Section 9 – Rotation in a Plane

Rotation is generally fairly complicated to study and usually involves yet another type of quantity beyond scalars and vectors, called *tensors*. However, we will restrict ourselves to rotation in a plane, usually the x-y plane, which should make things a bit easier.

The Center of Mass

Up to now, we have been treating objects as point masses. That is, if we were asked the location of an object, we could respond with as little as a single number, such as x = 3.576 meters. Extended objects, on the other hand, occupy many locations. A car may be said to be between x= 4.582m (at the tip of the front bumper) and x = 8.935 m (at the rear end bumper). Even that doesn't give many details. So, we often speak of the average position of a car. We don't mean that in the sense of a trip from Baltimore to Philly, where the average position is in Wilmington, but average in the sense of examining each particle composing the car and averaging their positions. We call this position the *center of mass*[1] of the object, and it is calculated in much the same way that the average on an exam is found.

DISCUSSION 9-1

> How do we find the average on an exam? If nineteen students each earn one hundred and one earns a zero, is the average fifty? If not, what is?

For an exam, we take each possible grade (G_n) from zero to one hundred and weight the importance of each with the number of students who earned that grade (N_n), then divide by the total number of students. Or, if we were to share all the earned points equally among all students, how many would each get?

$$G_{AVE} = \frac{\sum_n N_n G_n}{\sum_n N_n}$$

Instead, we look at every possible position and weight each by how much mass is located there. Clearly, there are many positions at which there is no mass, and we usually just skip them:

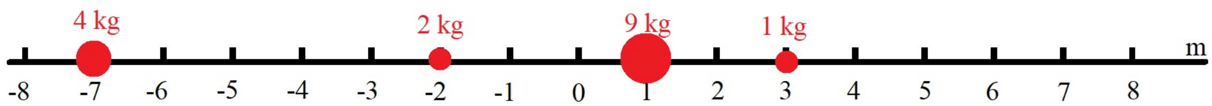
$$x_{CM} = \frac{\sum_n m_n x_n}{\sum_n m_n} \quad .$$

Because we like to ease into new things, let's start by finding the center of mass of a bunch of point masses.

EXAMPLE 9-1

[1] You may also hear the term *center of gravity*. So long as the gravitational field is uniform, these points are the same.

Find the center of mass of these four point masses.



$$x_{CM} = \frac{\sum_n m_n x_n}{\sum_n m_n} = \frac{4(-7) + 2(-2) + 9(+1) + 1(+3)}{4 + 2 + 9 + 3} = \frac{-20}{18} = -1.11\text{ m}\ .$$

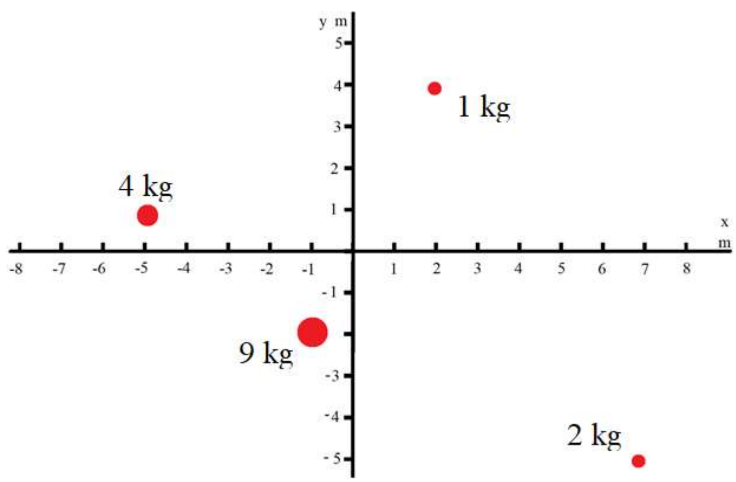
Of, course, not every object is composed of a linear arrangement of masses. In three dimensions, we can represent the location of each object by the location vector $\vec{r}$, so that

$$\vec{r}_{CM} = \frac{\sum_n m_n \vec{r}_n}{\sum_n m_n}\ ,$$

but in reality, this is just a way of writing three separate equations for x_{CM}, y_{CM}, and z_{CM}.

HOMEWORK 9-1

Find the center of mass of the four masses shown in the figure. The coördinates are the integers they appear to be.



What about objects that are not made up of discrete point masses, but rather a continuous structure? In principle, it's the same process, although the process of performing the calculations may be quite difficult (see the EXAMPLE below). Luckily, we don't need many shapes to get out points across in this course.

DISCUSSION 9-2

Consider, for example, a thin uniform rod of length L and mass M, like a meterstick. Where do you suppose is the center of mass? Can you make an argument to support your contention?

We can make use of the notion of *symmetry* to make an argument about a few simple shapes. If we initially suppose that the center of the rod is at its physical center, we see that for every bit of mass on the left side, there is an equal amount of mass on the right side at exactly the same distance out. The positions of these two masses will average out to the center of the rod, the average of the averages will of course also be at the center. We can use this argument for other shapes, such as a uniform hoop, uniform disk, and uniform sphere.

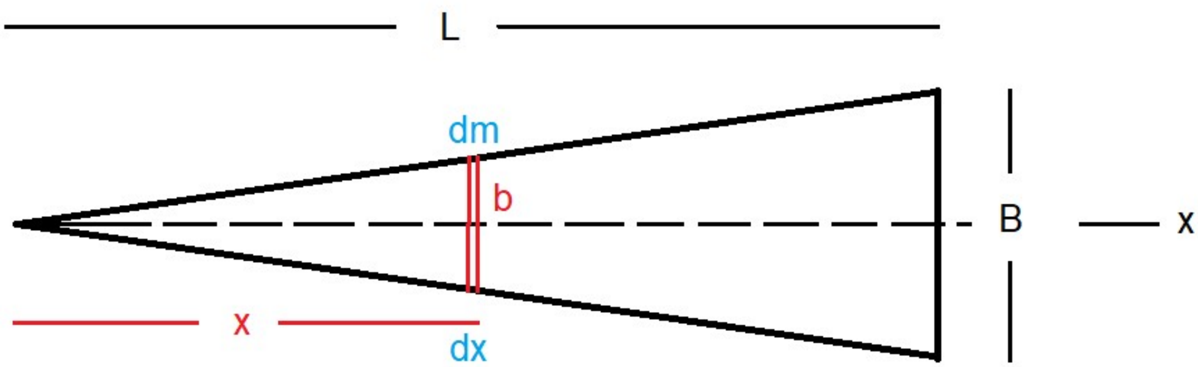
DISCUSSION 9-3

> What if the rod above were half made of maple and half made of pine? Where would the center of mass be? The density of maple is about 700 kg/m^3 while that of pine is 500 kg/m^3.

What if the mass distribution is more complicated?

EXAMPLE 9-X

Consider a triangular piece of some uniform material of density D. Where is the center of mass?



From a symmetry argument, we can assert that the center of mass is on the x axis. Since the mass is not in identifiable discreet point masses, we'll slice it up into small, thin strips of mass dm and length dx, so small we could consider them to be point masses. We'll set the origin at the left tip. Then,

$$x_{CM} = \frac{\sum_n m_n x_n}{\sum_n m_n} \rightarrow \frac{\int x\,dm}{\int dm}.$$

Let's figure out how big each dm is. We can assume that if the density is uniform, then the mass of each slice is proportional to its area. However, the height of each slice varies with x. In this case we can make use of the fact that the triangle of height L and base B is similar to the triangle of height x and base b, so that

$$\frac{b}{x} = \frac{B}{L} \quad \rightarrow \quad b = \frac{B}{L}\,x$$

and the area is

$$dA = b\,dx = \frac{B}{L}\,x \;\; dx\;,$$

and furthermore, the mass is

$$dm = D\,dA = \frac{DB}{L}\,x\;\;dx\;.$$

Finally,

$$x_{CM} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^L x\,\frac{DB}{L}\,x\;\;dx}{\int_0^L \frac{DB}{L}\,x\;\;dx} = \frac{\int_0^L x^2\;\;dx}{\int_0^L x\;\;dx} = \frac{\frac{L^3}{3}}{\frac{L^2}{2}} = \frac{2}{3}L\;.$$

One last thing: although we will be dealing with extended objects for the next few sections, we'll restrict ourselves to *rigid objects*, where the individual pieces always maintain the same distances to all of the other pieces in the object. So, a hammer would be considered a rigid object, but a bucket's worth of water thrown across the room would not. We should probably be a bit more careful with that definition, but it's good enough to get the idea across.

The center of mass has one especially remarkable property that makes life a bit easier for us.

DERVIATION 9-1

Consider a system of masses m_n with total mass $M = \Sigma_n\, m_n$:

$$\vec{r}_{CM} = \frac{\sum_n m_n \vec{r}_n}{\sum_n m_n}\;,$$

$$\left(\sum_n m_n\right)\vec{r}_{CM} = \sum_n m_n \vec{r}_n\;,$$

$$M\,\vec{r}_{CM} = \sum_n m_n \vec{r}_n\;.$$

Now, take the time derivative of each side twice. The first makes the positions into velocities and the second makes the velocities into accelerations.

$$M\,\vec{a}_{CM} = \sum_n m_n \vec{a}_n\;.$$

Now, let's consider all of the forces acting on any one of the masses. For the n^{th} one, we have

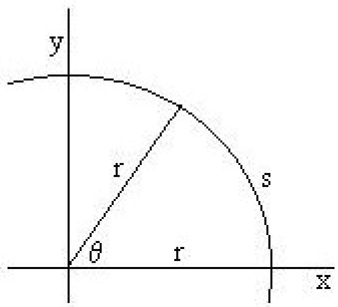
$$\sum_m \vec{F}_{n,m} = m_n \vec{a}_n\;.$$

If we add up all of those equations, we get

$$\sum_{n}\sum_{m}\vec{F}_{n,m} = \sum_{n} m_n\vec{a}_n = M\,\vec{a}_{CM}\,.$$

In other words, the forces acting on any of the parts of the collection of masses will accelerate the center of mass as if it were a single point particle of mass M. This is how we got away with the first half of the course.

