SECTION 6 – THE SECOND PICTURE

We've looked at the motions of objects using forces and accelerations, and if we were lucky enough to have constant accelerations, the kinematic equations. Now, we'll introduce a second picture which we may, or may not, find more convenient to use on certain classes of problems. Please note that this new picture is really nothing more than Newton’s second law with a few definitions thrown in; there is a tendency for students to thjnk of this material as disconnected from previous discussions, but it really just a re-arrangement of stuff you already know.

DISCUSSION 6-1

Consider a small toy car sitting on a table at a spot marked ‘X’; we’ll assume the wheels make its contact with the table frictionless. Observe the car closely. Now, observe the car as it travels through point X. Is it fair to say that the car possesses some quality or property in the latter case which it lacks in the former? How did the car acquire that property?

CHEESY EXPERIMENT 6-1 ~~VIDEO~~

After the experiment, we concluded/agreed on the following:

- We agreed that there is some quality the object possesses when it’s moving through X that it lacks when it’s stationary. For want of a better word, let's call that quality *energy* (E).
- Energy is transferred into the object by applying a force. However, the force must act through a displacement. Applying a force to a non-moving object transfers no energy.
- Transferring energy into (or out of) an object is a process; let us call the transfer of energy the *work* (W) done on the object. Work is not a form of energy, it is the transfer of energy. Let’s define the work on an object to be positive when energy enters the object and negative when it is removed (why not?)
- The bigger the force, the more energy is transferred: as F↑, W↑. We might even speculate that W is proportional to F. That would certainly be the simplest relationship consistent with our observations. We could be wrong, of course; perhaps $W \sim F^2$ or F^3. We’ll make the simplest assumption and see if there is a contradiction somewhere in our subsequent experiments.
- The greater the displacement over which the force acted, the more work is done: that is, as Δx ↑,W↑ .We might speculate that W is proportional to Δx.

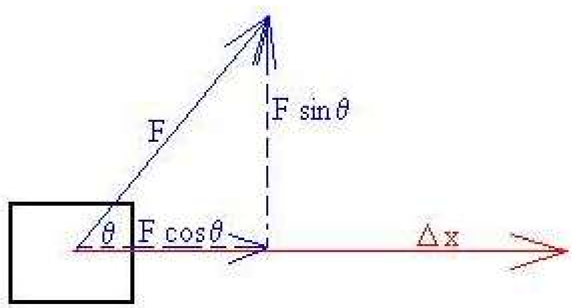
What's more, there is an effect due to the relative orientation of the force with the displacement. We saw that:

- If $\vec{F}$ and $\Delta\vec{x}$ are in the same direction, energy is transferred into the object and we say that positive work was done.
- If $\vec{F}$ and $\Delta\vec{x}$ are in the opposite directions, energy is transferred out of the object and we say that negative work was done.
- If $\vec{F}$ and $\Delta\vec{x}$ are perpendicular, no energy is transferred into the object and we say that no work was done.

DISCUSSION 6-2

> How can we express these last notions in a more mathematical way? Can you think of a function that will give us a positive value when two vectors are parallel, a negative value when they are anti-parallel, and zero when they are perpendicular?

Let's consider an object moving along, say, the x-axis while a force is applied at some angle away from the x-axis, as shown in the figure. When we talked about vectors and components, we said that the components of a vector can replace the original vector. Let's do so for this force. The component parallel to the displacement is F cos θ and the component perpendicular is F sin θ. The former should contribute to the work, while the latter does not. Sounds like just what we need. Indeed, if the angel were greater than 90°, the cosine would provide the negative sign required when the force and displacement are in generally opposite directions.



Let's synthesize these notions into a single mathematical expression, with the assumption that the universe works as simply as possible:

$$W = F\,\Delta x\,\cos\theta_{F,\Delta x} = \Delta E\,.$$

The unit for work is newtons times meters; we will define one *joule* (J) as the work done by one newton of force acting on an object while it displaces one meter in the same direction. This procedure will increase the energy of the object by one joule.[1]

Now, since the result for the work doesn't depend on the actual directions of the force or the displacement, but only on their relative directions, we might guess that the work is a scalar quantity. We'll confirm this in a page or so. As such, the work can be written as[2]

$$W = \vec{F} \cdot \Delta\vec{x}\ .$$

EXAMPLE 6-1

> Consider a box pulled 4 meters along the flat ground by a rope with tension 58 newtons which is at an angle of 54° above the horizontal. How much work does this force do?
>
> The diagram for this is close to the one above. The work would be

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[1] I like to use a bank account as an analogy. Work is like the deposits and withdrawals, while the amount of energy is like the balance. If there is a deposit of $19, the balance increases by $19.

[2] Revisit Section One to review the dot product of two vectors.

$$W = F\,\Delta x\,\cos\theta_{F,\Delta x} = 58\,(4)\cos(54^o) = +\,136.4\text{ J} \;.$$

HOMEWORK 6-1

A cowboy grabs a rope trailed by a runaway horse and applies a force of 1100 N as he is dragged 37 meters. How much work does the cowboy do on the horse? How much work does the horse do on the cowboy? How do the answers to this question depend on whether the horse stops, slows, or keeps running?

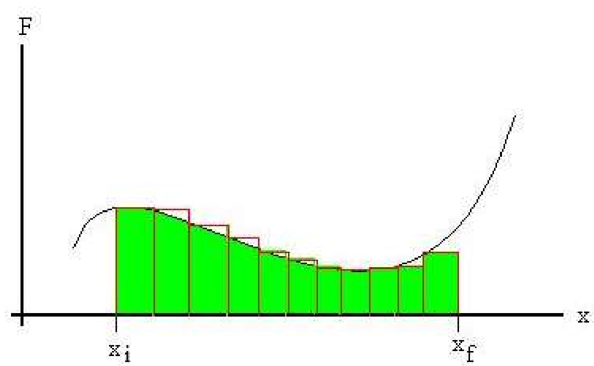
DISCUSSION 6-3

What if several forces act on the object simultaneously? Can you extend the analogy with the back account?

So, presumably, each force applied to the object transfers its own amount of energy. Think of your bank account with a number of deposits and withdrawals. Your balance will change by the total deposits (positive) plus the total withdrawals (negative):

$$W_{TOTAL} = \sum_j W_j \;.$$

Now let's make thigs a little harder. First, what if the force applied were not constant (or, a *variable force*)? Clearly, more work would be done in some displacement intervals than in others.



We need to break the overall displacement down into very many, very small displacements Δx_n, over which we can consider the force to be relatively constant at value $F_{\Delta x\,n}$; we then find the work done over each interval to be (approximately)

$$W_n = F_n\,\Delta x_n$$

so that the total work is

$$W = \sum_n F_{\Delta x_n}\,\Delta x_n \;.$$

To make the approximation more accurate, we'll make the displacements shorter and more numerous*, i.e.*, we'll make the Δx's go to zero:

$$W = \int F(x)\, dx \ .$$

What if the curve were to go below the axis? Then the work would be negative, since a 'negative' force would suggest one in the negative x-direction, opposite to the displacement. What if the displacement were negative? Remember that an integral from a larger limit value to a smaller limit value introduces a negative sign that would also switch the sign of the work.

Next, what if the object moved in three dimensions? The force could be written in components, F_x, F_y, and F_z. F_x would make no work contribution due to movement in the y or z directions (the force would be perpendicular to each of those displacements), F_y would make no contribution due to movement in the x or z directions, and F_z would make no contribution due to movements in the x or y directions, Therefore,

$$W = \sum_n F_x\,\Delta x + F_y\,\Delta y + F_z\,\Delta z \ \rightarrow \int F_x(x,y,z)\, dx + F_y(x,y,z)\, dy + F_z(x,y,z)\, dz$$

$$= \int \vec{F}(\vec{r}) \cdot d\vec{r} \ .$$

At this point, we're in a strange position. We've defined the work, but we can define anything we please to be whatever we please; what's the point? The definition is meaningful only if it is useful. Let's think back to the beginning of this discussion. We talked about the work as a transfer of energy, so that we should be able to say that the work corresponds to the <u>change</u> in the amount of this energy stuff that the object possesses, *i.e.*, W = ΔE. Keep this in mind as we do an important derivation:

DERIVATION 6-1

Consider the two definitions below from kinematics, where the acceleration is <u>not</u> constant:

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \vec{v} = \frac{d\vec{r}}{dt} \ .$$

Let's once again perform a dot product multiplication on each side to obtain

$$\vec{a} \cdot \frac{d\vec{r}}{dt} = \vec{v} \cdot \frac{d\vec{v}}{dt} \ .$$

$$\vec{a} \cdot d\vec{r} = \vec{v} \cdot d\vec{v} = v\, dv \ .$$ [3]

[3] We explained this dot product result back in Section Two.

