SECTION THREE – KINEMATICS IN TWO DIMENSIONS

In the last section, we discussed the kinematics of a point mass in one dimension. Again, kinematics describes the motion of an object without regard to the cause of that motion. In this section, we shall examine two special cases of two dimensional motion: projectile motion and uniform circular motion.

We need a way of keeping track of the motion of a particle. Luckily, we discussed this back in Section One, where we defined the *position vector* $\vec{\mathbf{r}}$ as

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}\ .$$

The displacement is then

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = (x_f\,\hat{\imath} + y_f\,\hat{\jmath}) - (x_i\,\hat{\imath} + y_i\,\hat{\jmath}) = (x_f - x_i)\,\hat{\imath} + (y_f - y_i)\,\hat{\jmath} = \Delta x\,\hat{\imath} + \Delta y\,\hat{\jmath}\ .$$

so that the displacement is the vector sum of the individual displacements in the x and y directions (no surprise there). The average velocity is

$$\vec{v}_{AVE} = \frac{d\vec{r}}{dt} = \frac{d(x\,\hat{\imath} + y\,\hat{\jmath})}{dt} = \frac{dx\,\hat{\imath} + \Delta y\,\hat{\jmath}}{dt} = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath} = v_{x\,AVE}\,\hat{\imath} + v_{y\,AVE}\,\hat{\jmath}\ .$$

The instantaneous velocity **v** is defined as before as

$$\vec{v}_{INST} = \lim_{\Delta t\to 0}\vec{v}_{AVE} = \lim_{\Delta t\to 0} v_{x\,AVE}\,\hat{\imath} + \lim_{\Delta t\to 0} v_{y\,AVE}\,\hat{\jmath} = v_{x\,INST}\,\hat{\imath} + v_{y\,INST}\,\hat{\jmath}\ .$$

And, of course, because the acceleration is to the velocity as the velocity is to the position, we can immediately write that

$$\vec{a}_{AVE} = a_{x\,AVE}\,\hat{\imath} + a_{y\,AVE}\,\hat{\jmath} \quad \text{and} \quad \vec{a}_{INST} = a_{x\,INST}\,\hat{\imath} + a_{y\,INST}\,\hat{\jmath}\ .$$

DISCUSSION

> Consider a ball whirled around on the end of a string at constant speed. Is the velocity of the ball constant? Is its acceleration?

PROJECTILE MOTION IN TWO DIMENSIONS

Projectile motion describes objects that are thrown, dropped, launched, tossed, pitched, hurled, catapulted, or chucked near the surface of a planet. Such objects are said to be in *free fall*. We shall assume the following for now:

The planet's gravitational field is uniform, *i.e.*, constant in direction and magnitude. Once an object is launched, the only agency acting on it is gravity; therefore its acceleration is a constant a_g downward.

This assumption leads us to suspect that the horizontal and vertical motions of an object are independent. We confirmed to some degree of satisfaction by observing a demonstration.

DEMONSTRATION 3-1

VIDEO

First, two balls were released from rest at the same time and allowed to fall toward the table; they arrived at the same time. Then, one ball was dropped while the other was launched horizontally from the same height at the same time; once again, they arrived at the same instant. This led us to an interesting conclusion, namely that the motions of the object in the horizontal and vertical direction will be independent of one another, thereby making a two-dimensional problem in fact two one-dimensional problems. Of course, there are many situations where this is not true. For example, if we were to account for *drag*, or as you probably know it, *air resistance*, this assumption could be false.

So, following our assumptions, we have two sets of kinematic equations, which I am simply copying from Section Two,

$v_{xf} = v_{xi} + a_x t$	$v_{yf} = v_{yi} + a_y t$
$v_{x\,AVE} = \dfrac{v_{xf} + v_{xi}}{2}$	$v_{y\,AVE} = \dfrac{v_{yf} + v_{yi}}{2}$
$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$	$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$
$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$	$v_{yf}^2 = v_{yi}^2 + 2a_y(y_f - y_i)$

with the time as the obvious connection between the two motions.

Before we start on examples, let me talk briefly about what I call Rule Number One,[1] which says that in problems in which there is acceleration, you should make one of the coördinate axes in the direction of the acceleration and the other, if necessary, perpendicular to that. The reason for this is to avoid breaking the acceleration into components, an action that generally makes the mathematics of solving a problem much more difficult. For projectile problems, this probably seems very natural; make horizontal the x-axis and vertical the y-axis. Remember, though, as the semester moves along, that the situation may change.

[1] Strictly speaking, it's a Really Strong Suggestion.

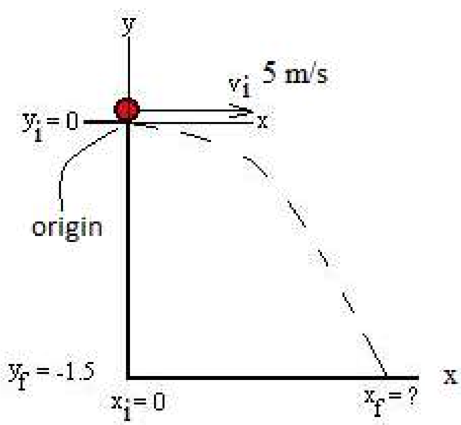
Here follows an example that you should use as the model for solving most projectile problems.