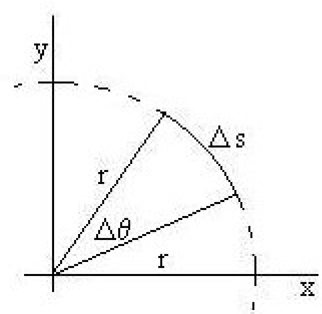
Rotational Kinematics



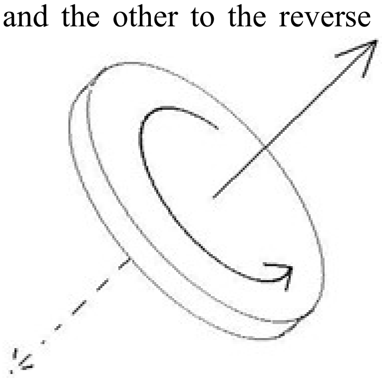
Before we proceed, let's review a bit from Section 3. Consider a point mass m free to move about a circle of radius r. First, we need to be able to specify the object's position. For this, we will return to our convention of measuring angles CCW from the x-axis. However, we will for now on think of such angles in *radians*, not degrees. A *radian* is the angle such that the *arclength* s subtended is equal to the radius r, or about 57.3°. Clearly, if we halve the angle, we also halve the distance along the arc, so that theta and s are proportional by the factor r:

$$s = r\,\theta\ .$$

So, there are then 2π radians in a circle, since the circumference is $2\pi r$.



We should next find a way of describing changes in the position, or the *angular displacement*, $\Delta\theta = \theta_f - \theta_i$, so that $\Delta s = \Delta\theta\ r$. Since linear displacement was a vector, we should require the angular displacement to be as well. The magnitude of $\Delta\vec{\theta}$ will of course indicate how much the object has turned. The direction of $\Delta\vec{\theta}$ will tell us two things: the plane in which the object rotated, and the direction in which it rotated. A plane can be defined by a vector that is perpendicular to the plane, and luckily, there are two directions, one of which we'll assign to rotation in one direction and the other to the reverse direction. The choice is arbitrary, but we'll want to match what everyone else does. We can remember which is which by using our right hands. Curl your fingers like little arrows in the direction of rotation, and your thumb will point in the direction of the vector $\Delta\vec{\theta}$. Be sure to use your right hands. Most of the problems you'll encounter here are with objects rotating in the plane of the page; in that case, $\Delta\vec{\theta}$ out of the paper (motion is CCW) is considered to be positive, and $\Delta\vec{\theta}$ into the page (CW motion) is considered to be negative. Of course, as always, you can change this for your convenience so long as you are clear and consistent.



We're going to work our way through the analogs of all the quantities we discussed in terms of linear motion. We continue with the *angular velocity*, the angular displacement *per* unit time:

$$\vec{\omega}_{AVE} = \frac{\Delta\vec{\theta}}{\Delta t} \;\; ; \;\; \vec{\omega}_{INST} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt} \;\; ; \;\; \text{The direction of } \vec{\omega} \text{ is the same as for } \Delta\vec{\theta} \;\; .$$

Looking back to Section 3, a point on the rotating object will possess a speed tangent to its path given by

$$v_T = \omega r \;\; .$$

Likewise, we can define the *angular acceleration* as the time rate of change of the angular velocity:

$$\vec{\alpha}_{AVE} = \frac{\Delta\vec{\omega}}{\Delta t} \;\; ; \;\; \vec{\alpha}_{INST} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} \;\; .$$

A piece of this rotating object will have a tangential acceleration, a_T, given by

$$a_T = \alpha r \;\; .$$

Determining the direction of the angular acceleration alpha is sometimes difficult. Remember what we said back in Section 2: if an object (moving in one dimensions) is speeding up, $\vec{v}$ and $\vec{a}$ are in the same direction, while if it is slowing down, $\vec{v}$ and $\vec{a}$ are opposite. Do the same for $\vec{\omega}$ and $\vec{\alpha}$. We'll leave problems when this is otherwise until your junior year Physics class.

Of course, there is also the angular jerk, the angular kick, and the angular lurch, *et c.*

If we assume that there are situations where the angular acceleration is constant, we can derive some kinematic relationships. Since θ, ω, and α share the same relationships as x, v, and a, we need not actually perform these derivations, but simply replace each linear quantity with the analogous rotational quantity:

$$\vec{\omega}_f = \vec{\omega}_i + \vec{\alpha}t$$

$$\vec{\omega}_{AVE} = \frac{\vec{\omega}_f + \vec{\omega}_i}{2}$$

$$\Delta\vec{\theta} = \vec{\omega}t + \tfrac{1}{2}\vec{\alpha}t^2$$

$$\omega_f^2 = \omega_i^2 + 2\vec{\alpha} \cdot \Delta\vec{\theta}$$

If we restrict ourselves to rotation in a single plane, we can use the same notation (+ or -) for the direction of each vector and drop the dot product in KEq 4.

EXAMPLE 9-2

A wheel starts from rest and starts to spin with an angular acceleration of 2.5 rad/s^2. After 34 seconds, what is the angular speed and through what angle has it turned?

We treat the problem the same as we did linear kinematic problems, by constructing a table. We weren't told which way the wheel is spinning, so let's just make that the positive direction.

$\theta_i = 0$ (make that the origin)
$\theta_f = ? \leftarrow$
$\omega_i = 0$ (starts from rest)
$\omega_f = ? \leftarrow$
$\alpha = +2.5$ rad/s^2
$t = 34$ sec

KEq 1 gives us the final velocity directly:

$$\omega_f = \omega_i + \alpha t$$

$$\omega_f = 0 + 2.5(34) = 85\,\frac{\text{rad}}{\text{s}}.$$

KEq 3 gives us the displacement directly:

$$\Delta\theta = \omega t + \tfrac{1}{2}\alpha t^2$$

$$\Delta\theta = 0(34) + \tfrac{1}{2}(2.5)(34^2) = 1445 \text{ radians}.$$

Seem familiar? Try this one.

EXERCISE 9-1

A wheel is turning at 30 rad/s but slows and reverses direction to 40 rad/s. It does so while turning through a net 500 revolutions. How much time did this take and what as the acceleration?

HOMEWORK 9-2

An object initially rotating at an angular speed of 1.8 rad/sec turns through 50 revolutions during the time it experienced an angular acceleration of 0.3 rad/s^2. For how much time did the acceleration last and what was the final angular speed?

Torque

DISCUSSION 9-4

Suppose you go home this evening, open the fridge, and take out a container of your favorite beverage. How will you open it? For some of you, applying a force will be sufficient, but would that work for everyone? What must the rest of you do?

Back in Section Five, we saw that Newton's second law of motion says that a net force is necessary in order for an object to have an acceleration. We might expect a similar necessary condition in order for an object to have an angular acceleration. Instead of a 'push' or 'pull,' it requires a 'twist.' Physics talk for a twist is *torque*, represented by the Greek letter τ (tau). So, we might guess that, in analogy with NII, that the net torque and the acceleration are proportional, and in the same direction:

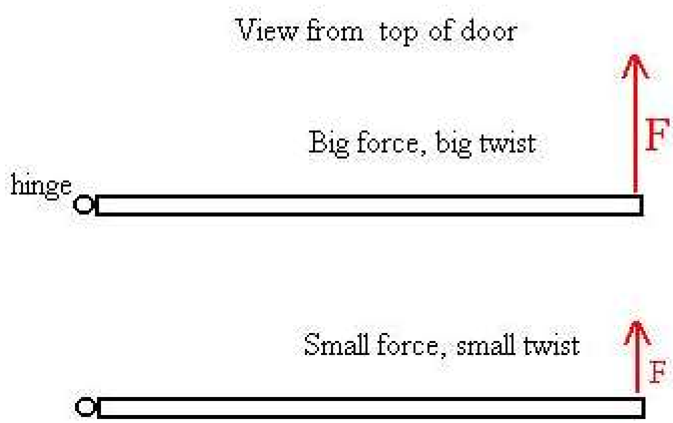
$$\sum_n \vec{\tau}_n \sim \vec{\alpha} \quad .$$

Let's choose a symbol to make this an equation,

$$\sum_n \vec{\tau}_n = I\,\vec{\alpha} \quad ,$$

where I is some measure of the object's rotational inertia (how hard it is to accelerate rotationally) in the same way that the mass m is a measure of its translational inertia (how hard it is to accelerate linearly). This quantity I may well be the mass, but let's not jump to any conclusion too soon.

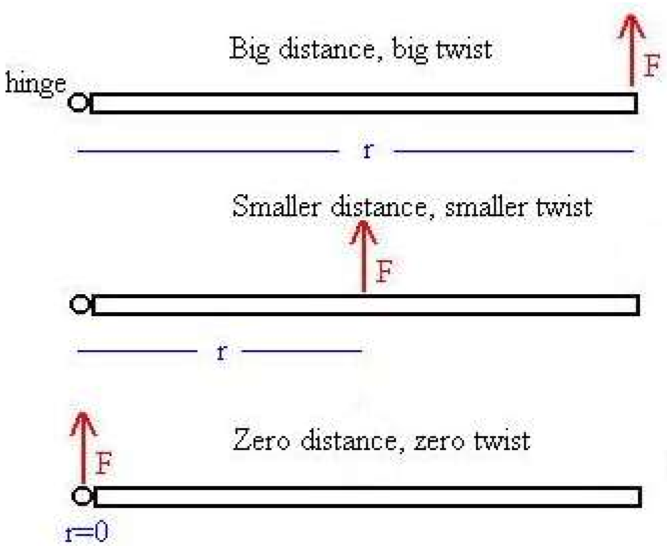
Before we try to justify this relationship, let's see if we can work out exactly what we mean by torque. Remember that we are the ones who get to define things, and if we're clever, what we define might actually be useful.



Get yourself a meter stick to play with, if you like, or better yet, walk over to a convenient door. Consider an object free to rotate around a particular axis, such as a door about its hinges. To get the door to begin to accelerate rotationally, it seems clear that a force must be applied. The larger the force, the bigger the twist applied. So, we might guess that

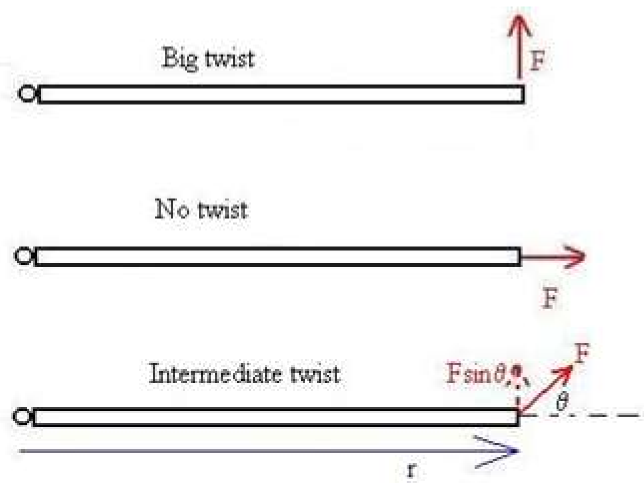
$\tau \sim F$.