We have usually assumed that the acceleration is a function of time, but it might just as well be a function of position instead.

$$\vec{a}(\vec{r}) \cdot d\vec{r} = v\,dv\,.$$

Let's make use of Newton's second law, $\sum \vec{F}_n = m\vec{a}$, to obtain

$$\vec{a}(\vec{r}) \cdot d\vec{r} = \frac{\sum \vec{F}_n(\vec{r})}{m} \cdot d\vec{r} = \frac{\sum \vec{F}_n(\vec{r}) \cdot d\vec{r}}{m} = v\,dv\,.$$

$$\sum_n W_n = \int_{\vec{r}_i}^{\vec{r}_f} \sum_n \vec{F}_n(\vec{r}) \cdot d\vec{r} = m \int_{v_i}^{v_f} v\,dv\,.$$

$$\sum_n W_n = \sum_n \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_n(\vec{r}) \cdot d\vec{r} = m\ \frac{v^2}{2}\Big|_{v_i}^{v_f}.$$

$$\sum_n W_n = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = \Delta\left(\tfrac{1}{2}mv^2\right)\ .$$

$$W_{TOTAL} = \Delta\left(\tfrac{1}{2}mv^2\right)\ .$$

Since we have previously asserted that work corresponds to a change in energy, we might jump to the conclusion that

$$E = \tfrac{1}{2}mv^2\ .$$

This is a little dangerous; just because two quantities have the same change in value doesn't mean that they have the same value. For example, there could be some constant term included in the energy that cancels out when calculating the change.[4] However, we have previously decided to go with the simplest explanations, until a contradiction is found. Historically, this was the definition of energy, but as we proceed through this section, we will introduce notions of other types of energy. Seeing as our object possesses this energy due to its motion, let's define this specifically to be the *kinetic energy*, K:

$$K = \tfrac{1}{2}mv^2\ .$$

DISSCUSSION

Is kinetic energy a scalar or vector quantity?

So, we finally end at the basic concept of the Second Picture, the *work-energy theorem*:

[4] Perhaps $E = ½\,mv^2 + mc^2$. The second term will always disappear when ΔE is calculated.

$$W_{TOTAL} = \Delta K.$$

Well, this is all well and good, but remember that it was based on some speculation about the specific manner in which energy is transferred *via* work. What we're going to do, as we did for the second law, is make some predictions about the real world and see if there are any inconsistencies with our notions.

Note that the work energy theorem is nothing more than Newton's second law mixed up with several of the definitions from kinematics, and as such, it is fundamentally the same as that law, in spite of its very different appearance. What we will find is that this formulation will be very useful for certain classes of problems that would have been very difficult to solve using NII and kinematics directly, particularly when the acceleration is not constant.

Net force	causes	change in velocity
Net work	causes	change in kinetic energy
?	causes	change in ?

HOMEWORK 6-2

At what speed would a 50 kg person have to run to have the same kinetic energy as a 2000kg auto traveling at 100 km/h?

DISCUSSION 6-4

Suppose a jogger of mass M is trotting along at speed v_o, and therefor has kinetic energy K_o. What kinetic energy (in terms of K_o) would he have if he doubled his pace? If his daughter with half his mass then runs beside him, how much kinetic energy would she have?

EXAMPLE 6-2

Throw a ball upward with an initial speed of 12 m/s. How high does it rise (H)?

Let's use the WE thm:

$$W_{TOTAL} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \ .$$

When we used Newton's second law, we put in all of our effort on the left side finding the forces, but the right side was always $m\vec{a}$. Here, we again put all the effort in on the left side finding the works, and the right side is always ΔK.

The only force acting on the ball is its weight, gm, downward. The displacement is H upward, so our angle between the force and the displacement is 180°. The work done is therefor

$$W_g = (gm)(H) \cos(180^o) = -gmH$$

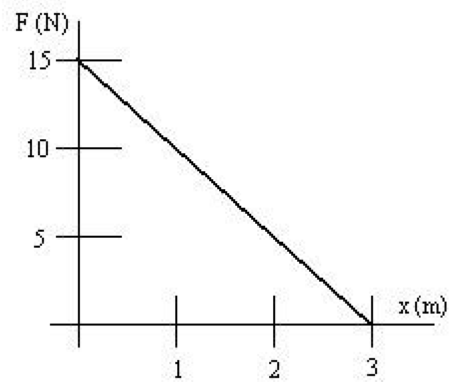
The ball stops at its highest altitude, so $v_f = 0$. Then,

$$-gmH = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \ .$$

$$H = \frac{v_i^2 - v_f^2}{2g} = \frac{12^2 - 0^2}{2(10)} = 7.2 \text{ m} \ .$$

HOMEWORK 6-3

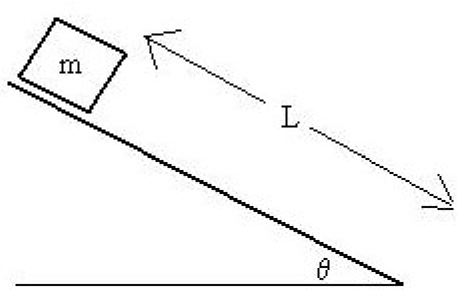
A 3 kg object initially at rest is acted on by a non-constant force which causes it to move 3 m. The force varies with position as shown in the graph.



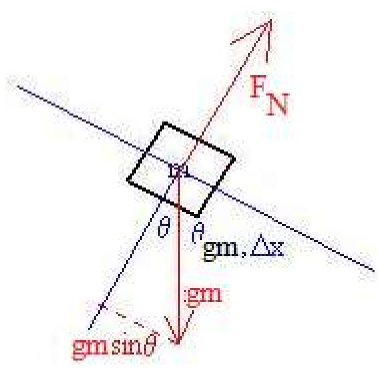
a) How much work is done on the object by this force?

b) What is the final speed of the object as it arrives at x = 3 m? Assume that the given force is the only force acting on the mass.

EXAMPLE 6-3



Here's a problem we've seen before to compare the solution methods of Picture One and Picture Two. Consider a block of mass m = 5 kg at the top of a frictionless ramp L = 2 meters long that is inclined at θ = 37° to the horizontal. If the mass starts from rest at the top, how quickly will it be moving when it reaches the bottom? The answer better be 4.9 m/s.



Draw a free-body diagram; the weight and a normal force are the only forces. One thing we don't need to do is choose a coördinate system. Everything is relative to the direction of the displacement. We'll use the WE theorem,

$$W_{TOTAL} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \ .$$

Let's look at the works:

$W_N = 0$, since the force is perpendicular to the displacement;

$W_g = (mg)\,(L)\,(\cos\theta_{mg,\Delta x})$.

What angle should we use for W_g? It's not 37°! We want the angle between the force and the displacement, 53°.

$$W_N + W_g = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2$$

$$0 + W_g = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2$$

$$v_f = \pm\sqrt{v_i^2 + \frac{2W_g}{m}} = \sqrt{v_i^2 + 2gL\cos(\theta)} = \sqrt{0^2 + 2(10)2\cos(53^o)} = 4.9\ \mathrm{m/s}.$$

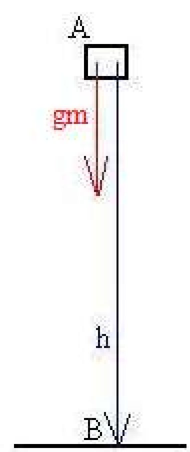
You may well point out that this problem is just as easily done with Newton's second and a kinematic equation, and you'd be correct. Eventually, we'll encounter problems where that will not be true.

DISCUSSION 6-5

Suppose that I drop an object from a given height, such as a pen onto the table. The force of gravity (the object's weight) does work and the kinetic energy of the object increases. Now, suppose instead that I slowly lower the object to the table from the same initial altitude. Compare the work done by gravity in the second case to the work done in the first case. Do you understand the difference between the work done by a force and the total work done by all forces on an object?