ADMONITION

When we do projectile problems, we remember that the problem runs from the moment just after the ball leaves the table to just before it hits the floor or ground. If the problem asks for the final velocity, do not assume it is zero because the ball hit the floor and presumably stopped! During the collision with the floor, there was another agency besides gravity acting on the ball, and so the acceleration was not constant and the kinematic equations are not valid.

EXAMPLE 3-1

A ball is rolled horizontally off a table 1.2 m in height at 5 m/s. How far from the base of the table will the ball strike the floor?



First, draw a figure to help visualize the situation, including a system of axes with an origin. Your choice of origin can be arbitrary, but in this problem, there are two obvious locations: the top edge of the table and the foot of the table. The top is a slightly better choice for reasons you are invited to work out on your own. But don't get hung up on it, the bottom will work out O.K., too. The axes are chosen to be x horizontal and y vertical, according to Rule Number One above. All these things are labeled in the diagram so that whoever is grading your paper can easily tell what you are doing. Here, I've added in a few other pieces of information as well.

Next, make your inventory of what you know, what you think you know, and what you want to know. This is pretty standard for every problem. We're interested in what's happening in the x-direction, so let's start there. I use question marks for quantities I don't know and arrows for the ones I don't know but want to know.

$x_i = 0$ m

$x_f = ? \leftarrow$

$v_{xi} = +5$ m/s
$v_{xf} = +5$ m/s (why?)
$a_x = 0$ m/s^2 (the acceleration is downward, and not at all horizontal, once the ball is in free fall)
$t = ?$

As we did in the last section, we'll try to find a kinematic equation, or a combination, that will give us what we want to know. Is there one?

Since there is not enough information on the x-side, we need to look to the y-side. Here I will give you what I call an 80% Rule.[2] Generally, it's the time that is common to both sides of the problem, so I will find the time for the y-side, if possible, then bring it back over to the x-side, at which point I may have enough information there to solve. Since the time features prominently in KEq. 3, I'll probably use that on both sides.

$y_i = 0$ m
$y_f = -1.2$ m (upward is positive and the ball moved downward from the origin)
$v_{yi} = 0$ m/s (the ball was travelling horizontally and not at all vertically as it left the table)
$v_{yf} = ?$
$a_y = -10$ m/s^2 (we chose upward to be positive)
$t = ?$

Next, we state which principle of Physics we are using, in this case, KEq. 3:

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

As discussed in Section Two, it's best to try to manipulate the symbols to secure a general abstract solution, but since we want to learn the time, KEq. 3 will become a quadratic equation in t, which is the exception to our rule. Inserting the numbers and re-arranging to the standard format leaves us with

$$(5)t^2 + (0)t + (-1.2) = 0\,,$$

which, it turns out, we can solve directly:

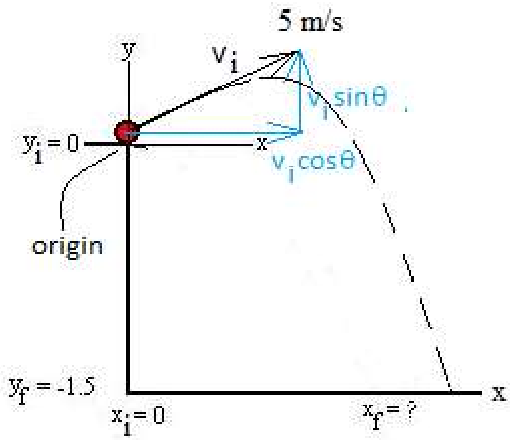
$$t = \pm\sqrt{\frac{1.2}{5}} = \pm\,0.49 \text{ seconds}\ .$$

Since the ball hits the floor after it leaves the table, the time must be positive, so t = 0.49 seconds. We'll take this backin KEq 3 on the x-side to find x_f.

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 5(0.49) + 0(0.49^2) = 2.45 \text{ m}\ .$$

EXAMPLE 3-2

[2] The 80% number is obviously made up, I merely mean that this will work a large *per centage* of the time and it's how I myself would start such a problem. If it doesn't work, then try something else.



Take the same ball as above and launch it from the edge of the table at 5 m/s at a 30º angle above the horizontal. How far from the foot of the table will it land?

What is the only way in which this problem is different from the previous example? How will you deal with that difference?

Use the same origin and coördinate system as above, because, well, why not?