Where the force is applied also seems to matter. Try pushing the door near the end, then with the same force near the center. See how the former results in more twist than the latter. Pushing near the hinge (axis) results in no twist at all. So, now we might think that $\tau \sim Fr$, where r represents the distance from the axis of rotation to the point of application of the force.



Lastly, we see that there is a dependence on the orientation of the force with respect to the door. Namely, if we pull or push along the length of the door, there is no twist, and we obtain the maximum twist when the force is at right angles to the door. At intermediate angles, it seems clear that we need to take the component of the force which is perpendicular to the r-vector, namely F sinθ, where θ is the angle as shown between the force vector and the r vector. So, perhaps $\tau \sim F\, r \sin\theta$.

We also need to define a direction for the torque (after all, we can twist a bottle cap on or off). Assume that the door in the figures starts from rest in the example above, then starts to turn CCW as a result of the applied force shown. Then, $\Delta\vec{\theta}$ is out of the page, $\vec{\omega}_{ave}$ is out of the page, and $\vec{\alpha}$ is out of the page. Since for Newtonian translational motion, the net force and the acceleration point in the same direction, we will require the net torque and the angular acceleration to do so as well. We see that we can get this result by defining the torque as the *cross-product* of $\vec{r}$ and $\vec{F}$:



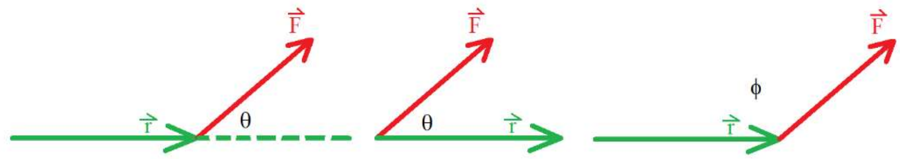
$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{or} \quad |\vec{\tau}| = |\vec{r}|\left|\vec{F}\right| \sin\theta_{r,F} \quad \text{(RHR)}\,.$$

Writing this definition as a cross product is really just shorthand; the second version above reminds you of what you actually need to do. Review the discussion of the right-hand rule (RHR) in Section 1. The order of the subscripts on theta tells you which finger to use for each vector.

EXAMPLE 9-3

Use the right-hand rule to confirm that it gives you the desired direction of the torque in the very last figure above.

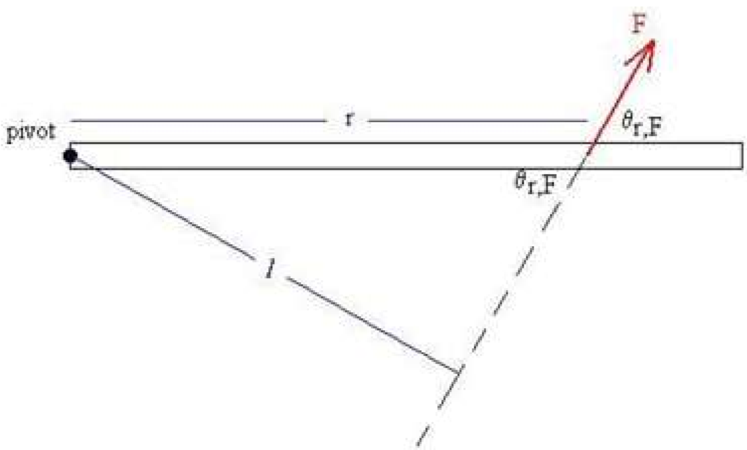
Your index finger should be pointed exactly to your right, your middle finger should be at an angle to the right and towards the top of the page, and your thumb should be pointing up out of the page.



Let's take a moment to clarify something that seems to confuse. Notice that I described the angle theta in the diagram above as being between $\vec{r}$ and $\vec{F}$. Doesn't look it though, does it? You have to remember that r and F there are not drawn tail to tail, but tail to tip. When drawn correctly, it becomes clearer, as in the middle drawing. What some students do is use the angle labeled phi, which is not the correct angle, although you can see that it more closely meets the expectation of being the angle 'between' the vectors. But here's the thing: theta and phi are supplementary and the sines of supplementary angles are the same. So, it really doesn't matter which angle is used, the numerical result will be the same. Since we've now discussed it, it's O.K. with me, just be clear in your solutions.

Now, like everything else we've discussed so far in this course, this result is still tentative, since although we think we know on what factors the torque depends, we don't know the exact dependence. Only testing of the usefulness of this definition will vindicate our effort here.

There is no special unit for torque; it's clear then that we can write it in terms of newton-meters. In the U. S. Customary System., the unit is the *pound-foot*, distinct from the *foot-pound*, the unit of work. This is an interesting point: the dimension of work and of torque are the same, but the quantities are themselves very different (vector *v*. scalar).



Occasionally, the torque will be expressed as the product of a force and its *lever arm*, *l*. A little bit of trigonometry shows that $l = r\sin\theta$, so this definition is equivalent to, and sometimes more useful than, the one given above. The lever arm is found by extending the line of action of the force and finding the perpendicular distance (the lever arm) from the pivot to this line. You must be careful when reading other sources; some define r as the lever arm and it is not.

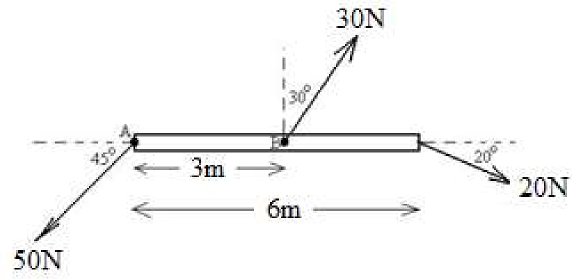
$$\vec{\tau} = \vec{l} \times \vec{F} \quad \rightarrow \quad |\vec{\tau}| = |\vec{l}||\vec{F}| \quad \text{(RHR)} \ .$$

HOMEWORK 9-2

A pendulum consists of a 2 kg bob at the end of a 1.2 m long light string, suspending the bob from a pivot. Calculate the net torque on the bob about the pivot when the string makes a 6° angle with the vertical. Indicate the direction of this net torque. Your answer to the direction will depend on how you draw your figure.

HOMEWORK 9-3

Calculate the net torque of these forces about an axis through Point A that is perpendicular to the length of the rod. Repeat for an axis through Point B.



The Second Law for Rotation and the Moment of Rotational Inertia

Following up on the notion of the existence of an analogy between linear and rotational motion, we might suspect that that there is a relationship similar to Newton's second law,

$$\sum_n \vec{F}_n = m\,\vec{a} \quad ,$$

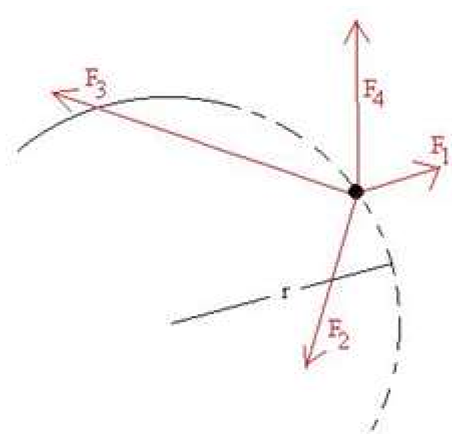
perhaps of the form

$$\sum_n \vec{\tau}_n = I\,\vec{\alpha} \quad ,$$

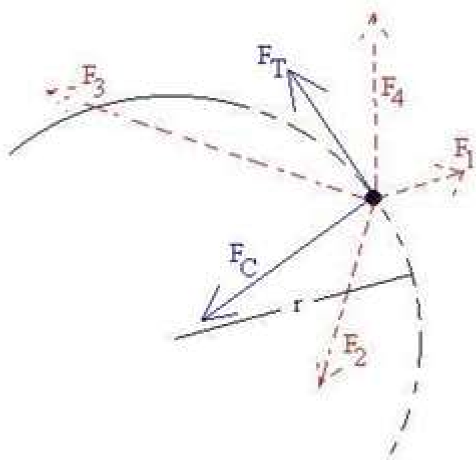
where I is a constant whose meaning we still need to divine, but which we suspect might be a measure of how difficult it is to accelerate rotationally some object, in the same way that an interpretation of the mass is as a measure of the difficulty of altering an object's linear velocity.

DERIVATION 9-2

Consider an object (point mass) constrained (for now) to move along a circular path, to which forces are applied. However many forces are applied, they can be added and resolved into components which are either centripetal or tangential, resulting in net force components as shown in the figure below. The centripetal component is what keeps the object moving in a circle and is of no particular interest to us just now. The tangential component, however, will accelerate the object <u>along</u> the circle, that is, tangentially:



$$\sum_n F_{Tn} = m\,a_T \quad .$$



Let's multiply both sides of the relationship by the radius of the circle, because, well why not?

$$r \sum_n F_{Tn} = m\, a_T\, r$$

Distribute the r to get

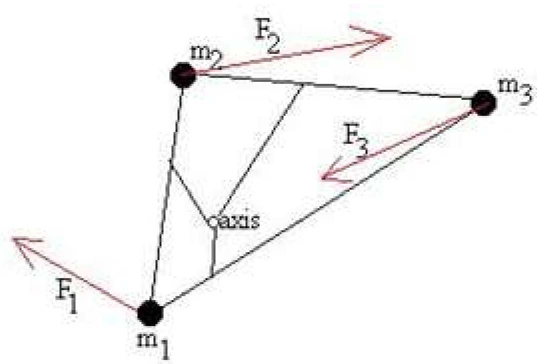
$$\sum_n r\, F_{Tn} = m\, a_T\, r \ .$$

Since every tangential force component is (by definition) perpendicular to the radius r, we recognize the terms in the sum to be the torques exerted by each of the forces, and we remember that $a_T = \alpha\, r$, so that

$$\sum_n \tau_n = m\,(\alpha r)\, r = mr^2\, \alpha \ .$$

So, <u>in this very special case</u>, we see that a rotational form of Newton's second law holds true if the proportionality constant is

$$I_{\text{Point Mass}} = mr^2 \ .$$

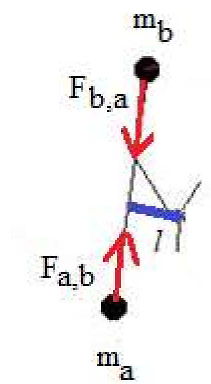


Note that this quantity depends not only on the mass, but on the <u>distribution</u> of the mass. This last comment should become clearer after the next discussion. Suppose we have an object that comprises several point masses which are somehow connected, perhaps with light rigid rods. I drew three, which is enough to make the point, but there could be as many as you like. Without bothering to calculate each torque explicitly, we can safely assume that there will be some torques applied to each object, including external torques due to the forces from other objects (make each the net force, if more than one force is desired), and also internal torques from the other objects, mediated through the rods. For each mass m_n, we can write that

$$\sum_m \vec{\tau}_{\text{EXT n m}} + \sum_m \vec{\tau}_{\text{INT n m}} = m_n r_n^2\, \vec{\alpha}_n \ .$$