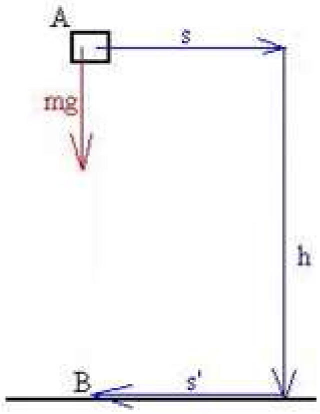
Conservative and non-Conservative Forces

Let's divide the realm of forces in to two categories: *conservative forces* and *non-conservative forces*. This may seem rather facile, in that I could divide forces in to red and non-red categories, and each force would have to fit into one of them. However, this is a distinction which we will find useful. What we find is that for some forces, the work they do on some object moving from any particular point A to any particular point B is independent of the path taken between A and B. We call this type of force a conservative force. There are a number of alternate ways to define what a conservative force is, but they are all equivalent to each other. Any force for which the work can depend on the path is a non-conservative force.



Let's take the weight of an object as a concrete example. Suppose that I lower a mass m from a height h above the table to the top of the table. I'm only interested at this point in what the weight does, not what any other force, such as from my hand, does. The force is gm downward, and the displacement is h downward, and those two vectors are parallel, so we have that

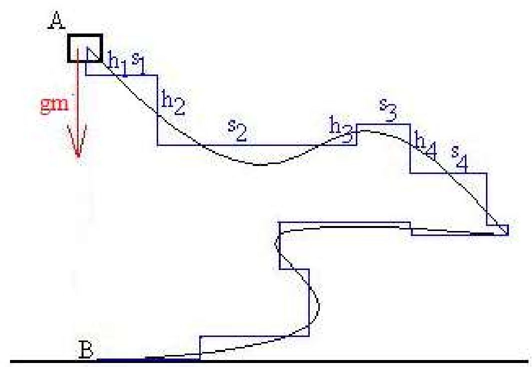
$$W_g = gm\,h\cos(0^o) = gmh.$$



Now, let's take the object on a little tour of the region. Move it horizontally a displacement s, then down h, then horizontally again s, back to point B. The work done will be

$$W_g = gm\, s \cos(90^o) + gm\, h \cos(0^o) + gm\, s' \cos(90^o) = gmh$$

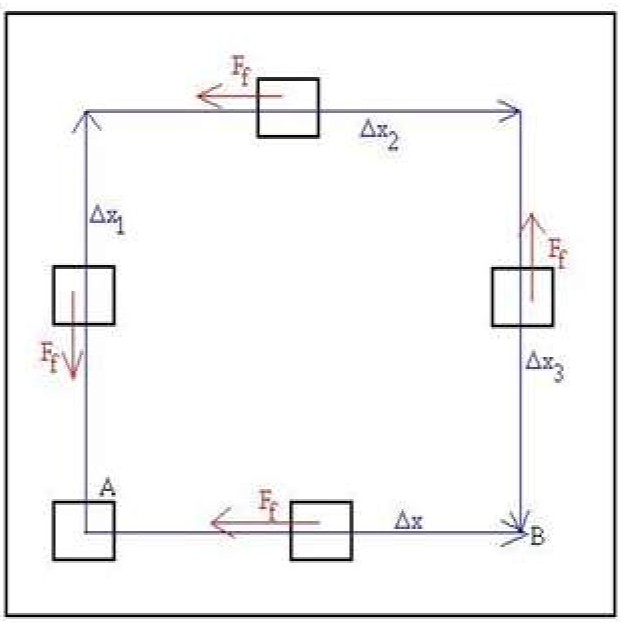
once again.



Let's pick a random path. You might be able to see that we can always approximate any path to an arbitrary degree of accuracy with these stepped horizontal and vertical movements. From previous discussion, we know that any horizontal movements will correspond to no work being done by gravity. The vertical displacements are each of magnitude h_n, some parallel to the weight and some anti-parallel, such that the work done by the weight during each vertical motion is

$$W_g = \sum_n gm\, h_n \cos(\theta_n) = gm \sum_n h_n \cos(\theta_n) \ ,$$

where cos (θ_n) = +1 if the displacement is downward (parallel to the force) and -1 if the displacement is upward (anti-parallel to the force). We realize that the last summation is simply h, so that the work done by the weight is gmh, as before, and work done by the weight throughout the whole trip is indeed independent of the path taken.



Next, let's consider an example of a non-conservative force: friction. Consider an object being slid across a table top along two paths (for simplicity, let all Δx's be the same magnitude). Remember that we are not concerned with the work done by any other force, such as that of the hand that pushes the block. The frictional force will be (not proven here):

$$F_{fK} = \mu_K\,(gm)$$

So that the work done by friction from Point A to Point B along the direct path is

$$W_{f\,\text{Direct Path}} = \mu_K\,(gm)\,\Delta x \ .$$

If instead, the object is pushed along the other three sides of the square, the same amount of work will be done by friction along <u>each</u> of the sides, so that

$$W_{f\,\text{Long Path}} = 3\,\mu_K\,(gm)\,\Delta x \neq W_{f\,\text{Direct Path}}\ .$$

So, we see that friction is <u>not</u> a conservative force. Just as clearly, neither is the force that pushed the object around on the table.

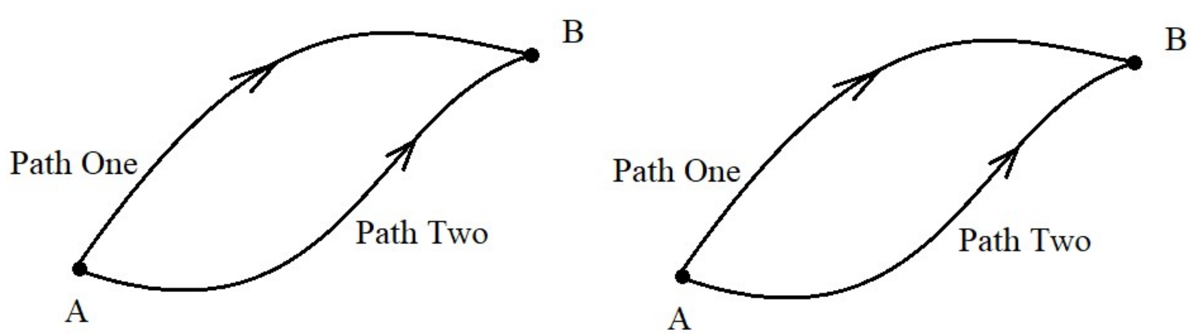
Potential Energy

A few pages ago, we defined energy as ½ mv^2, which is how it was originally defined. It was a few years later that the 'kinetic' was added. We make this distinction because we will introduce a second type of energy, although to my mind, it is only a bookkeeping trick to keep track of some work terms. I admit, though, that the concept of *potential energy* (U) can be extremely useful.

Let's consider the dropped pen again. We can say that during its fall, the pen is acted on only by the force of gravity, which does positive work, and thereby causes an increase in the pen's kinetic energy (work-energy theorem). We can develop an alternate notion, by saying that energy is somehow stored in the pen by virtue of its altitude above the table, and that this potential energy is then converted to kinetic energy as the pen falls. What we find is that any conservative force can have a potential energy function associated with it. For example, if a conservative force does positive work on an object so that the kinetic energy increases, we could alternatively say that the potential energy of the object is decreasing while the kinetic energy is increasing. So, for a given conservative force (F_C), we require that

$$W_C = -\Delta U\ .$$

We can do this only for conservative forces. Here's why.



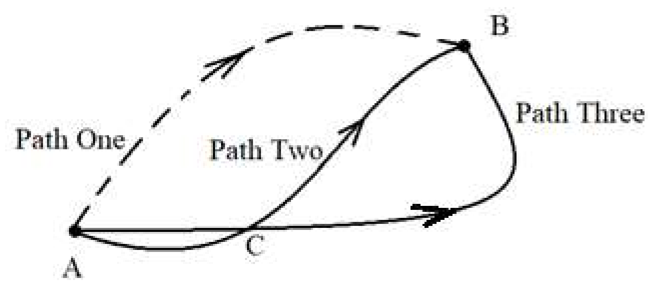
Conservative
$W_{1C} = W_{2C}$

non-Conservative
$W_{1nC} \neq W_{2nC}$

By definition for a conservative force, the work done by the force along any Path One from A to B is the same as along any other Path Two. Since the two paths have only points A and B in common, there must be some numbers associated with the object being at each of these points that provide sufficient information to determine the work. We call these values the potential energy of the object at A (U_A) and the potential energy of the object at B (U_B).[5] If we tried to do that with the non-conservative force, starting at point A, we would have to conclude that there are two different values associated with point B, or indeed, potentially an infinite number of such values, one for each possible path and amount of work done.

