SECTION 4 – RELATIVE MOTION

Relative Velocities

On occasion, it is useful to consider the motion of an object with respect to an origin/coördinate system which is itself in motion relative to some third reference frame. A simple example is that of the 'people mover' at the airport, a giant conveyor belt that carries weary passengers along the length of the concourse, while also providing those in a rush a little extra speed as they run down the walkway. For example, consider such a walkway (W) which moves with a velocity of +2 m/s with respect to the ground (G). We'll represent this with this notation: $v_{W,G}$ = +2 m/s; the first letter indicated which object we're examining, and the second what it is moving 'with respect to.' Now, think of a person (P) walking in the same direction at +1 m/s along the walkway: $v_{P,W}$ = +1 m/s. I don't think anyone would argue that the person's velocity relative to the ground is +3 m/s, which would imply that

$$\vec{v}_{P,G} = \vec{v}_{P,W} + \vec{v}_{W,G} \ .$$

Let's test this for some other scenarios. Suppose the person were to walk the wrong way on the walkway at -1 m/s. Then

$$\vec{v}_{P,G} = \vec{v}_{P,W} + \vec{v}_{W,G} = -1 + 2 = +1 \text{ m/s} .$$

The person would still be going in the same direction as before, although more slowly. Still happy?

One of the hardest aspects of relative velocity is to determine which two quantities get added to obtain the third. Let's look more closely at the notation. Two of the velocity terms have the P in the first position and two have the G in the second position. But, one has the W in the first position and the other has the W in the second position. It's the ones with the letters in different positions that are added to obtain the third. If you do it correctly, the outer subscript letters on the right side will match the subscript letters on the left side.

Here's another useful fact: $\vec{v}_{A,B} = -\vec{v}_{B,A}$. As an example, suppose that I'm driving I-83 to York, Penna at 120 kilometers *per* hour (kph) and pass someone parked on the shoulder. That person sees me going northward at 120 kph and himself as stationary, but I see myself as stationary and see him moving southward at 120 kph.

EXAMPLE 4-1

> An airplane with an *airspeed* of 400 kph files eastward with an 85 kph tailwind. What is the *ground speed* of the plane?
>
> Well, here we've introduced a couple of terms that may need explaining. Airspeed is the speed of the plane as measured relative to the air, and of course the ground speed is measured relative to the ground. The wind is measured relative to the ground, and a tailwind moves in the same direction as the plane.

We have three objects to consider: the plane; the ground; and the air. Let eastward be the positive direction. Then,

$v_{P,G} = ? \leftarrow$
$v_{P,A} = +400$ kph
$v_{A,G} = +85$ kph

We note that it is the A that changes position, so we write that

$$\vec{v}_{P,G} = \vec{v}_{P,A} + \vec{v}_{A,G} \ .$$

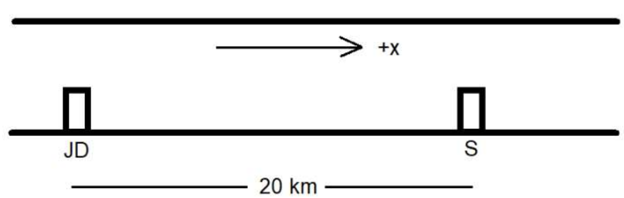
The solution is straightforward:

$$\vec{v}_{P,G} = \vec{v}_{P,A} + \vec{v}_{A,G} = 400 + 85 = 485 \text{ kph}.$$

EXERCISE 4-1

Back on the 'people mover,' Person A walks the correct way at a speed of 1 m/s, while Person B walks at 2 m/s the correct way on the return walkway. What is the relative speed between the brothers? Assume each walkway moves at 2 m/s relative to the ground.

EXAMPLE 4-2



Consider a river flowing in a straight course at 8 kph. Joe has a dock (labeled JD) and want to make a run to the store (S) 20 km downstream for 'supplies.' If the boat can travel 12 kph in still water, how long will it take Joe to make a round trip? We'll assume his order is already waiting for him, so he can immediately turn around.

There are again three objects to worry about: the boat; the water; and the ground. But also there are two parts to the problem: there (I) and back again (II). The velocities are $v_{B,G}$, $v_{B,W}$, and $v_{W,G}$. Since we want the time for each of these trips, we need to work in the displacement. So,

$$\vec{v}_{B,G} = \frac{\Delta\vec{x}_{B,G}}{t} = \vec{v}_{B,W} + \vec{v}_{W,G} \quad \rightarrow \quad t = \frac{\Delta x_{B,G}}{v_{B,W} + v_{W,G}} \ .$$

Let's make downstream the +x direction and use the notation of Sections Two and Three, *i.e.*, let the sign of the value indicate the direction.