If the objects rotate as a single object about a common axis, then all the α_n 's are the same. Let's add the equations.

$$\sum_{n}\sum_{m}\vec{\tau}_{\mathrm{EXT\,n\,m}} + \sum_{n}\sum_{m}\vec{\tau}_{\mathrm{INT\,n\,m}} = \left(\sum_{n} m_n r_n^2\right)\vec{\alpha} \ .$$



Let's concentrate on the internal torques term. We know from NIII that masses a and b exert forces on each other that are equal in magnitude and opposite in direction. If those forces act along the line between m_a and m_b,[2] then the lever arms associated with those forces about the axis are the same, and so the torques generated by those forces will also be equal but opposite in direction. Therefore, the sum of all the internal torques should be zero. The sum of the external torques is just the sum of the torques exerted on the masses as a unit, so we now have that

$$\sum_{q}\vec{\tau}_{\mathrm{EXT\,q}} = \left(\sum_{n} m_n r_n^2\right)\vec{\alpha} \ .$$

from which we see that <u>the moment of inertial of an extended, rigid object is the sum of the moments of its constituent parts</u>:

$$I_{\mathrm{TOTAL}} = \sum_{n} I_n = \sum_{n} m_n r_n^2 \ .$$

Because the value for the moment of inertia depends not only on the mass, but also on the <u>distribution</u> of the mass in an object, the value for I for a given object may well (and probably will) be different for different axes of rotation.

DISCUSSION 9-5 VIDEO

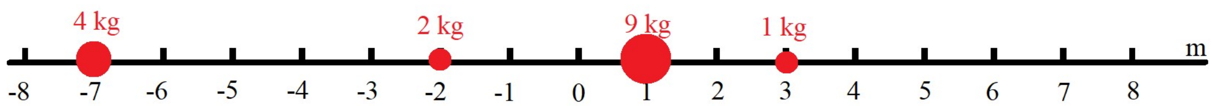
> Consider the two rods in the video. They have the same mass, but the blue one is much harder to twist around than the red one. Can you explain why?

DISCUSSION 9-6

> Pick up a meter stick at its center and try to twist it back and forth. Now try the same thing, but while holding the stick near the end. Which was harder to do? Why?

EXAMPLE 9-4

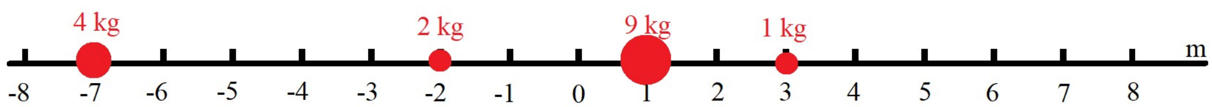
[2] This is a necessary condition for this argument to work. Looking ahead to Semester Two, the forces between atoms in real objects do indeed act along the line of connection, so we should be alright.



Find the moment of inertia of these masses about an axis passing through x = -4 m.

$$I = \sum_n m_n r_n^2 = 4(3^2) + 2(2^2) + 9(5^2) + 1(7^2) = 318 \text{ kg m}^2 \ .$$

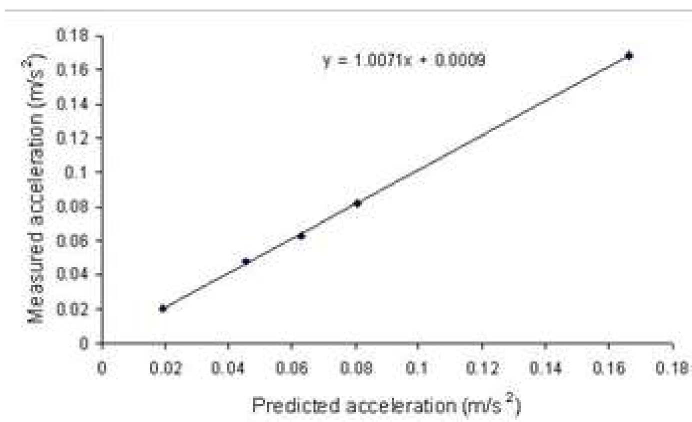
HOMEWORK 9-4



Find the moment of inertia of these masses about an axis passing through x = +1 m. What do you notice about your answer and the answer to Example 9-x?

EXPERIMENT 9-1

Here are the results of an experiment that should give us some confidence that the second law for rotation is true. Similarly to the experiment in Section 5, a hanging mass pulled a string wrapped around a horizontal disc of moment I. The data here are presented a bit differently. The linear acceleration of the falling mass is plotted against the predicted acceleration, based on the concepts discussed above. The results vary from prediction by less than 1 %.

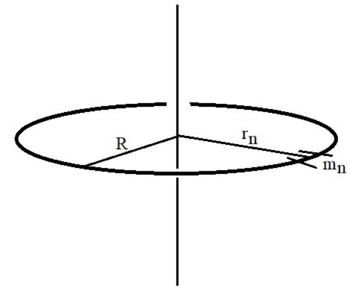


Finding the moment of inertia of an object may be conceptually easy, $I = \Sigma_n m_n r_n^2$, but actually performing this calculation can be quite difficult for continuous objects and is usually accomplished with calculus. In such a case, we break the object down into very small pieces of mass *d*m and add. the moment of inertia becomes

$$I = \sum_n m_n r_n^2 \ \rightarrow \int r^2 \, dm$$

EXAMPLE 9-5

Find the moment of inertia of a thin ring of radius R and mass M about an axis through the center perpendicular to the plane of the ring.



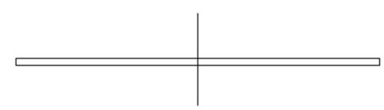
Let's break the ring up into very small masses *d*m, so small that they seem like point masses. Each is a distance r from the axis. We've shown that the moment of each point mass is *d*m r^2 and that the moment of an extended object is the sum of the moments of the individual parts. In this situation, all of the rs are equal to R, so

$$\mathrm{I} = \int \mathrm{r}^2\, d\mathrm{m} = \int \mathrm{R}^2 d\mathrm{m} = \mathrm{R}^2 \int d\mathrm{m} = \mathrm{MR}^2 \ .$$

Remember that this result is valid only for this particular axis of rotation.

EXAMPLE 9-8*

Find the moment of inertia of a uniform thin rod of mass M and length L about an axis through its center perpendicular to its length.



Let x = 0 correspond to the midpoint of the rod. Then, x = r, and we'll split the rod into small slices of length *d*x and mass *d*m:

$$\mathrm{I} = \int \mathrm{r}^2\, d\mathrm{m} = \int_{\frac{-\mathrm{L}}{2}}^{\frac{+\mathrm{L}}{2}} \mathrm{x}^2\, d\mathrm{m}\,.$$

But we now have two variables; we need to put x in terms of m, or better, m in terms of x. If the rod is uniform, then we have a proportion:

$$\frac{d\mathrm{m}}{\mathrm{M}} = \frac{d\mathrm{x}}{\mathrm{L}}\ .$$

Then,

$$\mathrm{I} = \int_{\frac{-\mathrm{L}}{2}}^{\frac{+\mathrm{L}}{2}} \mathrm{x}^2 \frac{M}{L} d\mathrm{x} = \frac{M}{L}\left(\frac{x^3}{3}\Bigg|_{\frac{-L}{2}}^{\frac{+L}{2}}\right) = \frac{1}{12} ML^2\ .$$

EXAMPLE 9-8*

Find the moment of inertia of a uniform thin rod of mass M and length L about an axis through its end perpendicular to its length.

Let x = 0 correspond to the left end of the rod. Then, x = r, and we'll split the rod into small slices of length dx and mass dm:

$$\mathrm{I} = \int \mathrm{r}^2 \, d\mathrm{m} = \int_0^{+L} \mathrm{x}^2 \, d\mathrm{m} \, .$$

Following the same path as in the previous example,

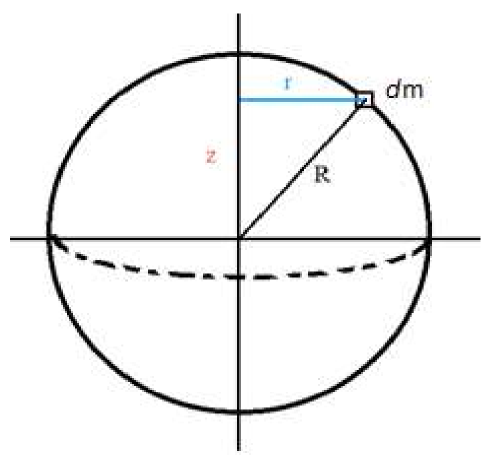
$$\frac{d\mathrm{m}}{\mathrm{M}} = \frac{d\mathrm{x}}{\mathrm{L}} \, .$$

Then,

$$\mathrm{I} = \int_0^{+\mathrm{L}} \mathrm{x}^2 \frac{M}{L} d\mathrm{x} = \frac{M}{L}\left(\frac{x^3}{3} \Big|_0^{+L} \right) = \frac{1}{3} M L^2 \, .$$

First of all, we see that this moment is larger than when the rod is rotated about its center; there is more mass located farther from the axis here than before. I want to use these two calculations in an example later.

EXAMPLE 9-X



Find the moment of inertia of a very thin hollow spherical shell of mass M and radius R about any one of its diameters.