DISCUSSION 6-6

One might point out that this argument regarding conservative forces is valid only when Path One and Path Two do not cross (they would have more than just two points in common). Of course, we can come up with any number of paths that do cross. Can you provide an argument that takes care of that omission?



Here we go. Let's start with the work-energy theorem, and divide the works on the left into two categories, depending on whether the associated forces are conservative (C) or non-conservative (NC):

$$W_{TOTAL} = \Delta K$$

$$W_C + W_{NC} = \Delta K \ .$$

We'll define the change in potential energy with $-\Delta U = W_{CONS}$, so that

$$-\Delta U + W_{NC} = \Delta K \ .$$

Since work and energy are not the same thing, and because I hate minus signs,

$$W_{NC} = \Delta K + \Delta U \ .$$

In the same way that I find $\Sigma F = ma$ to be more convenient than the conceptually better $a = \Sigma F/m$, I find this form of the work-energy theorem to be more convenient than the conceptually satisfying version of a few pages back.

Now, keep in mind that the potential energy term replaces the conservative work term; it's one or the other, but not both.

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[5] If you read this paragraph carefully, you should have noted that these values are not directly associated with the points A and B themselves, but with our object being located at A and at B. There is a quantity associated with the points themselves, regardless of whether there is an object there or not, but it is typically not covered in PHYS 1. Look for an analogous quantity though in PHYS 2!

Can we figure out what the potential energy function associated with a conservative force is? Not really. We can only figure out an expression for its <u>change</u>:

$$\Delta U = -\int \vec{F}_C(\vec{r}) \cdot d\vec{r} \quad .$$

If we instead know the potential energy expression, we can turn this around to find the force. In one dimension,

$$\Delta U = -W_C$$

$$\int dU = -\int F(x)\, dx \quad ,$$

Since this must be true of any path, however short, we can write

$$dU = -F(x)\, dx \quad \rightarrow \quad F(x) = -\frac{dU}{dx} \quad .$$

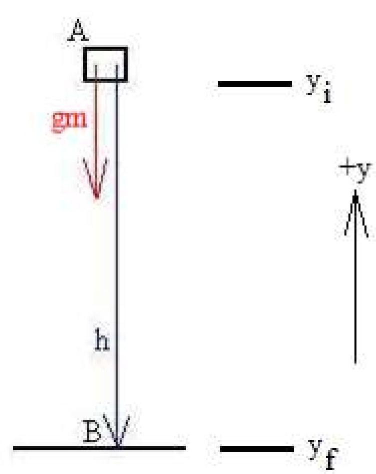
In three dimensions, this generalizes to[6,7]

$$\vec{F}(\vec{r}) = -\left(\frac{\partial U}{\partial x}\,\hat{\imath} + \frac{\partial U}{\partial y}\,\hat{\jmath} + \frac{\partial U}{\partial z}\,\hat{k}\right) = -\vec{\nabla}\, U \quad .$$

A way of determining if a force is conservative is to check if curl $\vec{F}(\vec{r}) = 0$.

DERIVATION 6-4

Consider the specific example of the pen discussed earlier that was lowered from a height H to the table below. We calculated that the work done on the pen by the weight was gmh. Now, to make this work consistently, it's necessary to give up a little freedom of choice; we will



[6] See the notes at the end of this section.
[7] ∇U is the *gradient* of function U.

require upward to be positive y.[8] Since h is a positive number (the magnitude of the displacement) and since $y_i > y_f$, we can instead write that

$$W_g = gmh = gm(y_i - y_f) = -(gmy_f - gmy_i) = -\Delta(gmy) \ .$$

If we keep in mind that we defined U such that

$$W_g = -\Delta U_g \ ,$$

we might just jump to the conclusion that

$$U(y) = gmy \ .$$

Now, of course, we still have the same problem we had with kinetic energy, that there may be some constant term we're missing that will cancel out when we find ΔU: $U(y) = gmy + U_o$. This time, though, we're going to take advantage of that. Where we pick our origin *(i.e.,* where y = 0) is entirely up to us, and so that is where we choose the potential energy to be zero. So, we'll make these choices to be as convenient for us as possible. Generally (80% Rule!) you will want to place y = 0 at the lowest level of a problem.[9]

But, let's consider. Suppose I raise a 2 kg object from a tabletop 1 m above the floor to 2 m above the floor. If the zero of potential energy is zero at floor level, I increased U from (10)(2)(1) = 20 joules to (10)(2)(2) = 40 joules. If the zero had been at table level, it went from 0 joules to (10(2)(1) = 20 joules. And if the 3 meter high ceiling had been U = 0, it went from (10)(2)(-2) = -40 joules to (10)(2)(-1) = -20 joules. In each case the change was the same (+20 joules) even if the assigned potential energy values were very different.

Some admonitions before we start examples. First, remember that you should not put the potential energy term on both sides of the relationship; it's <u>either</u> a work term on the left, <u>or</u> it's a potential energy change on the right. As I said, this is a bookkeeping trick. Second, remember that there may well be more than one conservative force operating on the object, which would require us to have more than one ΔU term. For this course, there are only three conservative forces; all others should be considered to be non-conservative.

Conservation of Mechanical Energy

Let's call the sum of an object's kinetic and potential energies its *mechanical energy*. Let's consider a special case in the absence of non-conservative forces, or at least a situation where no non-conservative forces do work:

[8] You may remember doing this when we required radial forces to be counted as positive when toward the center of the circle and negative when away.
[9] Of course, later, we'll see some exceptions*, i.e.* the other 20%!

$$W_{NC} = \Delta K + \Delta U$$

becomes

$$0 = \Delta K + \Delta U$$

$$0 = K_f - K_i + U_f - U_i$$

$$K_i + U_i = K_f + U_f \ .$$

This is interesting. It says that, in the absence of non-conservative forces (or at least of such forces which do any work), the total mechanical energy is *conserved*, or remains constant. There is, in physics, a great number of quantities that are conserved in the absence of outside agencies. In the present example, the energy may change from from kinetic to potential or *vice versa*, but it is neither created nor destroyed.

This concept of the conservation of mechanical energy is not the same as *conservation of total energy*, which you may have heard of in your other classes. This is a much more restricted form of that concept. For example, let's look once again at the dropped pen. Just after release, the pen has zero kinetic energy and mgh of potential energy (we'll let U = 0 at the tabletop). Just before hitting the table, U = 0 and K is not zero, and in fact equals numerically mgh. Now in a more general way, we can talk about the conservation of total energy, but only if we broaden the definition of energy. You may remember from your other classes that the molecules in solids can be modeled by balls connected by springs, and that the balls are constantly vibrating, possessing kinetic and potential energy. This kinetic energy is different (in a fashion) from the *translational* kinetic energy discussed above, in that for translational kinetic energy, every particle shared the same velocity vector, but for *vibrational* kinetic energy, the motions are more random. When the pen hit the table, shock waves went out from the impact through both the table and the pen, increasing the vibration of the molecules in each object. This increased *thermal energy* is observed macroscopically as an increase in the *temperatures*[10] of both the table and the pen. Other energy is carried away as *sound,* which eventually warms other objects it hits, such as your eardrum.