On the way downstream:

$v_{B,W} = +12$ kph
$v_{W,G} = +8$ kph
$\Delta x_{B,G} = +20$ km

$$t_I = \frac{\Delta x_{B,G}}{v_{B,W} + v_{W,G}} = \frac{+20}{+12+8} = 1 \text{ hour} \ .$$

On the way upstream:
$v_{B,W} = -12$ kph
$v_{W,G} = +8$ kph
$\Delta x_{B,G} = -20$ km

$$t_{II} = \frac{\Delta x_{B,G}}{v_{B,W} + v_{W,G}} = \frac{-20}{-12+8} = 5 \text{ hours} \ .$$

This is then a total of 6 hours.

HOMEWORK 4- 1

An escalator is 20 m long. If a person simply stands on the 'up' side, it takes 30 seconds to ride to the top. If a person walks up the escalator at a speed of 0.6 m/s relative to the escalator, how long will it take him to get to the top? If the same person walks down the 'up' side at the same relative speed as before, how long will it take him to arrive at the bottom?

HOMEWORK 4-2

An airplane is to fly from City A due west to City B to pick up cargo, then return to City A. It will take exactly one hour to load the plane at B, and this entire trip should be done in the shortest time possible. The plane has a maximum airspeed of 300 kph, and encounters an 80 kph westerly wind[1] for the entire trip. If the distance between A and B is 1800 km, how long does the entire trip require?

Well, that was one dimensional motion. Let's move on to two dimensional problems. We saw back in Section One that if

$$\vec{C} = \vec{A} + \vec{B} \ ,$$

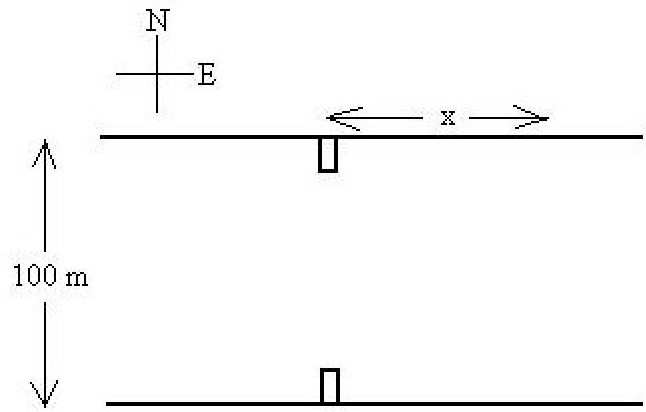
then

$$C_x = A_x + B_x \quad \text{and} \quad C_y = A_y + B_y \ ,$$

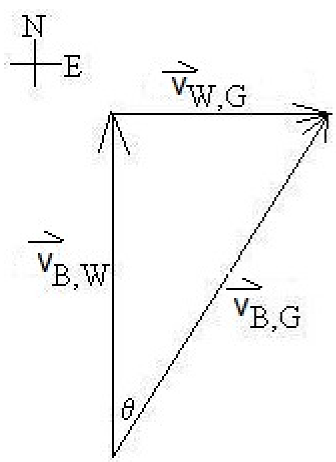
and we can treat a two dimensional problem as two one dimensional problems.

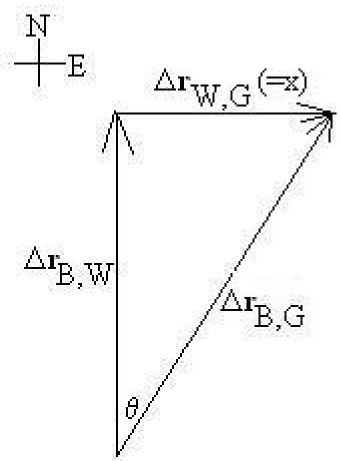
[1] A westerly wind blows from the west, towards the east.

EXAMPLE 4-3



Consider a boatman who wishes to cross a river (100 metres wide) from one dock north to another exactly opposite. His boat will make 10 m/s in calm water. The velocity of the water is 8 m/s eastward. He aims his boat exactly northward and sets off. How far downstream (x) will he actually land, in what compass direction did he actually travel, and how long will it take him to get there?