For this solution, let's make it the z-axis. Each small mass dm is a distance r from the z-axis. From the Pythagorean theorem, $z^2 + r^2 = R^2$, and so we can make a substitution as follows:

$$\mathrm{I_z} = \int \mathrm{r}^2 \, d\mathrm{m} = \int (\mathrm{R}^2 - \mathrm{z}^2) \, d\mathrm{m}$$

Now, here's the trick. Let's repeat this calculation for rotation around the x axis,

$$\mathrm{I_x} = \int (\mathrm{R}^2 - \mathrm{x}^2) \, d\mathrm{m}$$

and the y-axis,

$$I_y = \int r^2\, dm = \int (R^2 - y^2)\, dm$$

Next, two things. First, all of the points x, y, z, must be located where there is mass, *i.e*,. a distance R from the center of the sphere, such that $R^2 = x^2 + y^2 + z^2$. Second, each of the expressions above are, by symmetry, equal and individually what we're looking for: $I_x = I_y = I_z = I_{SPHERE}$. Let's add them together.

$$3I_{SPHERE} = I_x + I_y + I_z = \int (R^2 - x^2 + R^2 - y^2 + R^2 - z^2)\, dm$$
$$= \int (3R^2 - x^2 - y^2 - z^2)\, dm + \int (3R^2 - R^2)\, dm = 2R^2 \int dm$$
$$= 2MR^2$$

Finally,

$$I_{SPHERE} = \frac{2}{3}MR^2 \quad .$$

EXAMPLE 9-7*

Find the moment of inertia of a disc (mass M, radius R, and uniform areal density σ) about an axis through its center perpendicular to its plane.

In this case, we'll consider some very thin rings concentric with the disc's center. Each has area

$$dA = 2\pi r\, dr \ .$$

Once again, we'll make a proportion, given that the density is uniform:

$$\frac{dm}{M} = \frac{dA}{\pi R^2} \quad \rightarrow \quad dm = \frac{2M\, r\, dr}{R^2} \quad .$$

Then,

$$I = \int r^2\, dm = \int_0^R r^2 \frac{2M\, r\, dr}{R^2} = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\frac{R^4}{4}$$

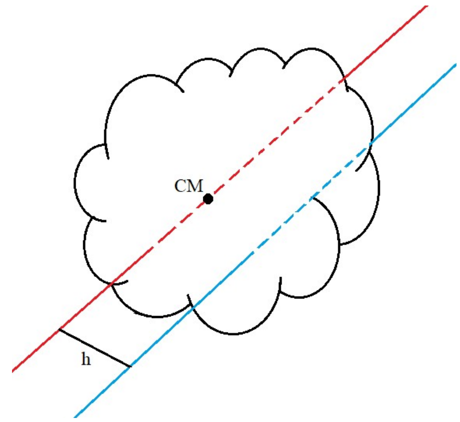
$$I_{DISC} = \frac{1}{2}MR^2.$$

Next, let's add to our toolbox with two derivations.

The *parallel axis theorem* states that, if one knows the moment of inertia of an object of mass M about an axis passing through the center of mass of an object (I_{CM}), then the moment about any other axis parallel to that one is given by

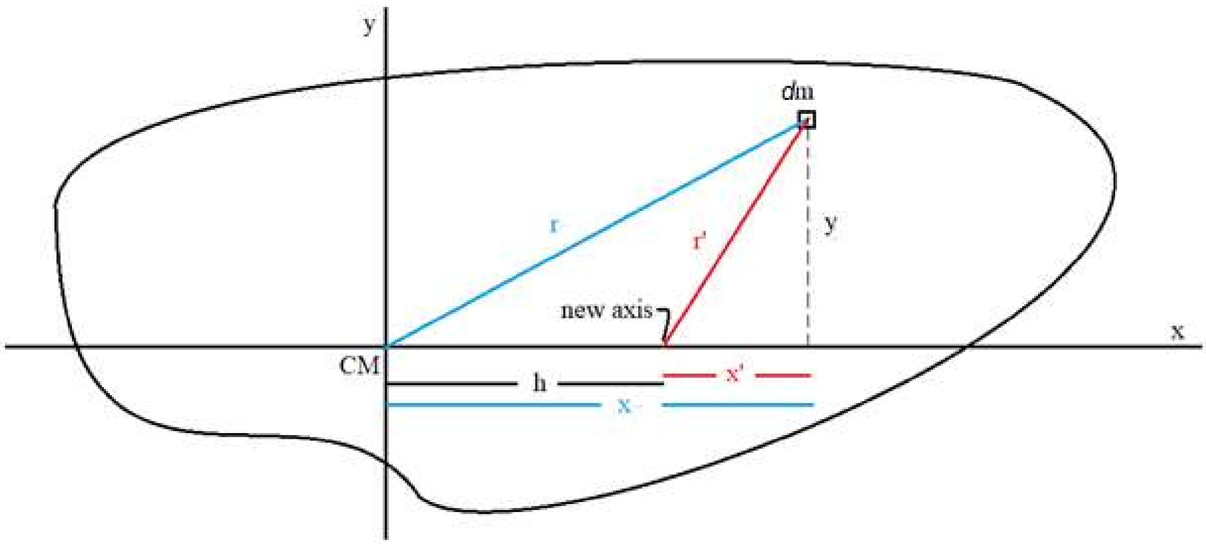
$$I_{PARALLEL} = I_{CM} + Mh^2 \quad ,$$

where h is the distance the second axis is displaced from the first.



DERIVATION 9-3

For simplicity of calculation, place the origin at the center of mass, let the original axis of rotation be the z-axis, and align the x axis along the direction of the displacement of the axis of rotation. That way, y = y'.



The moment about the center of mass is

$$I_{CM} = \int r^2\, dm = \int (x^2 + y^2)\, dm \quad .$$

The moment about the new axis is

$$I_{PARALLEL} = \int r'^2\, dm = \int (x'^2 + y'^2)\, dm = \int ((x-h)^2 + y^2)\, dm$$
$$= \int (x^2 - 2xh + h^2 + y^2)\, dm = \int (x^2 + y^2)\, dm - 2h \int x\, dm + h^2 \int dm$$

The first term we recognize as I_{CM}, the third is Mh^2, and the second is 2hM times the x coördinate of the center of mass, which we specified was at the origin, so that term is zero. So,

$$I_{PARALLEL} = I_{CM} + Mh^2 \quad .$$

To sum up, the parallel axis theorem is valid for any rigid object of any shape. One should know the moment of inertia about an axis through the center of mass, but that could be any such axis. The moment about any axis parallel to that original axis can be found with the relationship above.

EXAMPLE 9-X

Use the parallel axis theorem to find the moment of inertia of a thing rod of length L and mass M about an axis through one end perpendicular to its length (we did this above with calculus).

We presumably already know that the moment of inertia about the center axis is

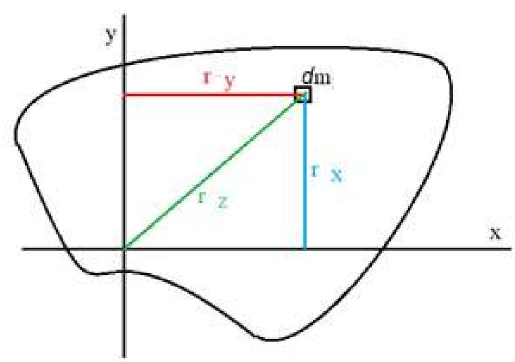
$$\mathrm{I}_{CM} = \frac{1}{12}ML^2 \,.$$

The moment about an axis perpendicular to that one offset by L/2 is then

$$\mathrm{I}_{\mathrm{END}} = \mathrm{I}_{\mathrm{CM}} + \mathrm{Mh}^2 = \frac{1}{12}ML^2 + \mathrm{M}\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

as before.

The *perpendicular axis theorem* is valid for conditions very different than for the parallel axis theorem. The object must be infinitesimally thin and flat. The location of the center of mass is irrelevant here. Choose a perpendicular set of x- and y- axes in the plane of the object; these axes do not even need to pass through the object. Suppose that we know the moments of inertia about each of the x- and y-axes, I_x and I_y. The moment of inertia about the z-axis, perpendicular to the plane of the object and intersecting the other two axes, is given by



$$\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y \quad .$$

DERIVATION 9-4

$$\mathrm{I}_x = \int \mathrm{r}_x^2\, dm = \int \mathrm{y}^2\, dm \qquad\qquad \mathrm{I}_y = \int \mathrm{r}_y^2\, dm = \int \mathrm{x}^2\, dm$$

$$\mathrm{I}_z = \int \mathrm{r}_z^2\, dm = \int (\mathrm{x}^2 + \mathrm{y}^2)\, dm = \int \mathrm{y}^2\, dm + \int \mathrm{x}^2\, dm = \mathrm{I}_y + \mathrm{I}_x \quad .$$

EXAMPLE 9-9

Suppose that we want to know the moment of inertia about the diameter of a hoop. We already know the moment about an axis through the center, perpendicular to the hoop, is MR^2. We'll make use of the perpendicular axis theorem 'in reverse' to solve this problem.

Let the x-axis be a diameter, and let the y-axis be the diameter perpendicular to the first. By symmetry, we can assert that $I_x = I_y$. Then,

$$I_{PERP} = I_z = I_x + I_y = 2I_x = 2I_{DIAMETER} \quad \rightarrow \quad I_{DIAMETER} = \tfrac{1}{2}I_{PERP} = \tfrac{1}{2}MR^2 \ .$$

EXAMPLE 9-10

Find the moment of inertia of a disk (mass M and radius R) about an axis in the plane of the disc, passing tangentially through the rim of the disc.

This time, we'll make use of both theorems. First, use the perpendicular axis theorem to find the moment of inertial about a diameter; the method is similar to that used in the preceding example. Then use the parallel axis theorem to slide the axis over to the edge.

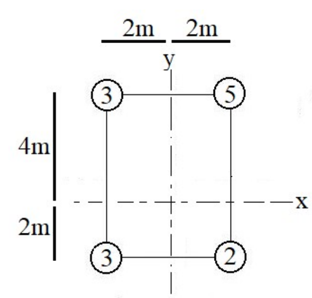
$$I_{PERP} = I_z = I_x + I_y = 2I_x = 2I_{DIAMETER} \quad \rightarrow \quad I_{DIAMETER} = \tfrac{1}{2}I_{PERP} = \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)$$
$$= \tfrac{1}{4}MR^2 \ .$$

Since a diameter of a disc passes through the center of mass, we are O.K. with using the parallel axis theorem with $h = R$:

$$I_{TANGENTIAL} = I_{DIAMETER} + Mh^2 = \tfrac{1}{4}MR^2 + MR^2 = \tfrac{5}{4}MR^2 \ .$$

HOMEWORK 9-5

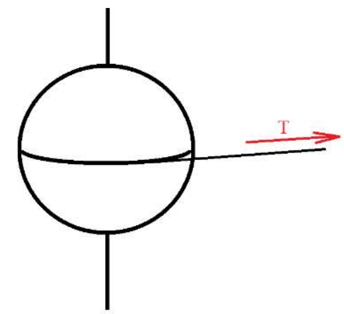
Four masses are connected by very light stiff rods, as shown in the figure. Find the moment of inertia of the four masses about the x-axis, then about the y-axis, then about the z-axis (out of the page, intersecting the other two). The masses are in kilograms. Are your results consistent with the perpendicular axis theorem?