EXPERIMENT 6-2

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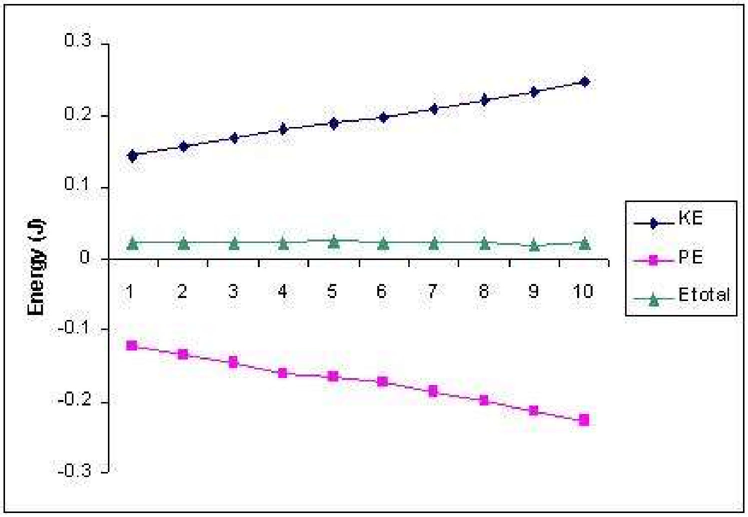
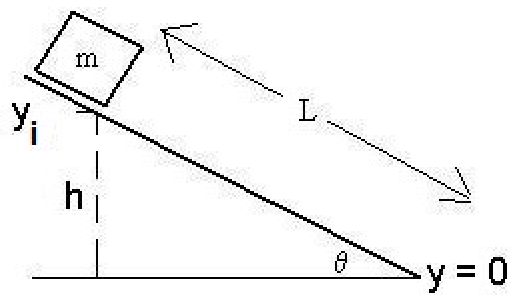
[10] While you may have a general idea of what temperature is, we'll define it carefully later in the course.



Everything we've done in this section up to this point was based on the conjecture that the work is found by multiplying the displacement of an object by the parallel component of the force acting on it. While the rest of the section has a fairly firm basis, if the original conjecture is incorrect, all that followed may be just as incorrect. So we need some evidence to support the conjecture, and that is usually accomplished by performing an experiment. Here are the results of an experiment measuring the potential and kinetic energies of an object as it slides down a frictionless incline. Note that, as the potential energy U decreases, the kinetic energy K correspondingly increases, but that the total energy (U + K) remains constant as predicted (to within experimental error). In this experiment, the maximum deviation from the average is 0.8%.

EXAMPLE 6-3

You.ve seen this one before. Consider a block of mass m = 5 kg at the top of a frictionless ramp L = 2 meters long, which is inclined at θ = 37° to the horizontal. If the mass starts from rest at the top, how quickly will it be moving when it reaches the bottom?

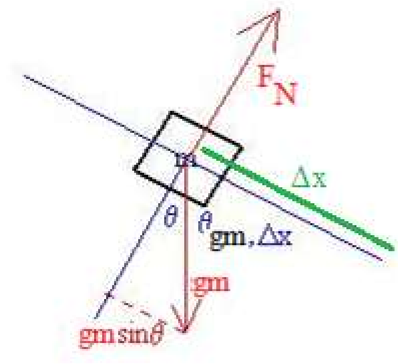


As usual, draw a sketch and a free-body diagram. There are two forces acting on the mass: the weight and the normal force. Let's start with the more recent and more useful version of the work-energy theorem:

$$W_{NC} = \Delta K + \Delta U$$

$$W_{NC} = \frac{1}{2} m\, v_f^2 - \frac{1}{2} m\, v_i^2 + gmy_f - gmy_i$$

Next, let's consider the works done:



$W_N = 0$ – the force is perpendicular to the path and so the cosine term is always zero.
W_g – this is a conservative force and will be dealt with on the right side of the equation.

There is one piece of information we will need: the initial altitude of the object. Since the $\sin\theta = y_i/L$, y_i is $L \sin(37^o) = 1.2$ m.

So then

$$0 = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2 + gmy_f - gmy_i \ ,$$

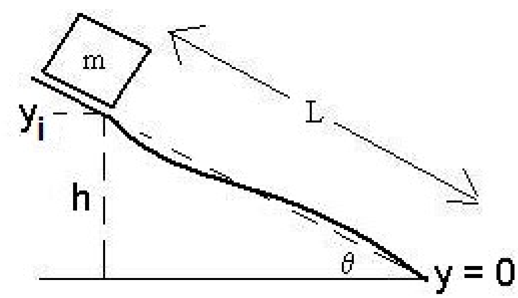
which is always nice.

Following a brilliant suggestion I read somewhere, I'll put y = 0 at the bottom of the ramp. I also realize that the object starts from rest at the top, and so I'll simplify here with justification.[11]

$$0 = \tfrac{1}{2} m v_f^2 - \underbrace{\tfrac{1}{2} m v_i^2}_{\text{starts from rest}} + \underbrace{gmy_f}_{y_f = 0} - gmy_i$$

$$\tfrac{1}{2} m v_f^2 = gmy_i$$

$$v_f = \sqrt{2\,gy_i} = \sqrt{2(10)(1.2)} = 4.9\text{m/s} \,.$$



At this time, it is legitimate to ask, "Gee, Dr Baum, we've learned how to do this problem three ways, none of which seems any easier than the others. What's the point?" And here it comes. Suppose that instead of a straight frictionless surface, the incline had instead possessed a 'wavy' surface, as shown in the figure. Here, the mass does not slide uniformly down a straight surface. Let's think about doing this with Newton's second law. Let x be the direction down the incline. The weight will have a constant component along the dotted line shown in the figure, but the normal force will have a varying component in that direction, sometimes down the incline, sometimes up the incline, and sometimes zero, depending on the exact shape of the surface. Since the x-acceleration is not constant, the kinematic equations are not valid. We could proablyAnd that's an oversimplification. If we try to use the original form of the work-energy theorem (following the wavy line), it's certainly true that the normal force does

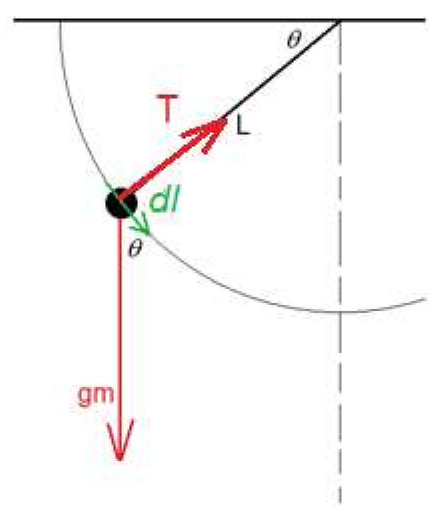
[11] In this section, I'll indicate quantities that are zero in red, and justifying why in the line directly underneath.

no work, but the angle that the weight makes with each small interval of displacement as the object slides down will vary. Either way, we would have to know a lot of very specific data about the shape of the curves ramp and do a horrendous calculation. However, because the weight is a conservative force, the work done on the object does not depend on the exact path taken; all we need to know are the potential energies at the start and end of the trip. Here is a perhaps clearer example to illustrate the usefulness of potential energy.

EXAMPLE 6-4

Consider a small ball (mass m) attached to the end of a (magic) string of length L = 1.5 m. The ball is held up at 90° to the vertical and released. How quickly is it moving when it reaches the bottom of its swing?

As a demonstration of the usefulness of the concept of potential energy, we're going to do this problem twice. First, the long way. We'll use the original form of the work-energy theorem. There are two forces acting on the mass, the ball's weight and the tension in the string.



$W_T = 0$ (the tension is always perpendicular to the path). Wg is tough. In general, the work is F Δx cos $\theta_{F,\Delta x}$. That works if the force is constant and the displacement is along a straight line. But here, the angle between the weight and the direction of motion is continually changing. We must break the path down into very many very small lengths *d*s, find the work for each displacement, then add them all up.

$$W_g = \int_0^{\pi/2} gm\ ds\cos(\theta)$$

Here, we have two variables; we need to change one into the other so that we can perform the integral. Using the arclength relationship, we can write that

$$ds = L\ d\theta$$

and substitute to get

$$W_g = \int_0^{\pi/2} gm\ L\ d\theta\cos(\theta) = gm\ L\int_0^{\pi/2}\cos(\theta)\ d\theta\ .$$

This is pretty straightforward:

$$W_g = -gm\ L\sin\theta\ \Big|_0^{\frac{\pi}{2}} = -gm\ L\ (0-1) = gm\ L\ .$$

Then,

$$W_T + W_g = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$
$$W_T = 0 \qquad \text{starts from rest}$$

$$gmL = \frac{1}{2}mv_f^2$$

$$v_f = \sqrt{2gL} = \sqrt{2(10)1.5} = 5.48\ \text{m/s}\ .$$

Now, the shorter way:

$W_T = 0$ (the tension is always perpendicular to the path)
Wg – conservative force, treat as potential energy terms

Set U = 0 at the bottom of the problem, so $y_f = 0$ and $y_i = L$.

$$0 = \frac{1}{2}m\,v_f^2 - \frac{1}{2}m\,v_i^2 + gmy_f - gmy_i$$
$$\text{starts from rest} \qquad y_f \text{ set to zero}$$

$$\frac{1}{2}m\,v_f^2 = gmL$$

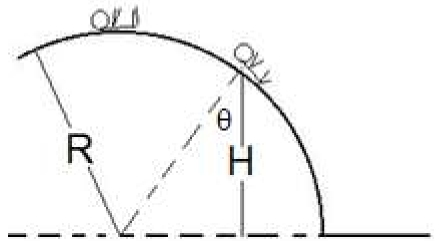
$$v_f = \sqrt{2gL} = \sqrt{2(10)1.5} = 5.48\ \text{m/s}\ .$$

Generally, you will find that using potential energy is never harder than finding the work directly, and usually much easier.

