrm{r}}) = \frac{d}{d\mathrm{t}}(\mathrm{r}^2) = 2\mathrm{r}\frac{d\mathrm{r}}{d\mathrm{t}} \quad\rightarrow\quad \vec{\mathrm{r}}\cdot\frac{d\vec{\mathrm{r}}}{d\mathrm{t}} = \mathrm{r}\frac{d\mathrm{r}}{d\mathrm{t}}\ ,$$

and

$$\frac{d\hat{\mathrm{r}}}{d\mathrm{t}} = \frac{d}{d\mathrm{t}}\left(\frac{\vec{\mathrm{r}}}{\mathrm{r}}\right) = \frac{1}{\mathrm{r}}\frac{d\vec{\mathrm{r}}}{d\mathrm{t}} - \frac{1}{\mathrm{r}^2}\frac{d\mathrm{r}}{d\mathrm{t}}\,\vec{\mathrm{r}}\,.$$

So,

$$-\frac{1}{\mathrm{r}^3}\left(\left(\mathrm{r}\frac{d\mathrm{r}}{d\mathrm{t}}\right)\vec{\mathrm{r}} - \mathrm{r}^2\frac{d\vec{\mathrm{r}}}{d\mathrm{t}}\right) = \frac{1}{\mathrm{r}}\frac{d\vec{\mathrm{r}}}{d\mathrm{t}} - \frac{1}{\mathrm{r}^2}\frac{d\mathrm{r}}{d\mathrm{t}}\,\vec{\mathrm{r}}\ ,$$

$$\frac{1}{\mathrm{r}}\frac{d\vec{\mathrm{r}}}{d\mathrm{t}} - \frac{1}{\mathrm{r}^2}\frac{d\mathrm{r}}{d\mathrm{t}}\vec{\mathrm{r}} = \frac{1}{\mathrm{r}}\frac{d\vec{\mathrm{r}}}{d\mathrm{t}} - \frac{1}{\mathrm{r}^2}\frac{d\mathrm{r}}{d\mathrm{t}}\,\vec{\mathrm{r}}\ ,$$

which is clearly a true statement.