We have three objects to consider: the boat; the ground; and the water. The velocity equation is then

$$\vec{v}_{B,G} = \vec{v}_{B,W} + \vec{v}_{W,G} \ .$$

In this case, the quickest solution may be to realize that, if the velocities are all constant, a displacement component diagram can be constructed where each term is parallel to the corresponding velocity term. The two triangles so formed are then similar, and so there is a proportionality of the lengths of the sides:

$$\frac{x}{100\ \text{m}} = \frac{8\ \text{m/s}}{10\ \text{m/s}} \quad \rightarrow \quad x = \frac{100\ (8)}{10} = 80\ \text{m}\ .$$

The direction traveled can be found using the tangent of the angle θ:

$$\tan\theta = \frac{8}{10} = 0.8 \quad \rightarrow \quad \theta = 38.7^{o}\ .$$

For the time, we consider that the motion northward (in this case) is independent of the motion eastward; it would take 10 seconds to cover 100 meters at 10 m/s. Or, it would take 10 seconds to cover 80 meters at 8 m/s.

EXERCISE 4-2

Now, suppose that, having learned his lesson, he tries again to cross directly to the other side. In what direction should he aim his boat (relative to north) to arrive exactly at the other dock, and how long will it take him?

HOMEWORK 4-3

A plane needs to leave City A on time and arrive at City B on time exactly seven hours later. City B is 1500 km due east of City A. There is a southerly wind blowing at 120 kph. With what airspeed and in what direction should the pilot head the plane?

HOMEWORK 4-4

A plane needs to leave City A on time and arrive at City B on time exactly eight hours later. City B is 2500 km due east of City A. There is a wind blowing at 120 kph toward 37 degrees west of north. With what airspeed and in what direction should the pilot head the plane?

Relative Accelerations*

We've just discussed relative velocities, and of course we introduced vector addition as 'relative displacements,' so is there such a thing as *relative acceleration*? Well, you betcha.

EXAMPLE 4-4

A heavy two-meter stick (S) with a vertical orientation is dropped from rest over a tall cliff. At that same instant, an ant (A) starts to accelerate up the stick from the bottom end. From the markings on the meterstick and on his wristwatch, he sees that he covers 0.8 meters in 1.1 seconds. At that time, how far has the ant fallen with respect to the cliff face (C)?

Let upward be positive x. The ant's acceleration relative to the stick ($a_{A,S}$) is found from KEq 3:

$x_{ASi} = 0$ m
$x_{A,Sf} = 0.8$ m
$v_{A,Si} = 0$ m/s
$v_{A,Sf} = ?$
$a_{A,S} = ?$ ←
$t = 1.1$ sec

$$x = x_i + v_i t + \frac{1}{2}at^2$$

$$a_{A,S} = 2\frac{x - x_i - v_i t}{t^2} = 2\frac{0.8 - 0 - 0(1.1)}{1.1^2} = 1.32 \text{ m/s}^2 \quad .$$

The acceleration of the meter stick[2] is -10 m/s^2. Then, the acceleration of the ant with regard to the cliff face is

$$\vec{a}_{A,C} = \vec{a}_{A,S} + \vec{a}_{S,C} \ .$$

$$a_{A,C} = 1.32 + (-10) = -8.68 \ \text{m/s}^2 \ .$$

Then,

$x_{ACi} = 0$ m
$x_{A,Cf} = ?$ ←
$v_{A,Ci} = 0$ m/s
$v_{A,Sf} = ?$
$a_{A,S} = -8.68$ m/s^2
$t = 1.1$ sec

$$x = x_i + v_i t + \frac{1}{2} a t^2$$

$$x_{A,C} = 0 + 0(1.1) + \frac{1}{2}(-8.68)(1.1^2) = -5.25 \text{ m} \ .$$

DISCUSSION 4-1

Suppose that a passenger is sitting in a train waiting at the station. You are standing on the platform spying on him. Suddenly, a frog jumps straight upward from the floor of the train car. What shape path will the frog seem to follow as seen by you? As seen by the passenger?

Suppose that a passenger is sitting in a train that is traveling through the station at constant velocity. You watch as the train passes. Suddenly, a frog jumps as before. What shape path will the frog seem to follow as seen by you? As seen by the passenger?

Suppose that a passenger is sitting in a train that is traveling through the station with a constant acceleration a_T, specifically the train is speeding up. You watch as the train passes. Suddenly, a frog jumps as before. What shape path will the frog seem to follow as seen by you? As seen by the passenger?