$x_i = 0$ m

$x_f = ?$ ←

$v_{xi} = v_i \cos\theta = 5 \cos(30^o) = +4.33$ m/s (find the x-component of the initial velocity)

$v_{xf} = +4.33$ m/s

$a_x = 0$ m/s^2

$t = ?$

Well, we don't have any more information about the x-motion this time around than we did the last, so our plan should be the same as for the previous example. Let's move on to inventory the y-side:

$y_i = 0$ m

$y_f = -1.2$ m (upward is positive and the ball moved downward from the origin)

$v_{yi} = v_i \sin\theta = 5 \sin(30^o) = +2.5$ m/s

$v_{yf} = ?$

$a_y = -10$ m/s^2 (we chose upward to be positive)

$t = ?$

KEq. 3:

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

This will be quadratic, so insert the numbers and re-arrange:

$$(5)t^2 + (-2.5)t + (-1.2) = 0\,,$$

$$t = \frac{-(-2.5) \pm \sqrt{(-2.5)^2 - 4(5)(-1.2)}}{2(5)} = -0.3 \text{ OR } +0.8 \text{ seconds}\ .$$

Taking the positive time back to KEq.3 in the x-side:

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 = 0 + 4.33(0.8) + 0(0.8^2) = 3.46 \text{ m} \; .$$

HOMEWORK 3-1

A ball is thrown horizontally from the top of a 26 m tall building and hits the ground 12 meters from the base of the building. With what initial speed was the ball thrown?

EXERCISE 3-1

A classic problem involves a hunter on the ground trying to shoot a monkey at the top of a tree. The hunter is 30 m away for the base of the tree, the tree is 40 m high, and the speed of the arrow, once off the bow, is 35 m/s. Not having taken Physics, the hunter aims directly at the monkey and shoots. The monkey, however, sees the hunter shoot, and figures the quickest escape is simply to fall immediately from the tree towards the ground. Show that, in spite of this, the hunter hits the monkey after all. You should ignore the hunter's height, that is, the arrow starts at ground level.

EXAMPLE 3-3

Let's try an example where we don't use the 80% Rule. An object is thrown horizontally from the top of a building of height H and hits the ground below four seconds later at a 45° angle. How tall is the building and with what speed was it launched? How far from the base of the building did the object land?

You should draw the figure for this. Let's put the origin at the base of the building and make positive x be horizontal to the right and positive y be upward. What do we know?

$x_i = 0$ m	$y_i = ? \leftarrow$ (this is the height H)
$x_f = ? \leftarrow$	$y_f = 0$
$v_{xi} = ? \leftarrow$	$v_{yi} = 0$ m/s (launched horizontally)
$v_{xf} = v_{xi}$	$v_{yf} = ?$
$a_x = 0$ m/s^2	$a_y = -10$ m/s^2

$t = 4$ seconds

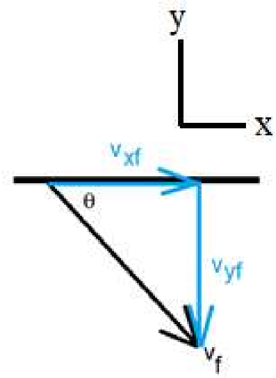
Finding the height of the building is straightforward with KEq. 3:

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$H = y_i = y_f - v_{yi}t - \frac{1}{2}a_y t^2 = 0 - 0(4) - (-5)(4^2) = 80 \text{ m} \; .$$

Now for the x values. Looking at the x-side, we see that even knowing the time does us no good. So, what else links the two sides? We know something about the final velocity components. The angle in the diagram is -45° (below the x-axis) and the ratio of the final velocity components is

$$\frac{v_{yf}}{v_{xf}} = \tan\theta = \tan(-45^o) = -1 \quad \rightarrow \quad v_{xf} = -v_{yf} \ .$$



Since the final x velocity is the same as the initial, KEq. 1 tells us that

$$v_{xi} = v_{xf} = -v_{yf} = -\left(v_{yi} + a_y t\right) = -\left(0 + (-10)(4)\right)$$

$$= 40 \text{ m/s} \ .$$

Lastly, x_f is given by KEq. 3 as

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 40(4) + 0 = 160 \text{ m} \ .$$

DISCUSSION

Students, if asked, often guess that the object lands 80 meters from the base of the building; after all, it hit the ground at a 45° angle, and the building is 80 m tall. This would seem to imply that the object followed a straight line from the top of the building after having made an abrupt change of direction immediately after launch. In a moment, we'll discuss the path actually taken by the object.

EXERCISE 3-2

Repeat Example 3-3 if the object had hit instead at an angle 53 degrees below the horizontal.

Shape of a Projectile's Path

There are some interesting ideas circulating about the shape of the path (the *trajectory*) taken by a thrown object. As mentioned, some students assume that the object of the previous example follows a straight line path from the top of the building to the ground. On the other hand, cartoon physics says that a coyote running horizontally off a cliff continues horizontally, until he realizes his predicament, then falls straight downward. Let's try to determine the actual type of path a projectile will take through space near the surface of the earth, that is, we want y as a function of x.

DERIVATION 3-1

Start once again with the kinematic equations; we'll use our 80% Rule. Call the starting point the origin, and let upward be +y and horizontal direction of motion be +x. Then,[3]

[3] Notice that the initial speed is labelled v_o here. A 'nought' subscript denotes a specific value that isn't actually known.