HOMEWORK 9-6

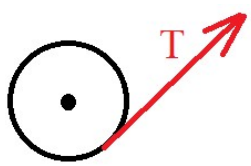
For the object in the preceding problem what magnitude toque must be applied to give it an angular acceleration of 3.5 rad/s^2 about the x-axis? The y-axis? The z-axis?

EXAMPLE 9-11



Consider a solid sphere with a light string wrapped around its 'equator.' The radius of the sphere is 3 kg and its radius 0.2 meters. If I pull the string in the plane of the equator with a force of 45 N, what will be the angular acceleration of the sphere?

The second figure is as seen from above the sphere. There is the weight of the sphere is downward (into the page) and there is a normal force of some kind holding the sphere up. These forces exert no torque because they are exerted at the axis and so their rs are zero.

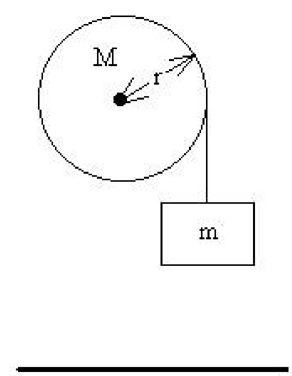


$$\sum_n \tau_n = I\alpha$$

$$(0)(gm)\sin(?) + (0)(F_N)\sin(?) + R(T)\sin(90^o) = \frac{2}{5}MR^2\alpha$$

I insert the question marks for a number of reasons. The angles themselves are undefined because there is no measurable angle between the force and a zero vector (r). Secondly, it maintains the format of the terms in the calculation, and so you are less likely to make an error. Lastly, this format tells me right away that you know that the torque term is zero and why its zero. Continuing,

$$T = \frac{2}{5}MR\alpha \quad \rightarrow \quad \alpha = \frac{5T}{2MR} = \frac{5(45)}{2(3)(0.2)} = 188\ \mathrm{rad/s^2}$$



HOMEWORK 9-7

A uniform disc (r = 0.6 m, M = 1.8 kg) is suspended vertically from a frictionless axle as shown in the figure. A string is wrapped around the wheel and is connected to a mass (m = 0.5 kg) as shown. If the mass m is released from rest, what is the linear acceleration of the mass and the tension in the string?

Rotational Kinetic Energy

Continuing with the notion of there being quantities in rotational motion which are analogous to quantities in translational motion, we might expect that there is such a thing as *rotational kinetic energy*.

DISCUSSION 9-7

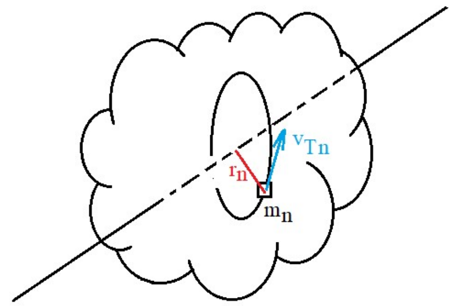
Can you guess the formula for rotational kinetic energy? In rotation, what takes the place of linear speed? What takes the place of the mass? Does your guess have the correct dimension?

DERIVATION 9-5

Consider a rigid object rotating about some stationary axis. That is, the object is rotating, but not translating. Each particle of the object, *d*m, will have kinetic energy by virtue of its motion, and the total K will be the sum of the individual Ks:

$$K_{ROT} = \int \frac{1}{2} v^2 \, dm \quad .$$

As seen from the axis of rotation, these **v**s are tangential velocities, $\mathbf{v}_T$. We saw previously that there is a relationship between the angular velocity and the tangential velocity,



$$v_T = \omega r \ ,$$

so we can substitute

$$K_{ROT} = \int \frac{1}{2} v_T^2 \, dm = \int \frac{1}{2} \omega^2 r^2 \, dm \quad .$$

But all the ωs are the same, since it's a rigid body, so factor it (and the half) out of the sum:

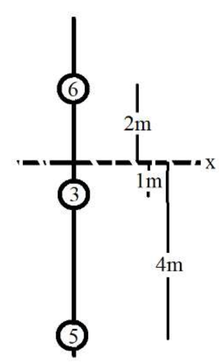
$$K_{ROT} = \frac{1}{2} \, \omega^2 \int r^2 \, dm \quad .$$

The remaining integral we recognize as the moment of inertia for the object, and so

$$K_{ROT} = \frac{1}{2} I\omega^2 \quad .$$

as expected. The unit of rotational kinetic energy is still the joule. Note that, like many of the things we discuss, this is a bookkeeping thing; we can think of this energy as the sum of the translational kinetic energies of the individual particles, or as the rotational kinetic energy of the object as a whole. Don’t double count!

HOMEWORK 9-7



In the discussion, we noted that rotational kinetic energy is just a convenient way of keeping track of the individual kinetic energies of all the small particles making up an object.

Three masses (labeled in kg) are connected in a line by strong light rods. They rotate at angular speed 6 rad/s^2. Find the following:

A) The moment of inertia about the x-axis
B) The kinetic energy using $^1/_2 I\omega^2$.
C) The tangential speed v_T of each mass as it moves in its circle.

D) The kinetic energy from $\Sigma_n \frac{1}{2} m_n v_{Tn}^2$.

How do the results from Parts B and D compare?

What happens when an object is rotating in addition to an overall translational motion? We'll consider a common case in which the axis of rotation maintains its orientation. In other words, the object rotates but doesn't tumble.

DERIVATION 9-6*

Each particle of mass m_n will have a velocity vector $\vec{v}_n$, as seen by some outside observer, so that

$$K = \int \tfrac{1}{2} v^2 \, dm = \int \tfrac{1}{2} \vec{v}_n \cdot \vec{v}_n \, dm \quad .$$

Now, we can use the concept of relative velocities to write $\vec{v} = \vec{v}_{RA} + \vec{v}_T$ for each piece dm , where $\vec{v}_{RA}$ is the velocity of the rotational axis as seen by our bystander and $\vec{v}_T$ is the tangential velocity of dm relative to an observer riding along with the rotational axis.

$$\begin{aligned} K &= \int \tfrac{1}{2} (\vec{v}_{RA} + \vec{v}_{Tn}) \cdot (\vec{v}_{RA} + \vec{v}_{Tn}) \, dm = \int \tfrac{1}{2} (\vec{v}_{RA} \cdot \vec{v}_{RA} + 2\vec{v}_{RA} \cdot \vec{v}_T + \vec{v}_T \cdot \vec{v}_T) \, dm \\ &= \int \tfrac{1}{2} v_{RA}^2 \, dm + \vec{v}_{RA} \cdot \int \vec{v}_T \, dm + \int \tfrac{1}{2} v_T^2 \, dm \quad . \end{aligned}$$

The last term we recognize from just above as the rotational kinetic energy of the object as if it were not translating. The first term is the object's kinetic energy as if it were not rotating. The middle term is tough. Remember that the velocity $\vec{v}$ is the instantaneous time rate of change of the position, $\vec{r}$. The masses of course do not change.

$$\int \vec{v}_T \, dm = \int \frac{d\vec{r}_{RA}}{dt} \, dm = \frac{d}{dt}\left(\int \vec{r}_{RA} \, dm \right) = \frac{d(M \, \vec{r}_{CM,RA})}{dt} = M \, \frac{d(\vec{r}_{CM,RA})}{dt} = M \, \vec{v}_{CM,RA} \quad .$$

This is the velocity of the object's center of mass relative to the rotational axis. Finally, we obtain

$$K = \tfrac{1}{2} M v_{RA}^2 + M \, \vec{v}_{RA} \cdot \vec{v}_{CM,RA} + \tfrac{1}{2} I_{RA} \omega^2 \quad .$$

Now, let's consider a very common special case, that of an object which is translating while at the same time rotating about an axis passing through the center of mass. In that case, $\vec{v}_{CM,RA} = 0$ and $\vec{v}_{RA} = \vec{v}_{CM,RA}$, so that this reduces to:

$$K = \tfrac{1}{2} M v_{CM}^2 + \tfrac{1}{2} I_{CM} \omega^2 \quad ,$$

that is, the total kinetic energy is the sum of the translational kinetic energy as if the object were not rotating and the rotational kinetic energy as if the object were not translating.

DISCUSSION 9-8

How do we transfer energy into or out of a rotating (or rotatable) object? What did we need to do to transfer energy in Section 6? Can you think of a relationship based on our analogies between linear and angular motions?

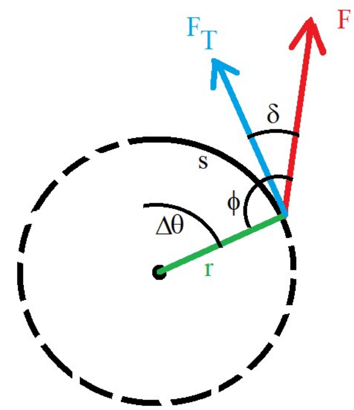
DERIVATION 9-7

We know that work involves forces, so let's apply a force F to an object at a distance r from the axis of rotation. The point of application of the force moves a distance s along a circle as the object rotates by an angle theta. We're interested in the component of the force tangent to the circle, that is, parallel to the motion of the point of application. See for example HOMEWORK 6-X.

$$\mathrm{W} = \mathrm{F}_{\|}\,\mathrm{s} = \mathrm{F}_{\mathrm{T}}\,\mathrm{s} = \mathrm{F}\cos(\delta)\,\mathrm{s}.$$

Since r and F_T are perpendicular, the cosine of delta equals the sine of phi and s = r Δθ. Substituting,

$$\mathrm{W} = \mathrm{F}\cos(\theta)\,\mathrm{s} = \mathrm{F}\sin(\phi)\,\mathrm{r}\,\Delta\theta = \tau\,\Delta\,\theta = \vec{\tau}\cdot\Delta\vec{\theta}\,.$$



Since both the toque and the angular displacement are vectors, directions matter. If the torque and displacement are in the same direction, either into or out of the page, then the work is positive and if they are in opposite directions, then the work is negative (remember, we're dealing only with rotations in a plane).