HOMEWORK 6-4

A pitcher hurls a 0.35 kg sportsball around a vertical circular path of radius 0.6 m, applying a tangential force of 30 N, before releasing it at the bottom of the circle (underhand pitch). If the speed of the ball at the top of the circle was 12 m/s, what will be the speed just after it's released?

EXERCISE 6-1



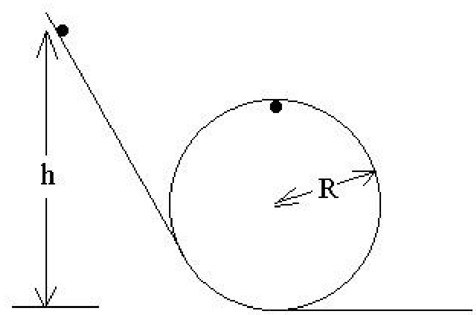
Another classic. Consider a child perched at the top of an igloo, which we will consider to be a hemisphere of radius R covered in slippery snow. He starts with an almost zero speed from the top and travels down the side. At what vertical distance H from the ground will he become airborne?

HOMEWORK 6-5

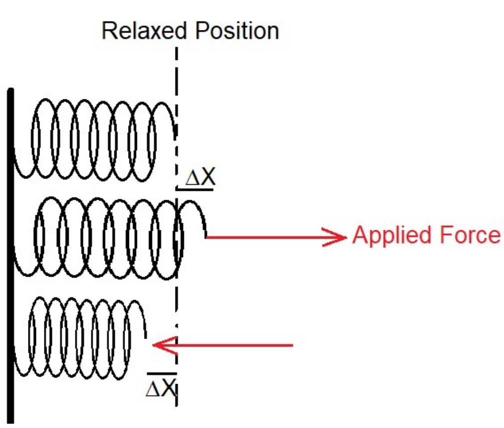
Tarzan swings on a 25 m long vine that was initially inclined at an angle of 25° from the (downward) vertical. What is his speed at the bottom of his swing if he pushed off his branch with an initial speed of 3 m/s?

HOMEWORK 6-6

A point mass block slides without friction on the loop-de-loop track of radius R as shown. From what height h must it be released from rest in order to make it around the loop without leaving the track?



Springs



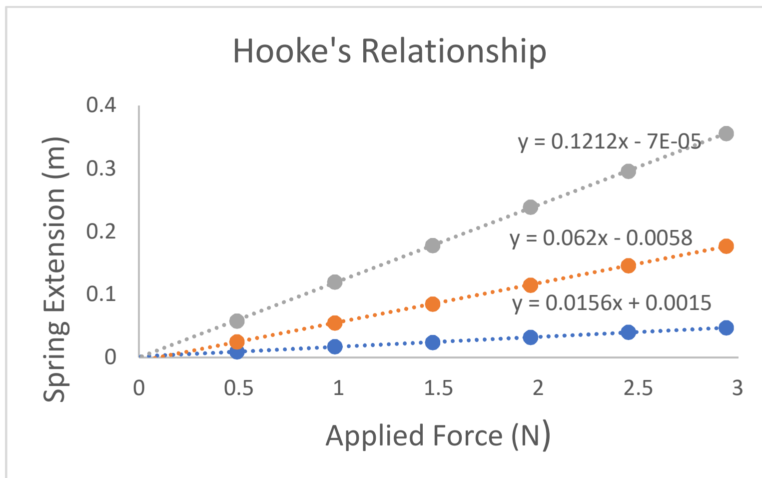
Let's next consider our second conservative force. If I take a spring and simply toss it onto the table, you may notice that it always assumes the same length, regardless of whether I compress it or stretch it before I toss it. Let's refer to this as the spring's relaxed condition and the corresponding length its *relaxed length*. In order to stretch or compress the spring, I must apply some force. In this course, at least for now, we shall assume that all springs obey *Hooke's relationship:* the force necessary to stretch (or compress) a spring from its relaxed state is proportional to the amount of stretching (or compression). In more mathematical terms:

$$F_{Applied} = k\,\Delta X \;.$$

The symbol k represents the *spring constant* of the spring, the number of newtons required to stretch (or compress) the spring one meter, and is given in N/m. A high value of k means that the spring is stiff, while a low value implies the spring is flexible. Notice that I am using a capital X to describe the position of the end of the spring; the reason for this should become apparent later in the discussion.

DISCUSSION 6-7

HookesLaw.mp4



In the figure, I've applied known forces to three springs over an admittedly small range of stiffness. Is the amount each spring is stretched indeed proportional to the applied force? What property of the data in the graph would indicate that? How is the spring constant k found for each curve? Which is the independent variable and which the dependent variable in this experiment? How should Hooke's relationship be arranged to match the equation of a line?

We need to be a bit careful about signs. The relationship above is the force which needs to be applied to the spring to stretch (compress) it, and that force needs to be in the direction of the displacement of the end of the spring. We do not expect this force to be conservative, as it may be provided by a hand or other such agency. However, we are often interested in the force applied by the spring to some object to which it is attached. By the third law, this spring force would be in the opposite direction:

$$F_{Spring} = -\,k\,\Delta X \;.$$

However, this is further complicated by our habit of writing down the magnitudes of forces and adding in the appropriate directional signs as necessary. As a result, I shall write this relationship this way,

$$F_{Spring} = (-)\,k\,\Delta X$$

with the minus sign there in parentheses to remind you that the force exerted by the spring is in the direction opposite to that in which the spring is stretched, but not to be taken literally. You must determine the correct sign for each specific problem encountered. Occasionally, the 'delta' is dropped as well, if it is understood that the relaxed position is at X = 0:

$$F_{Spring} = (-)k\,X\,.$$

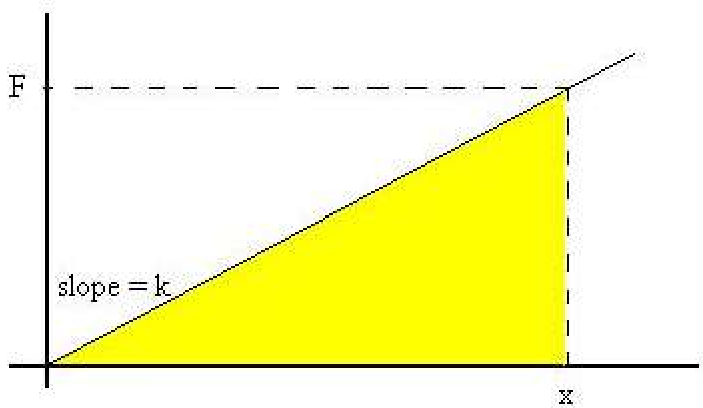
One last assumption: unless told otherwise, springs, like strings, will be considered to be massless.

DISCUSSION 6-8

Is the spring force conservative? Can you make a quick argument that it is? Suppose that we were to stretch the spring from X = 0 to point A. Repeat from X = 0 to A to a point B beyond point A, then back to B?

Since the force exerted by the spring depends only on the position of the end of the spring X (we'll assume that the other end is fixed), reversing the displacement back over already covered ground simply undoes the work done the first time (by flipping the sign of the cosine term), so that the net work done depends only on the initial and final positions of the end of the spring.