$x_i = 0$ $\quad$ $y_i = 0$
$x_f = ?$ $\quad$ $y_f = ?$
$v_{xi} = v_o \cos\theta_o$ $\quad$ $v_{yi} = v_o \sin\theta_o$
$v_{xf} = ?$ $\quad$ $v_{yf} = ?$
$a_x = 0$ $\quad$ $a_y = a_g$
$t = ?$

This time, we'll start with the x-side and find the time:

$$x = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$x = 0 + v_o\cos\theta_o t + 0 \quad \rightarrow \quad t = \frac{x}{v_o\cos\theta_o} \ .$$

Now to the y-side:

$$y = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

$$y = 0 + v_o\sin\theta_o \frac{x}{v_o\cos\theta_o} + \tfrac{1}{2}a_g\left(\frac{x}{v_o\cos\theta_o}\right)^2 = (\tan\theta_o)\,x + \left(\frac{a_g}{2v_o^2\cos^2\theta_o}\right) x^2 \ .$$

This looks messy, but that's O.K., because we don't care at the moment about most of it. For any given launch of an object, v_o and θ_o are fixed. That is, we can't go back and change their values midway through the trip. Let's replace the tangent term with a generic positive constant, A. Then, lump all the constants in the x^2 term together and call them negative constant B (remember that a_g is negative here):

$$y(x) = Ax + Bx^2 \ .$$

You should I hope recognize this form of curve; it is an example of a *parabola*, specifically one 'open down' and symmetric around a vertical axis.

So, so long as we meet the conditions outlined at the beginning of this section, any object thrown near a planet's surface should follow a parabolic path. Later in the semester, we'll see what happens when the object gets away from the earth's surface.

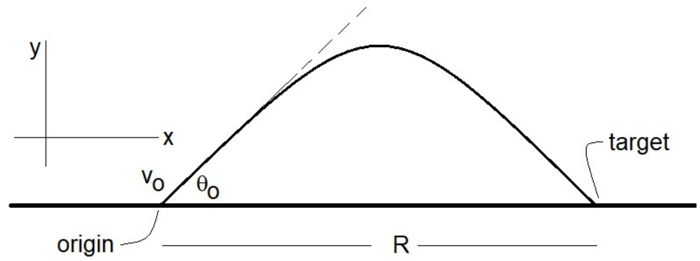
The Range Equation

Let's discuss a special case of projectile motion which is of historical interest. In the 17th and 18th century, being a physicist usually meant being an artillery officer. As is usual, we will consider a special case.

DERIVATION 3-2

Consider a flat horizontal plain (which is also a plane) on which are located a battery and a target. Given an initial projection angle θ_o (*elevation*) and launch speed v_o (*muzzle velocity,* for guns or cannon), how far will the projectile land from the gun (*range*, R)?

Because the plane is horizontal and flat, $y_o = y_f = 0$. We'll launch our projectile from the origin, for convenience.



$x_i = 0$	$y_i = 0$
$x_f = R \leftarrow$	$y_f = 0$
$v_{xi} = v_o \cos\theta_o$	$v_{yi} = v_o \sin\theta_o$
$v_{xf} = ?$	$v_{yf} = ?$
$a_x = 0$	$a_y = a_g$
$t = ?$	

Once again, we'll try our 80% Rule, starting on the y-side to find the time with KEq 3:

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 \ .$$

Insert some values and substitutions:

$$0 = 0 + v_o \sin\theta_o\, t + \tfrac{1}{2}a_g\, t^2 \,,$$

$$0 = \left(v_o \sin\theta_o + \tfrac{1}{2}a_g\, t\right) t \,.$$

For the right side here to equal zero, either t = 0 (which is uninteresting; we already know the object was on the ground at the start of the problem), or

$$v_o \sin\theta_o + \tfrac{1}{2}a_g\, t = 0,$$

in which case

$$t = \frac{-2\, v_o \sin\theta_o}{a_g} \ .$$

This may look a bit strange. Is the time actually negative? Did we hit the target before we launched the projectile? No, we're O.K. because a_g is negative. Having said that, I hate negative signs, so I'll take the absolute value of the negative acceleration and that will cancel the negative sign in the numerator:

$$t = \frac{2\,v_o \sin\theta_o}{|a_g|} \quad .$$

So, this is the time for the entire trip. How far does the projectile travel horizontally in that time. Back to KEq. 3.

$$x = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$R = 0 + v_o\cos\theta_o t + 0 = v_o\cos\theta_o \frac{2\,v_o\sin\theta_o}{|a_g|} = \frac{v_o^2(2\sin\theta_o\cos\theta_o)}{|a_g|} \; .$$

Finally, we'll use a trig identity to make this prettier: 2 sinα cosα = sin(2α). This brings us to the final result of

$$R = \frac{v_o^2\sin(2\theta_o)}{|a_g|} \; .$$

Remember that this result is valid only when the assumed conditions are met, particularly that the launching and landing spots must be at the same altitude. Otherwise, you will need to treat this as a projectile motion problem to be solved from scratch.

DISCUSSION

In 'real life,' we would also worry about a number of effects that would make the result above invalid, particularly for large ranges. Can you think of at least three?

EXAMPLE 3-4

A ball is thrown at 20 m/s at an angle of 25° above the horizontal over a flat surface. How far from the launch point will the ball land?