The instantaneous power can be written as

$$\mathrm{P}_{\mathrm{INST}} = \vec{\tau}\cdot\vec{\omega}$$

We might also be able to define a potential energy associated with rotation. An example is that of a torsional spring. Consider a wire or string which exerts a torque proportional to the angle through which its end has been twisted and in the opposite direction of that angular displacement:

$$\vec{\tau}_{\mathrm{TORSION}} = -\,\kappa\,\Delta\vec{\theta}\quad.$$

Then we would without hesitation assume that there is a corresponding potential energy given by

$$U_{TORSION} = \frac{1}{2}\kappa(\Delta\theta)^2\quad.$$

What about the units? Well, κ is in N m/radians (yet <u>another</u> quantity with the same dimension as energy!) and the $U_{TORSION}$ is in (Nm) rad^2 or Nm, so this looks O.K. dimensionally.

DISCUSSION 9-9

Now we have three types of potential energy and two types of kinetic energy. Can energy be redistributed from any of these to any other?

HOMEWORK 9-8

Consider the situation of Homework 9-x. Using conservation of mechanical energy, find the speed of the hanging mass after it has fallen a distance of 3 meters. Assume both masses are initially motionless.

A special example of an object translating and rotating is one which 'rolls without slipping.' In that case, there is a nice relationship between the angular velocity and the translational velocity of the center of mass. First, let's show that.

DERIVATION 9-8*

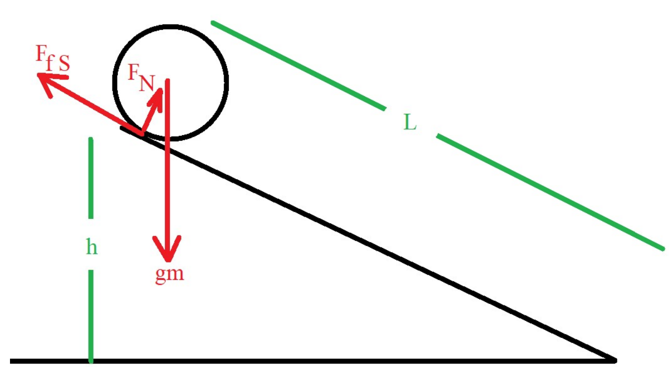
Consider a uniform circular wheel or something similar with radius R rolling without slipping across a horizontal floor. The center of mass has velocity $\vec{v}_{CM}$ as seen by an outside observer. Relative to the center of mass, a point on the outside edge where the object touches the floor will have an angular speed omega and a tangential velocity $\vec{v}_{T,CM}$. such that $v_{T,CM} = \omega R$. But, since that point is at the moment not moving,

$$\vec{v}_{CM} + \vec{v}_{T,CM} = 0 \quad \rightarrow \quad v_{CM} = \omega R \ .$$

Remember that if the object does slip in the surface, then this relationship is almost certainly invalid.

EXAMPLE 9-12

Let's repeat an example we've already done several times. A disk of mass M = 5 kg and radius R = 2 cm rests at the top of an incline (height h = 1.2 m, length L = 2 m). It's released and rolls without slipping down the incline. What is the disk's speed when it arrives at the foot of the incline? Will it be 4.9 m/s?



Let's try using conservation of mechanical energy. Set y = 0 at the bottom of the incline.

$$W_{NC} = \frac{1}{2}Mv_{CMf}^2 - \frac{1}{2}Mv_{CMi}^2 + \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 + gMy_f - gMy_i$$

What forces act on the disk and how much work does each do?

$W_N = 0$ (force is perpendicular to the path)
W_g – conservative
$W_f = 0$ – We're going to justify this after we're done. Be patient!

Then,

$$0 = \frac{1}{2}Mv_{CMf}^2 - \underbrace{\frac{1}{2}Mv_{CMi}^2}_{starts\ from\ rest} + \frac{1}{2}I\omega_f^2 - \underbrace{\frac{1}{2}I\omega_i^2}_{starts\ from\ rest} + \underbrace{gMy_f}_{y=0\ at\ bottom} - gMy_i$$

$$gMy_i = \frac{1}{2}Mv_{CMf}^2 + \frac{1}{2}I\omega_f^2$$

For a disk rotating about its central axis, I = ½ MR^2. Since it rolls without slipping, we can make use of the relationship $v_{CM} = R\omega$. Lastly, we'll replace y_i with h to obtain:

$$\mathrm{gMh} = \frac{1}{2}\mathrm{M}v_{\mathrm{CMf}}^2 + \frac{1}{2}\left(\frac{1}{2}\mathrm{MR}^2\right)\left(\frac{v_{\mathrm{CMf}}}{\mathrm{R}}\right)^2.$$

Now some interesting developments. First, the mass drops out, so our answer is independent of the mass of the disk. Also, R drops out, so the result is independent of the size of the disk.

$$\mathrm{gh} = \frac{1}{2}v_{\mathrm{CMf}}^2 + \frac{1}{4}v_{\mathrm{CMf}}^2 = \frac{3}{4}v_{\mathrm{CMf}}^2$$

$$v_{\mathrm{CMf}} = \sqrt{\frac{4\mathrm{gh}}{3}} = \sqrt{\frac{4(10)1.2}{3}} = 4\ \mathrm{m/s}.$$

DISCUSSION 9-10

When we did this example for a block sliding down a frictionless incline, the result was 4.9 m/s. Why is this result different? The block and the disk started with the same potential energies. What happened to that energy? Which energy determines how quickly an object moves? Does one have more of that kind than the other and if so, where did the rest of the potential energy go? Would the result be different if this were a solid sphere instead of a disk?

In the Section 6 example, gravitational potential energy was converted into translational kinetic energy. Here, however, there are two types of kinetic energy, translational and rotational. The potential energy must be split between these two categories. How much goes into each category depends on the shape of the object. For example, repeating the example above with a solid sphere where the moment of inertia is $^2/_5MR^2$ changes the final velocity to 4.14 m/s. Less energy converted to rotational kinetic energy means more available for translational energy.

Shape	Fraction before MR^2	*Per cent* Translational K	*Per Cent* Rotational K
Hoop	1	50%	50%
Hollow Sphere	2/3	60%	40%
Disk	1/2	66.7%	33.3%
Solid Sphere	2/5	71.4%	26.6%

The final velocities of these objects down the ramp depend only on the fraction in front of the moment of inertial term.

DISCUSSION 9-11

Let's run a race but placing two shapes at the top of an incline and releasing them simultaneously. Which will arrive first?

Shapes1.mp4

Consider a disk and a hoop with the same masses and radiuses. Which will win a race rolling down an incline?

Shapes2.mp4

Consider two disks of the same mass, but C has half the radius of A. Which will win a race rolling down an incline?

Shapes5.mp4

Consider two disks with D having both a radius and a mass much less than A. Which will win a race rolling down an incline?

Shapes4.mp4

Consider sphere F which has the same radius and mass as hoop E. Which will win a race rolling down an incline?

Consider Hoop G and Hoop E with G having both a radius and a mass much less than A. Which will win a race rolling down an incline?

Shapes7.mp4

Were any of the results seen in the film different than what you expected? Can you explain why the expected results were not obtained?

JUSTIFICATION 9-1*

Let's clean up the question about work done by static friction. Consider a ball rolling on a flat horizontal surface. It has translational kinetic energy $\frac{1}{2} m v_{CM}^2$ and rotational kinetic energy $\frac{1}{2} I\omega^2$. If it is simply rolling without slipping, not being driven by any agency, then there is a relationship between v_{CM} and ω, namely that $v_{CM} = \omega R$. The friction, if any, will be static, but due to the synchonization of the two types of motion, there is no tendency to slip, and the static frictional force, which is only as large as it needs to be, will be zero.

But what about an object on an inclined plane? Well, we know that the rolling object will have a lower speed at the bottom of the incline than will the frictionlessly sliding object, so friction must have done some negative work on the rolling object. It's easy enough to calculate for the example above:

$$W_{fs} = F_{fs} \, L \cos(180^o) = -F_{fs} \, L \, .$$

The frictional force also exerts a torque on the object about its center that points into the page,

$$\tau_{fs} = R \, F_{fs} \, \sin(90^o) = -R \, F_{fs} \; .$$

Since the angular displacement $\Delta\theta$ is also into the page, the work done in terms of rotation is

$$W_{fs} = \tau_{fs} \, \Delta\theta \, \cos 0^o = +R \, F_{fs} \, \Delta\theta == +F_{fs} \, (R \, \Delta\theta) = +F_{fs} \, L \, .$$

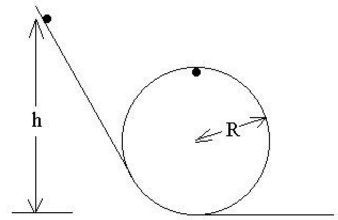
So the <u>total</u> work done by the friction is zero.

EXERCISE 9-3*

Here's a classic problem. Consider a bowling ball that is released with initial translational velocity v_o sliding down the lane but not initially rotating. Calculate the velocity of the ball and how far down the alley it is when it begins to roll without slipping. HINT: What condition is met when the ball rolls without slipping?

HOMEWORK 9-14

Consider the loop-de-loop track. A small round object with radius r<<R is placed on the track at altitude h and released. It rolls without slipping along the track and just barely makes it around the top of the loop. Find h if the object were a



A) solid sphere.
B) hollow sphere.
C) disk.
D) hoop.

HINT: If you represent the fraction before the mr^2 by some symbol, you can do almost all of the problems at once.