DERIVATION 6-5



How much work is necessary to stretch (or compress) a spring distance X from its relaxed position? We can use the graphical representation showing $F_{on\ spring}$ as a linear function of X with slope k:

We showed above that the work done by any variable force is represented by the area under the force *vs* position curve.[12] Since this is a triangle, the area is one-half the base times the height:

$$W_{on\ Spring} = \tfrac{1}{2}\,X\,F = \tfrac{1}{2}\,X\,(kX) = \tfrac{1}{2}\,kX^2$$

Now we have to do a couple of flip-flops. The work done <u>on</u> the spring is $^1/_2kX^2$, the work done <u>by</u> the spring is $-^1/_2kX^2$ (the forces are in opposite directions), and the change in the potential energy of the spring is the negative of that, or

[12] So, we're avoiding doing an 'official' integral, but this is an integral all the same.

$$\Delta U_{Spring} = -W_{by\ Spring} = -\left(-W_{on\ Spring}\right) = +\tfrac{1}{2}\,kX^2\ ,$$

$$U(X) - U(0) = +\tfrac{1}{2}\,kX^2\ .$$

It would seem extremely convenient to make U(0) = 0, so that

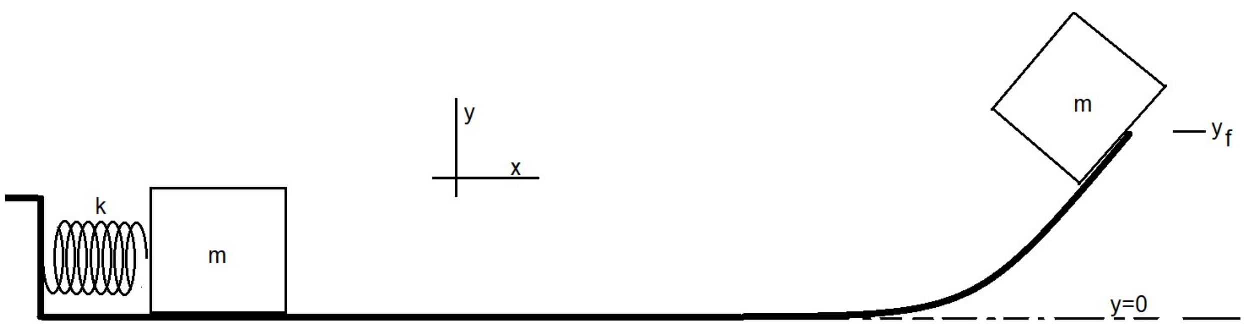
$$U_{Spring}(X) = +\tfrac{1}{2}\,kX^2\ .$$

Note that in this version, we have given up some freedom again. We will be assuming the spring's potential energy is zero when the spring is relaxed. Do we have to do this? No, but things will be much easier if we do. We will also see that maintaining this zero of potential energy supersedes our choice of where to make the gravitational energy zero, again for mathematical exigency.

Also, note that the potential energy of a spring depends on the square of the extension or compression. That is, for U(X), it really doesn't matter if we make compression or extension positive or negative; the potential energy increases either way.

EXAMPLE 6-5

Consider the frictionless surface shown. On the left is an ideal spring of constant k = 30 N/m. A mass of 5 kg is pushup against the spring, compressing it 0.2 m. When the mass is released, it is pushed to the right, slides across the surface, and travels up the incline. What is the mass's altitude y_f when it stops?



There are three forces acting on the mass at one time or another. Use the work-energy theorem.

$W_N = 0$ (the normal force is always perpendicular to the path).
W_{Sp} – conservative
W_g – conservative

Let's define y = 0 to be at the bottom of the problem. Then,

$$W_{NC} = \Delta K + \Delta U_g + \Delta U_{Sp}\ .$$

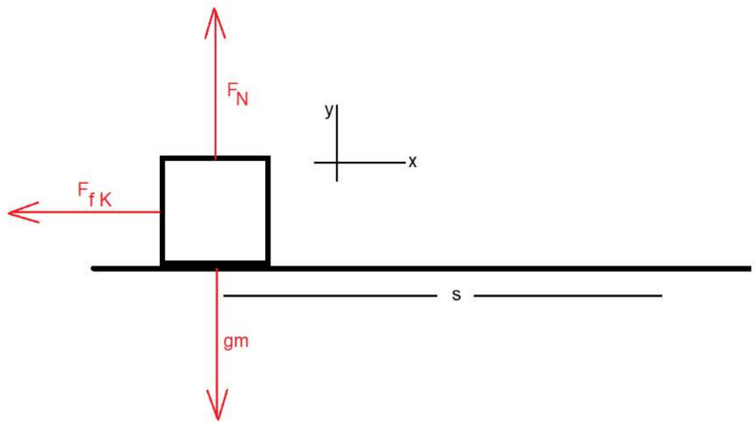
$$0 = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 + gmy_f - gmy_i + \frac{1}{2} kX_f^2 - \frac{1}{2} kX_i^2$$

stops — starts from rest — $y_i = 0$ — spring is relaxed

$$gmy_f = \frac{1}{2} kX_i^2 \quad \rightarrow \quad y_f = \frac{kX_i^2}{2gm} = \frac{30(0.2^2)}{2(10)5} = 0.012\text{ m}$$

This illustrates why I used X; it represents the location of the end of the spring, not the location of the mass.

EXAMPLE 6-6



Show that an object of mass m moving at speed v_o across a rough horizontal floor will slide a distance $s = (v_o)^2/2\mu_K g$.

There are three forces acting on the object. Consider the work each does.

$W_N = 0$ (the normal force is always perpendicular to the path).

Wg – conservative

$W_f = F_f\, s \cos\theta_{F,s}$. We need to find the frictional force, which unfortunately requires a trip back to Newton's second law land. Although the result here may seem obvious to you, it is important to actually show the effort. From NII in the y-direction,

$$+F_N - gm = ma_y = 0 \quad \rightarrow \quad F_N = gm$$

$$F_{fK} = \mu_K F_N = \mu_K gm\ .$$

Since the displacement and frictional forces are in opposite directions, we'll be taking the cosine of 180°, so

$$W_f = (\mu_K gm)\, s\, (-1) = -\mu_K gms\ .$$

The work-energy theorem then results in

$$-\mu_K gms = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 + gmy_f - gmy_i$$

comes to a stop — $y_f = y_i$

Finish up with

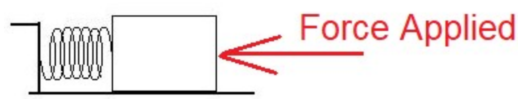
$$\mu_K g s = \frac{1}{2} v_o^2$$

$$s = \frac{v_o^2}{2\mu_K g} \ .$$

HOMEWORK 6-7

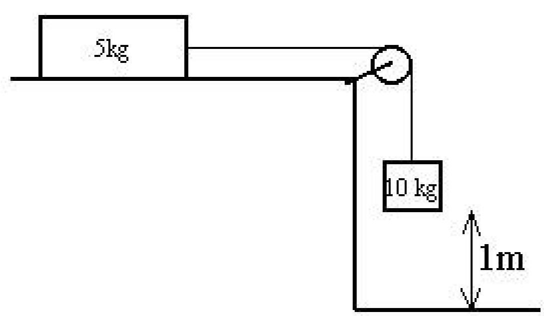
Two identical massless springs of constant k = 400 N/m are fixed at opposite ends of a level track, as shown. A 12 kg block is pressed against the left spring, compressing it by 0.3m. The block is then released from rest. The entire track is frictionless except for the region of length 0.2 m between points A and B, where $\mu_K = 0.08$. What is the maximum compression of the spring on the right?

HOMEWORK 6-8



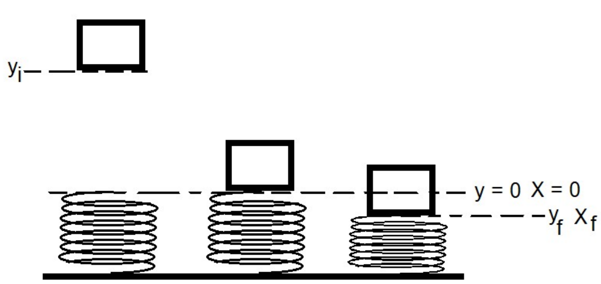
Consider a 13 kg block sitting at rest on a rough surface. The coëfficient of friction between the block and surface is 0.7. The block is barely touching a relaxed spring of constant k = 300 N/m. How much work would a hand or other such agency need to do to push the block very slowly 0.2 m against the spring?