This is straight plug-and-chug:

$$R = \frac{v_o^2\sin(2\theta_o)}{|a_g|} = \frac{20^2\,\sin(2\times 25^o)}{10} = \frac{20^2\,\sin(50^o)}{10} = 30.6\text{ m} \; .$$

EXAMPLE 3-5

Let's go the other way. The launch speed is 50 m/s and we wish to hit a target 160 m away on a flat surface. At what angle should the object be launched?

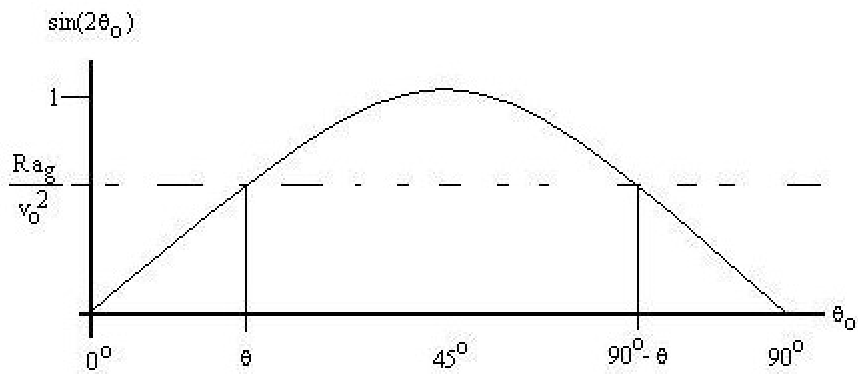
Re-arranging the Range Equation for theta,

$$\theta_o = \frac{1}{2}\arcsin\left(\frac{R|\,a_g\,|}{v_o^2}\right) = \frac{1}{2}\arcsin\left(\frac{160 \times 10}{50^2}\right) = \frac{1}{2}\arcsin(0.64) = \frac{1}{2}(40^o) = 20^o\,.$$

DISCUSSION

Examine the Range Equation again. For a given launch speed v_o, what launch angle will result in the largest range?

If we start at 0^o, the range will be zero; the projectile will just hit the ground right away. As we increase the elevation angle, the range will increase until the sine function maxes out at 1. What launch angle θ_o does that correspond to? If that angle results in the maximum range, what happens when we go above that angle?



Let's plot the sine of twice the launch angle against the launch angle, θ_o. Solving the example above meant finding an angle such that

$$\sin(2\theta_o) = \frac{R|\,a_g\,|}{v_o^2}\,,$$

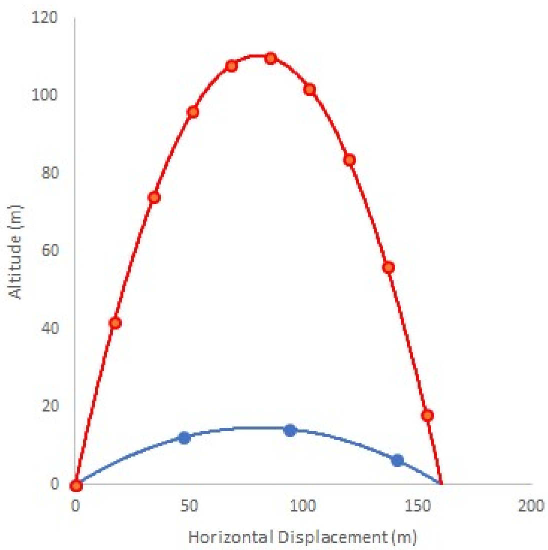
as can be seen on the graph; the solution is at the intersection of the two curves. But, if you follow the dotted line over to the right, you will see that there is a <u>second angle</u> that fulfills this requirement. Since this curve is symmetric, the second larger angle should be the *complement* of the smaller one. So, there are actually two answers to the example above, the 20^o we found, OR 70^o, the complement of 20^o. Except of course for 45^o, which is its own complement, there should be two answers to these problems.

So, how is this possible? At a low angle, the projectile is not in the air long, but it has a high x-component of velocity, while at a high angle, the projectile spends a lot of time in the air, but has a correspondingly lower x-component of velocity. These two effects combine to give the same final x displacement as for the low angle case. In the same way, the lower angle launch has a smaller initial y velocity component than the higher angle launch, and so will not reach as high an altitude. The lower angle is useful in tank warfare, where it is important to hit the other guy before he gets off a shot at you, while the second is good if there are obstacles around your target. In the example above, the travel times of the two paths are

$$t = \frac{2\,v_o \sin\theta_o}{|a_g|}$$

$$t_{20^o} = \frac{2(50)\sin(20^o)}{10} = 3.42 \text{ seconds}$$

$$t_{70^o} = \frac{2(50)\sin(70^o)}{10} = 9.40 \text{ seconds}\ .$$



The figure at left shows the trajectories of this object for each of the angles, 20° and 70°. The dots represent intervals of one second. One can see that if two such objects were launched simultaneously, the one launched at 70° would still be rising when the other arrived at its target.

HOMEWORK 3-2

Derive an expression (that is, start with the kinematic equations) in terms of v_o, θ_o, and a_g for the maximum altitude H reached by a projectile. Use this result to calculate the maximum altitudes for the object launched in Example 3-5 for each angle (20° and 70°). Check your results against the graph above.