EXERCISE 4-3*

Show that the path of the frog as seen by the accelerating passenger is parabolic and is consistent with the notion of relative accelerations. The most general form of the equation of a parabola is

[2] Well, it's actually just a little bit higher; we'll deal with that in a later section.

$$A^2y^2 + B^2x^2 - 2(f_x(A^2+B^2) + AC)x - 2\big(f_y(A^2+B^2) + BC\big)y - 2(AB)xy$$
$$+ \Big((A^2+B^2)\big(f_x^2 + f_y^2\big) + C^2\Big) = 0 \quad ,$$

where (f_x, f_y) are the coördinates of the focus and the line given by

$$Ax + By + C = 0 \quad \rightarrow \quad y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$$

is the *directrix*. For our purposes, we need remember only that the directrix is perpendicular to the axis of symmetry of the parabola.

EXERCISE 4-1 Solution

Here we have five objects to keep track of: Person A, Person A's walkway (AW), Person B, Person B's walkway (BW), and of course the ground. Let's let A's direction be positive x. We then know

$\vec{v}_{A,AW}$ = +1 m/s
$\vec{v}_{AW,G}$ = +2 m/s
$\vec{v}_{B,BW}$ = -2 m/s
$\vec{v}_{BW,G}$ = -2 m/s

Let's do each person relative to the ground, as we did in the example.

$$\vec{v}_{A,G} = \vec{v}_{A,AW} + \vec{v}_{AW,G} = +1 + (+2) = +3\ \mathrm{m/s}\,.$$

$$\vec{v}_{B,G} = \vec{v}_{B,BW} + \vec{v}_{BW,G} = (-2) + (-2) = -4\ \mathrm{m/s}\,.$$

We want to know $v_{A,B}$. Be careful.

$$\vec{v}_{A,B} = \vec{v}_{A,G} + \vec{v}_{G,B} = \vec{v}_{A,G} - \vec{v}_{B,G} = (+3) - (-4) = +7\mathrm{m/s}\,.$$

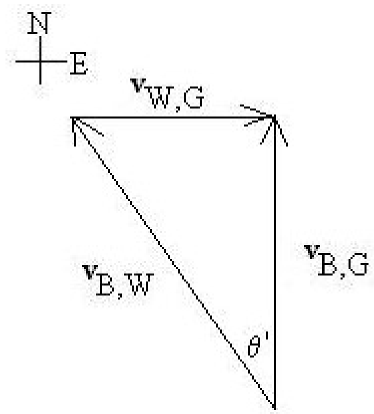
We actually could have done this in one go:

$$\vec{v}_{A,B} = \vec{v}_{A,AW} + \vec{v}_{AW,G} + \vec{v}_{G,BW} + \vec{v}_{BW,B} = \vec{v}_{A,AW} + \vec{v}_{AW,G} - \vec{v}_{BW,G} - \vec{v}_{B,BW}$$
$$= +1 + 2 - (-2) - (-2) = 7\mathrm{m/s}\,.$$

EXERCISE 4-1 Solution

The common error here is simply to flip the triangle over. But what should be done is to deform the triangle by sliding $\mathbf{v}_{W,G}$ over until the sum, $\mathbf{v}_{B,G}$, is pointing due north:

Before, the two short sides of the right triangle were $\mathbf{v}_{B,W}$ and $\mathbf{v}_{W,G}$, but now those vectors are the <u>hypotenuse</u> and a short side, respectively. So,



$\theta' = \arcsin(8/10)$ and $|\mathbf{v}_{B,G}| = 10\cos\theta'$.

Finish the calculations yourself.

EXERCISE 4-3 Solution

At the moment of launch, the frog is moving forward (say, the +x direction) at speed v_T because of the motion of the train, and upward at speed v_U. Once the frog leaves the floor, the only agency acting on it is gravity, and we've already worked out the trajectory for your point of view in Section Three:

$$y = 0 + v_o\sin\theta_o \frac{x}{v_o\cos\theta_o} + \tfrac{1}{2}a_g\left(\frac{x}{v_o\cos\theta_o}\right)^2 = \frac{v_U}{v_T}x + \frac{a_g}{2v_T^2}x^2 \ .$$

However, since the passenger is accelerating with respect to the frog, he sees (or thinks he sees) the frog accelerating backwards, since $\vec{a}_{B,M} = -\vec{a}_{M,B}$. To find the trajectory as seen by the passenger, we need to start from scratch.