HOMEWORK 6-9



The 10 kg mass is released from rest at a height of 1m above the floor. If the coëfficient of kinetic friction between the 5 kg mass and the table is 0.8, what will be the speed of the 10 kg mass just before it hits the floor?

EXAMPLE 6-7



Let's look at a problem where it is not a good idea to make the lowest point the zero of gravitational potential energy. Let's drop a box of mass M onto a vertical spring. How far is the spring compressed when the box comes to a stop? The figure shows three points in the process. Because the spring's potential energy is quadradic while the gravitational potential energy is linear, things will go much easier mathematically if we set the springs relaxed position to be the zero for both. Note then that the final values for y and for X will be the same. Let's pick some numbers: $y_i = 12$ m; M = 8 kg; k = 120 N/m.

Use the work-energy theorem:

There are two forces acting on the box, the weight and the spring force.
W_g – conservative
W_{Spring} – conservative

So, we have that happy situation when $W_{NC} = 0$.

$$0 = \frac{1}{2} m\, v_f^2 - \frac{1}{2} m\, v_i^2 + gmy_f - gmy_i + \frac{1}{2} kX_f^2 - \frac{1}{2} kX_i^2$$

The box begins and ends at rest   the spring is initally relaxed

Remember that $y_f = X_f$.

$$0 = gmy_f - gmy_i + \frac{1}{2} k\, y_f^2$$

This is a quadratic equation, so let's insert the values now and re-arrange for solution. We'll also divide both sides by 20 to make the numbers smaller.

$$0 = 10(8)y_f - 10(8)y_i + \frac{1}{2}(120)\, y_f^2$$

$$3\, y_f^2 + 4y_f - 4y_i = 0$$

$$y_f = \frac{-4 \pm \sqrt{4^2 - 4(3)(-4)}}{2(3)} = +0.67\text{m or} - 2\text{ m} \; .$$

This time, we want the negative root because we know the final position will be below the y = 0 level.

DISCUSSION 6-9

Why does the equation give us two values? What condition did we impose on the locations of the box and the end of the spring? What situation does the other root correspond to?

HOMEWORK 6-10

A 0.85 kg bunch of bananas depresses the pan of a spring balance at Wegman's 3.0 cm when resting on it. If the bananas were dropped onto the pan from a height of 0.3 m above the empty pan, how far will the pan be depressed before starting to return upward?

Power

Sometimes, we're interested in the rate at which energy is put into, or removed from, an object, or the rate at which work is done, the *power*:

$$P_{AVE} = \frac{W}{t} \ .$$

The instantaneous power is of course

$$P_{INST} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} \ .$$

If we consider forces acting on an object during a short interval of time (and displacement), we obtain an interesting result:

$$P_{INST} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{F \, \Delta x \cos(\theta_{F,\Delta x})}{t} \right) = F \lim_{\Delta t \to 0} \left(\frac{\Delta x}{t} \right) \cos(\theta_{F,\Delta x}) = F \, v \, \cos\left(\theta_{F,v}\right) = \vec{F} \cdot \vec{v} \ .$$

If one joule of work is performed in one second, we say that the power is one watt (symbol W). There is an alternate unit for power that is still commonly used in the U.S., the horsepower (hp). The hp has been redefined as exactly 750 W.

DISCUSSION 6-10

> What is the power rating of a typical incandescent light bulb?[13] What is the power rating of a corresponding diode light bulb? What is the power rating of the engine in your car? How many light bulbs could your car engine presumably light up at once?

EXAMPLE 6-8

> Suppose you're late for your next class. You need to run up a flight of stairs as quickly as possible. What power output is required?
>
> The result depends on the values picked and will of course vary from person to person. Let's assume that the floors of your building are 6 m apart (fairly typical for an office building). The average adult male American masses in at 90 kg. The part no one ever agrees on is the amount of time required to run up a flight of stairs. Let's call it twelve seconds. If we can agree that all of the work goes into increasing the person's potential energy (he's running the same speed at the top and at the bottom), then we have
>
> $$P = \frac{W}{t} = \frac{\Delta U}{t} = \frac{gmH}{t} = \frac{10(90)6}{7} = 771 \text{ W} = 1.03 \text{ hp}.$$

---

[13] This is not the same power that we discussed; it is the rate of conversion of electrical energy to thermal energy. But, let's continue anyway.

DISCUSSION 6-11

The horsepower was originally defined, loosely, as the power a draught house could supply while drawing a plough. From the previous result, it appears you could do this job. What's the difference between your ability and the horse's ability to draw a plough?

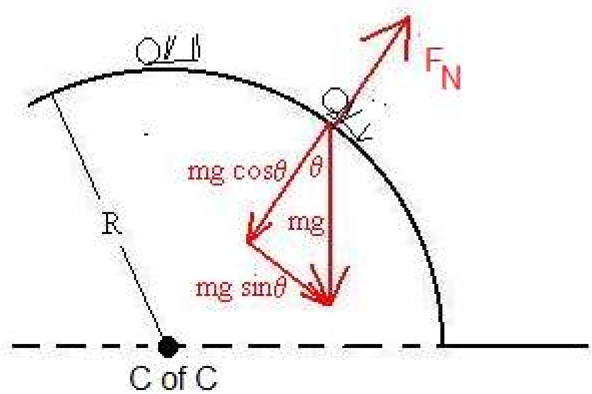
EXERCISE 6-1 Solution

Start as usual with a free body diagram. There are two forces acting on the child, his weight and the normal force.

$W_N = 0$ (the normal force is always perpendicular to the path).
W_g – conservative

So, we have again that $W_{NC} = 0$, and

$$0 = \frac{1}{2}\,m\,v_f^2 - \frac{1}{2}\,m\,v_i^2 + gmy_f - gmy_i\ .$$



The child starts from rest (or close to it) and we will set ground level as y = 0. The problem ends not at the ground, but when the child leaves the igloo surface at y = H. The original altitude is y = R, the radius of the igloo.

$$0 = \frac{1}{2}\,m\,v_f^2 + gmH - gmR\ .$$

The trouble we have here is that there are two unknowns. We need more information. We might notice that the child is moving in a circle, and we know a lot about things moving in circles. Let's return to Newton's second law:

$$+\,gm\cos(\theta) - F_N = ma_C = \frac{mv^2}{r}\ .$$

The radius of the circle is of course R. The normal force goes to zero when the child loses contact with the surface, and at that moment, $v = v_f$.

$$gm\cos(\theta) = \frac{mv_f^2}{R}\ .$$

We may notice that the cosine of theta is H/R, so that

$$\text{gm}\frac{\text{H}}{\text{R}} = \frac{\text{mv}_\text{f}^2}{\text{R}} \quad \rightarrow \quad \text{gmH} = \text{mv}_\text{f}^2 \ .$$

Returning to the energy equation,

$$0 = \tfrac{1}{2}\text{gmH} + \text{gmH} - \text{gmR} \ ;$$
$$\frac{3}{2}\text{H} = \text{R} \ ;$$
$$\text{H} = \frac{2}{3}\text{R} \ .$$

O.K., so this isn't a Calc III Course. Here are a couple of quick explanations of these ideas.

The first is solid enough. Previously, we noted that

$$\Delta\text{U} = -\text{W}_\text{C} = -\int \text{F}_x \, dx + \text{F}_y \, dy + \text{F}_z \, dz \, .$$

If the force is conservative, the integral has the same value regardless of the path taken, so let's move a very short distance *d*x along the x-axis first, resulting in a change of potential energy dU_1,

$$d\text{U}_1 = -\text{F}_x \, dx \quad \rightarrow \quad F_x = -\frac{d\text{U}}{dx} \quad .$$

We repeat in the y and z directions:

$$d\text{U}_2 = -\text{F}_y \, dy \quad \rightarrow \quad F_y = -\frac{d\text{U}}{dy}$$

$$d\text{U}_3 = -\text{F}_z \, dz \quad \rightarrow \quad F_z = -\frac{d\text{U}}{dz} \quad .$$

Then,

$$\vec{F} = F_x \, \hat{\imath} + F_y \, \hat{\jmath} + F_z \, \hat{k} = -\frac{d\text{U}}{dx}\hat{\imath} - \frac{d\text{U}}{dy}\hat{\jmath} - \frac{d\text{U}}{dz}\hat{k}$$