EXERCISE 3-3

Our target is 350 meters away along a flat surface. Our launcher will throw the projectile with an initial speed of 55 m/s. At what angle (or angles) could we launch in order to hit the target?

HOMEWORK 3-4

You're playing golf on a flat fairway. The green is 150 m away, and you can send the ball away at 60 m/s. At what angle or angles could you hit the ball for a hole-in-one?

HOMEWORK 3-5

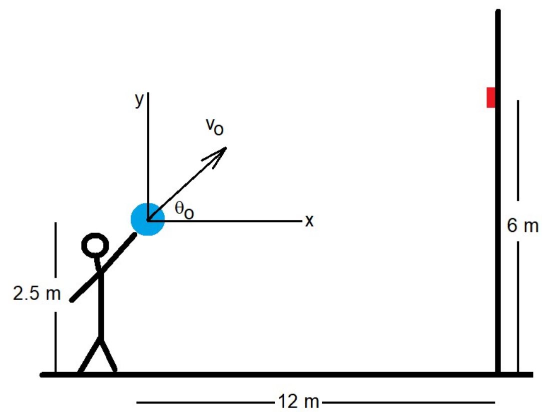
Show that, for a projectile thrown at an initial angle θ_o above a flat horizontal plane, the maximum altitude H is related to the range by

$$H = \frac{R \tan \theta_o}{4}\ .$$

HOMEWORK 3-6

To score in Sportsball, you must successfully throw the sportsball at a target on the wall from a distance of 12 m. The target is 6 meters above the floor, and you release the ball with a speed of 16 m/s at an altitude of 2.5 m above the floor. At what angle or angles θ_o from the horizontal should you throw the ball. You may find this relationship useful:

$$tan^2\,\theta + 1 = \frac{1}{cos^2\,\theta}\,.$$



CIRCULAR MOTION

Consider an object moving at constant speed v in a circle of radius r; forget about gravity for now.

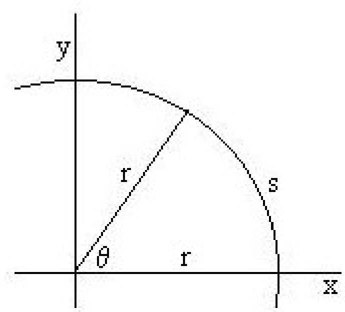
DISCUSSION

Does this object have a constant velocity? Which kind of a quantity is velocity? What are the two parts of velocity? Do they both need to be constant for the velocity to be constant? This means that the object is doing what?

Let's find that quantity. We'll do it two ways, one we could almost call' traditional,' and the other a bit less straightforward, but which will leave us with some additional useful relationships.

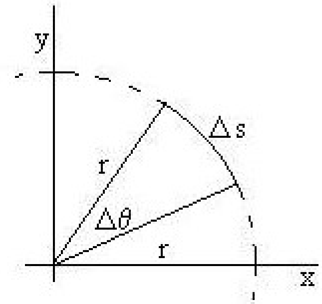
Before we start, let's define a quantity we will find useful through the rest of this course. Consider our object moving in a circle. Suppose that it has moved a distance s along the circumference of a circle of radius r, where arc s subtends an angle θ. The *arclength* relationship tells us that

$$s = r\,\theta_{RADIANS}\ .$$



A *radian* is the angle such that the arclength s is equal to the radius r, or about 57.3°. Clearly, if we halve the angle, we also halve the distance along the arc, so that θ and s are proportional by the factor r. As an extreme example, there are 2π radians in a circle, since the circumference (the arclength all the way around) is 2πr.

We should next find a way of describing changes in the object's position, or the *angular distance* $\Delta\theta$, so that



$$\Delta s = r\,\Delta\theta \ .$$

If we consider the instantaneous time rate of change of each side of the equation above, we obtain

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \ ,$$

since the radius is a constant. The left side we recognize as the speed v_T, We add the 'T' subscript because the velocity is tangent to the circle. On the right side we will define the *angular speed* ω (omega), the angular distance *per* unit time:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \ .$$

This gives us a choice in describing the motion of the object, in terms of either its speed around the circle or its angular position as seen from the center of the circle:

$$\boxed{v_T = r\,\omega \ .}$$

EXAMPLE 3-6

Consider a race car moving around a circular track at 70 m/s. If the radius of the track is 300 meters, what is the car's angular speed as seen from the center of the curve?

$$v_T = r\,\omega \ \rightarrow \ \omega = \frac{v_T}{r} = \frac{70}{300} = 0.23 \text{ radians/second}.$$

DERIVATION 3-XX

Let an object move at some speed around a circular path; the speed does not have to be constant. Consider a particular point in the object's path; let $\vec{r}$ be the position vector for the object, which points from the center of the circle to the object's location, and $\vec{v}$ be the velocity vector, which is tangent to the circle and therefor perpendicular to $\vec{r}$. We can then write that

$$\vec{r} \cdot \vec{v} = 0 \ .$$

Then,

$$\frac{d(\vec{r} \cdot \vec{v})}{dt} = \frac{d\vec{r}}{dt} \cdot \vec{v} + \vec{r} \cdot \frac{d\vec{v}}{dt} = 0 \ ,$$

and so

$$\vec{v}\cdot\vec{v} + \vec{r}\cdot\vec{a} = 0 \quad ,$$

$$v^2 + \vec{r}\cdot\vec{a} = 0 \quad ,$$

Now, $\vec{a}$ can have a radial component (parallel or anti-parallel to $\vec{r}$, positive outward) and a tangential component (perpendicular to $\vec{r}$); it's the first of these we're interested in today. We'll deal with the other later in the course.