$x_i = 0$ (why not?)	$y_i = 0$ (same here)
$x_f = ?$	$y_f = ? \leftarrow$
$v_{xi} = 0$ (as seen by the man)	$v_{yi} = v_U$
$v_{xf} = ?$	$v_{yf} = ?$
$a_x = -a_T$	$a_y = a_g$
$t = ?$	

Once again, we'll start with the x-side and find the time:

$$x = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$x = 0 + 0 + \tfrac{1}{2}(-a_T)t^2 \quad \rightarrow \quad t = \sqrt{\frac{2x}{-a_T}} \ .$$

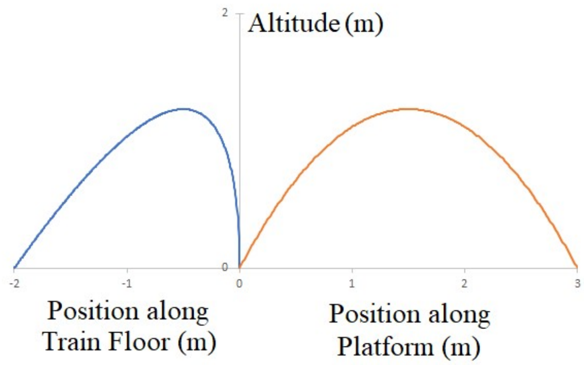
This result is O.K. because x will become negative as the frog 'falls behind' the passenger. Now to the y-side and to substitute in for the time:

$$y = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

$$y = 0 + v_U\sqrt{\frac{2x}{-a_T}} + \frac{a_g}{-a_T}\,x \ .$$

$$y + \frac{a_g}{a_T}x = v_U\sqrt{\frac{2x}{-a_T}} \ .$$

Here is a sample curve for some representative values of $a_T = 4$ m/s^2 , $v_{iT} = 3$ m/s, and $v_U = 5$ m/s for the trajectory of the frog as seen from the passenger's point of view and from your point of view standing on the platform.



Square both sides to get

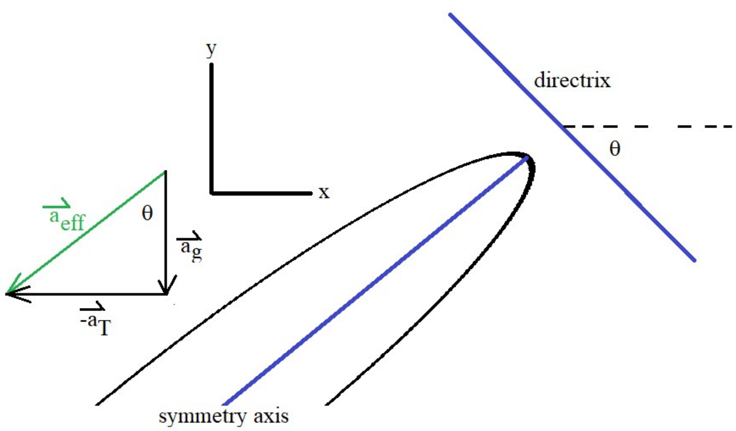
$$y^2 + 2\,\frac{a_g}{a_T}\,xy + \left(\frac{a_g}{a_T}\right)^2 x^2 + \frac{2v_U^2}{a_T}\,x = 0 \ .$$

First, is this a parabola? This does match the general form of the equation for a parabola as given.[3] The difference is that its axis of symmetry is not vertical but tilted. Continuing,

$$A^2y^2 + B^2x^2 - 2(f_x(A^2+B^2) + AC)x - 2\big(f_y(A^2+B^2) + BC\big)y - 2(AB)xy + \Big((A^2+B^2)\big(f_x^2+f_y^2\big) + C^2\Big) = 0 \ .$$

By comparing the coëfficients of each power of x and y here with those in our trajectory solution, we can determine the values of A, B, C, f_x, and f_y. Luckily, we need only A and B today:

$$A = 1 \ , \ \ B = -\,\frac{a_g}{a_T} \ .$$



The directrix is then a line with a slope of

$$\frac{-A}{B} = -\frac{-a_T}{a_g} = \frac{a_T}{a_g},$$

and that indicates a parabola with its symmetry line tilted from the y-direction by an angle θ with tangent $(-a_T/a_g)$.

How might we interpret this? The passenger would observe the direction of acceleration of the

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[3] Strictly speaking, we showed that the square of the trajectory function is a parabola; since that equation puts the same restrictions on the values of x and y as the original (there are no sign changes), that too is parabolic.

flying frog to be at an angle θ from the horizontal, but he also sees that the free-fall acceleration magnitude would be given by

$$a_{eff} = \sqrt{a_T^2 + a_g^2} \ .$$

For the specific example above, the gravitational acceleration would seem to be directed at an angle of 22° 'behind' the vertical and be 10.8 m/s^2.