$$\vec{a} = \vec{a}_R + \vec{a}_T \quad ,$$

and so

$$v^2 + \vec{r}\cdot(\vec{a}_R + \vec{a}_T) = v^2 + \vec{r}\cdot\vec{a}_R + \vec{r}\cdot\vec{a}_T = 0 \quad .$$

Since $\vec{r}$ and $\vec{a}_T$ are perpendicular, that dot product is zero, and

$$v^2 = -\vec{r}\cdot\vec{a}_R \quad .$$

Since v^2 can't be negative, the vectors $\vec{r}$ and $\vec{a}_R$ must point in opposite directions, *i.e.* the radial acceleration component points towards the center of the circle. Let's rename this the *centripetal acceleration* and make it positive inward so that

$$v^2 = r\,a_C \quad ,$$

$$a_C = \frac{v^2}{r} \quad .$$

For our purposes, centripetal means 'toward the center.'

To summarize, an object moving in a circle experiences an acceleration component toward the center of the circle with a magnitude equal to the square of its speed divided by the radius of the circle. Note that we have not made any claims regarding the tangential acceleration component.

EXAMPLE 3-7

What is the acceleration of a car that starts from rest and attains a speed of 35 m/s while traveling in a straight line for 100 m? What is the acceleration of a car travelling at a constant 35 m/s while driving in a circle of radius of 100 m? In which case do you think the tires would be more likely to slip?

Let's let the direction of motion be along the x-axis. From Section 2,

$x_i = 0$ m
$x_f = 100$ m
$v_{xi} = 0$ m/s (starts from rest)
$v_{xf} = 35$ m/s

$a_x = ? \leftarrow$
$t = ?$

Looks like KEq 4 may work.

$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{35^2 - 0^2}{2(100 - 0)} = 6.1\ m/s^2\,.$$

For the circular motion,

$$a_C = \frac{v^2}{r} = \frac{35^2}{100} = 12.3\ m/s^2\,.$$

We might well assume that the situation with the higher acceleration would be the one more likely to have the tires slip.

EXAMPLE 3-8

Suppose you're on a roller coaster with a loop-de-loop of radius 45 m. As you go over the top while upside-down, you notice that your bottom has just barely lost contact with your seat. How quickly is the roller coaster car moving at the top of the loop?

If there is no other agency than gravity acting on you at that point, your acceleration will be 10 m/s^2 downward, which at this point is toward the center of the circle. Then,

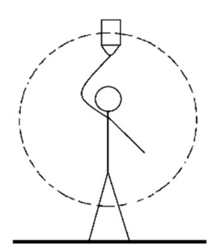
$$a_C = \frac{v^2}{r} \quad \rightarrow \quad v = \sqrt{a_C r} = \sqrt{10(45)} = 21.2\frac{m}{s}\,.$$

EXERCISE 3-5

Suppose that the moon were a perfect sphere of radius 1740 km. The gravitational field strength g_{MOON} on the surface of the moon is about 1/6 that at the surface of the earth (We know this because we've been there.). How quickly would you need to launch a satellite so that it just skims along the surface of the moon?

HOMEWORK 3-7

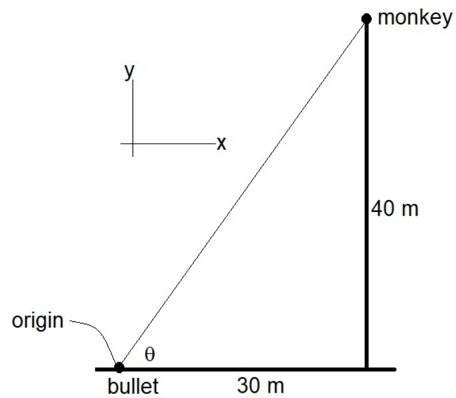
This was a demonstration when I took PHYS I. The professor took a pail of water and swung it in a vertical circle with the intent that the water would stay in the bucket, even when the bucket was inverted. That actually didn't work out well for him. What is the <u>minimum</u> number of revolutions *per* second necessary for Professor Buechner to stay dry?



We shall return for further discussion of centripetal acceleration in a later section.

EXERCISE 3-1 Solution

The idea here is that we need to show that, at some point, the bullet and the monkey are in the same place at the same time. The angle theta will be arctan(40/30) = 53°. We have two objects, and so we need a corresponding number of kinematic equations. The monkey is a bit easier, so let's do that first.



Monkey

$x_{Mi} = +30$ m	$y_{Mi} = +40$ m
$x_{Mf} = +30$ m	$y_{Mf} = y_{Af} = ?$
$v_{Mxi} = 0$ m/s	$v_{Myi} = 0$ m/s
$v_{Mxf} = 0$ m/s	$v_{Myf} = ?$
$a_{Mx} = 0$ m/s^2	$a_{My} = -10$ m/s^2
$t = ?$	

Arrow

$x_{Ai} = 0$ m	$y_{Ai} = 0$ m
$x_{Af} = +30$ m	$y_{Af} = y_{Mf} = ?$
$v_{Axi} = v_o \cos(\theta) = 35 \cos(53°) = +21$ m/s	$v_{Ayi} = v_o \sin(\theta) = 35 \sin(53°) = +28$ m/s
$v_{Axf} = +21$ m/s	$v_{Ayf} = ?$
$a_{Ax} = 0$ m/s^2	$a_{Ay} = -10$ m/s^2

We can easily find the time required for the arrow to travel 30 m horizontally by using KEq. 3:

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$30 = 0 + 21t + 0\,t^2 \quad \rightarrow \quad t = \frac{30}{21} = 1.43 \text{ seconds} \ .$$

At this time, both the monkey and the bullet are at x = +30 m. Now at that same time, are they at the same altitude? For the monkey,

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 = 40 + 0\,(1.43) + \tfrac{1}{2}(-10)(1.43)^2 = 29.8 \text{ m} \ .$$

For the bullet,

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 = 0 + 28\,(1.43) + \tfrac{1}{2}(-10)(1.43)^2 = 29.8 \text{ m} \ .$$

And so, yes, the arrow hits the monkey anyway.

EXERCISE 3-2 Solution

If the time is still four seconds, then the building is still 80 m tall.

$$\frac{v_{yf}}{v_{xf}} = \tan\theta = \tan(-53^o) = -1.33 \quad \rightarrow \quad v_{xf} = -0.75\, v_{yf} \ .$$

And, since the final x velocity is the same as the initial, KEq. 1 tells us that

$$v_{xi} = v_{xf} = -0.75\, v_{yf} = -0.75\left(v_{yi} + a_y t\right) = -0.75\left(0 + (-10)(4)\right) = 30 \text{ m/s} .$$

Lastly, x_f is given by KEq. 3 as

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 = 0 + 30(4) + 0 = 120 \text{ m} \ .$$

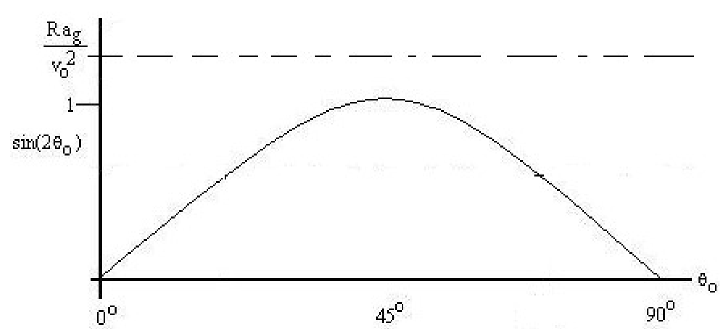
EXERCISE 3-3 Solution

The problem meets the conditions for using the Range Equation, so let's go for it.

$$R = \frac{v_o^2 \sin(2\theta_o)}{|\, a_g\, |}$$

$$\theta_o = \tfrac{1}{2} arcsin\left(\frac{R|\, a_g\, |}{v_o^2}\right) = \tfrac{1}{2} arcsin\left(\frac{350 \times 10}{55^2}\right) = \tfrac{1}{2} arcsin(1.16) = (ERROR)$$

How many times did you retry taking the arcsine? You didn't make a mistake. What angle has a sine of 1.16? Graphically, you're trying to find the intersection of these two curves, and it isn't happening. The physical interpretation of this is that it is impossible to hit the target under these conditions.



EXERCISE 3-4 Solution

For this, $\alpha = \omega t$ and $\beta = 90^o$. Then,

$$\cos(\omega t + 90^o) = \cos\omega t \cos 90^o - \sin\omega t \sin 90^o = \cos\omega t\ (0) - \sin\omega t\ (1) = -\sin\omega t \ .$$

$$\sin(\omega t + 90^o) = \sin\omega t \cos 90^o + \cos\omega t \sin 90^o = \sin\omega t\ (0) + \cos\omega t\ (1) = \cos\omega t \ .$$

EXERCISE 3-5 Solution

The acceleration of an object at the earth's surface is 10 m/s^2 towards the earth's center. If the moon's gravity is the only agency acting on the satellite, then we might assume that this satellite's acceleration will be $a_g = 10/6 = 1.7$ m/s^2 downward, towards the center of its circular orbit. Don't forget to convert the moon's radius into meters:

$$1740\ \mathrm{km}\ \times \frac{1000\ \mathrm{m}}{1\ \mathrm{km}} = 1.74\ \times 10^6 \mathrm{m}\ .$$

Then,

$$a_C = \frac{v^2}{r} \quad \rightarrow \quad v = \sqrt{a_C r} = \sqrt{1.7(1.74 \times 10^6)} = 1720 \frac{\mathrm{m}}{\mathrm{s}}\,.$$