Angular Momentum

Again as an analogy with linear motion, we might suspect that there is such a thing as *angular momentum* ($\vec{L}$), and we might guess that it is defined as $I\vec{\omega}$ (analogous to $\vec{p} = m\vec{v}$). Let's see:

Starting from the rotational form of the Second Law,

$$\vec{\tau}_{EXT} = I\vec{\alpha} \quad ,$$

we'll substitute the definition of angular acceleration (and assume that I is constant!) to get

$$\vec{\tau}_{EXT} = I\frac{\Delta\vec{\omega}}{\Delta t} \quad ,$$

$$\vec{\tau}_{EXT}\,\Delta t = I\Delta\vec{\omega} = \Delta(I\vec{\omega}) = \Delta\vec{L} \ .$$

The left-hand side of the preceding relationship is the rotational equivalent of impulse, and we can see that, in the absence of any external rotational impulses, the total amount of angular momentum is constant, or conserved. Our result suggests that the angular momentum points in the same directions as does the angular velocity.[3]

Several observations. First, for linear momentum, we expected that the masses of objects could not change, so that any changes in momentum $\vec{p}$ were due to changes in velocity. For angular momentum, we see that a change in angular momentum can be effected by changing either $\vec{\omega}$ or I or both. Secondly, and more interestingly, we remember the constant, droning repetition that all three of the pictures we developed in linear motion (force and acceleration, work and kinetic energy, and impulse and momentum) were not only equally valid, but derivable from each other. We might expect the same from the three pictures developed for rotational motion, namely torque and angular acceleration, work and rotational kinetic energy, rotational 'impulse' and

[3] Most of our derivations have worked out that way. For example, $\vec{J} = \Delta\vec{p} = \Delta(m\vec{v})$, so we assume that $\vec{p} = m\vec{v}$.

angular momentum. In the classical world we are studying this semester this is so, but in the real world, we find the suggestion that angular momentum is somewhat more fundamental as a concept than the other two. In your chemistry courses, you may have come across the notion that angular momentum is *quantized*, that is, that only certain numerical values are allowed; this can be true of energies also, but the values allowed depend on the exact system. Angular momentum may well be the most important topic we cover in this course, and the one we spend the least amount of time on.

DISCUSSION 9-11

VIDEO

Rotating student with barbells. By pulling the barbells in towards his body, he reduces the moment of inertia, I. If there are no external torques, the angular velocity correspondingly increases. This is the same effect used by figure skaters and high divers.

Student with bicycle wheel. A non-rotating student holds a wheel that is rotating so as to have (say) one unit of angular momentum, pointing upward (call this +1). Inverting the wheel causes the student to begin rotating. In the absence of external torques, the total angular momentum must remain +1. Inverting the wheel changes its angular momentum to -1, and the student then acquires angular momentum +2, so that the sum remains +1. How does the student magically acquire just the right amount of angular momentum? Inverting the wheel required that the student apply a torque, and so, by the third law, a torque equal in magnitude but opposite in direction was applied by the wheel on the student.

We can derive analogous relations for the final angular velocities for totally inelastic 'collisions' and for totally elastic 'collisions' by substituting moments of inertia for masses and angular velocities for linear velocities, although there are some restrictions on when these will be valid (the Is should be constant, for example!).

EXAMPLE 9-13

A 10" LP of mass 110 grams is dropped down the spindle onto a freely turning 12" turntable platter of mass 1 kg initially turning at $33^1/_3$ revolutions *per* minute (rpm). What is the final speed of the turntable in rpm?

HINT: Assume that both the LP and the platter are discs.

This is like a totally inelastic collision in Section 7 since the two objects have a common final angular velocity. The two objects share a common axis, so the third law of motion is valid for torques. Since the platter is freely turning, there are no external torques and we can use conservation of angular momentum:

$$L_{TOTAL\,i} = I_R\omega_{Ri} + I_P\omega_{Pi} = I_R\omega_{Rf} + I_P\omega_{Pf} = L_f \quad ,$$

$$I_R\omega_{Pi} = (I_R + I_P)\omega_f \ ,$$

$$\omega_f = \frac{I_R \omega_{Pi}}{I_R + I_P} = \frac{\frac{1}{2} m_P r_P^2}{\frac{1}{2} m_R r_R^2 + \frac{1}{2} m_P r_P^2} \omega_{Ri} = \frac{(1)12^2}{(.11)7^2 + (1)12^2} 33.3 = 32.1 \text{ rpm} \quad .$$

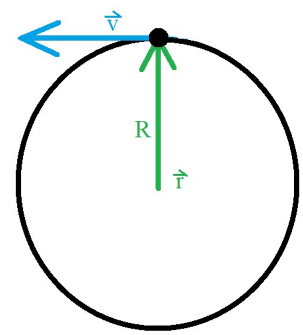
HOMEWORK 9-9

Many schools have a lab practical to cap off their physics courses. You are given a closed box with a shaft extending from one side. You are told that the shaft is attached to the center of a round symmetrical flywheel of mass 7 kg and radius 0.4 m. When you attach a constant torque motor (11.73 Nm), the system goes from rest to 600 rpm in 3 seconds. What shape or shapes could the flywheel be?

DERIVATION 9-9*

Is there a relationship between linear and angular momentum? Consider a special case of an object of mass m moving in a circle of radius r (location vector $\vec{r}$) with angular velocity $\vec{\omega}$. From our definition above,

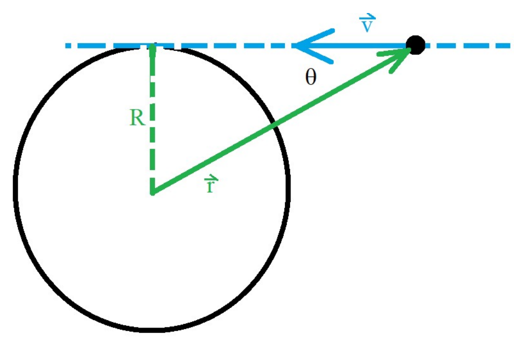
$$L = I\omega = mR^2 \frac{v}{R} = Rmv \quad .$$



Magnitude-wise, this looks promising. Direction-wise, we want L to point out of the paper towards us, parallel to omega. We can do that with a cross product. We can try $\vec{v} \times \vec{r}$, but that points into the paper, so we'll make it $\vec{r} \times \vec{v}$. Since in our example, $\vec{v}$ and $\vec{r}$ are perpendicular, the angle between them is 90° and we have

$$\vec{L} = m\,\vec{r} \times \vec{v} = \vec{r} \times \vec{p} \quad \rightarrow \quad |\vec{L}| = m\,r\,v \sin\theta_{r,v} = r\,p\,\sin\theta_{r,p} \quad (\text{RHR}) \,.$$

But what if $\vec{r}$ and $\vec{v}$ are not perpendicular? Well, since the object is moving in a plane, it's certain that at some time the situation will appear as above, but for most of the time, it will look like this figure. What does our proposed relationship give us in that situation? Since R = r sinθ, the result is the same, L = Rmv out of the page. So, it seems we have a nice way of writing angular momentum in terms of vectors.



HOMEWORK 9-x

Jimmy runs 2 m/s tangentially to a frictionless playground merry-go-round and jumps on. If Jimmy's mass is 30 kg and the platform has mass 100 kg and radius 2 m, what is the final angular speed of Jimmy and the platform?

HOMEWORK 9-10

VIDEO

A professor stands on a freely rotating platform like the one in the demonstration. With his arms outstretched, he has an angular speed of 2 radians/second. Once his arms are drawn inward next to his chest, his speed becomes 6 rad/sec. What is the ratio of his final kinetic energy to his initial kinetic energy?

EXERCISE 9-1 Solution

$\theta_i = 0$ (make that the origin)
$\theta_f = -500$ revolutions = - 3141.6 radians Why is it negative?
$\omega_i = +30$ rad/s
$\omega_f = -40$ rad/s reversed direction
$\alpha = ? \leftarrow$
$t = ? \leftarrow$

Try KEq 4:

$$\omega_f^2 = \omega_i^2 + 2\vec{\alpha} \cdot \Delta\vec{\theta}$$

$$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta} = \frac{(-40)^3 - 30^2}{2(-3141.6)} = -0.11 \text{ rad/s}^2 \cdot$$

Then KEq 1:

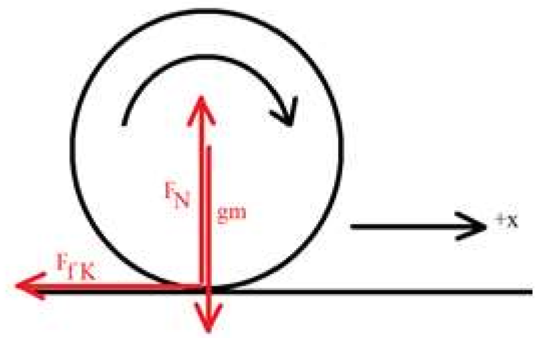
$$\vec{\omega}_f = \vec{\omega}_i + \vec{\alpha}t \rightarrow \quad t = \frac{\omega_f - \omega_i}{\alpha} = \frac{-40 - 30}{-0.11} = 636.4 \text{ seconds} \ .$$

EXERCISE 9-3 Solution

There are three forces acting on the bowling ball: the weight, the normal force from the floor, and the kinetic frictional force from the floor. Using the second law of motion,

$$+F_N - gm = ma_y = 0 \quad ; \quad -F_{fK} = ma_x \quad ; \quad F_{fK} = \mu_K F_N \quad \rightarrow \quad a_x = -\mu_K g \ .$$

In terms of torques about the center axis, F_N has angle 180º (or 0º, depending on how you measure the angle) and the weight has a lever arm of zero, since we have previously demonstrated that it can be considered to be applied at the center of mass. That leaves the friction:



$$\tau = RF_{fK}\sin(90^o) = I\alpha$$

$$R\mu_K gM = \tfrac{2}{5}MR^2\alpha$$

$$\alpha = \frac{5\ \mu_K g}{2R}.$$

We'll decide that CW rotation is positive to match the positive x-motion to the right (that is, into the page will be positive). Now we might consider setting $\Delta x = R\ \Delta\theta$, but that's not true because of the skidding. It's also temping to think that, when the ball stops skidding, $a = \alpha R$; this is a true statement, but both are zero (the friction goes from kinetic to static!) and so that's not very useful. We have to look at $v_{CM} = r\omega$ and use KEq 1 for both linear and rotational motions. Let t be the time from ball's launch to when it rolls without slipping.

$$v_f = R\omega_f$$

$$v_o + at = R(\omega_o + \alpha t)$$

$$v_o - \mu_K g\, t = R\left(\frac{5\ \mu_K g}{2R}\right)t$$

$$v_o = \frac{7\ \mu_K g}{2}t \quad \rightarrow \quad t = \frac{2v_o}{7\mu_K g}.$$

Then,

$$v_f = v_o + (-\mu_K g)\left(\frac{2v_o}{7\mu_K g}\right) = \frac{5}{7}\ v_o$$

and

$$x_f = x_i + v_i t + \tfrac{1}{2}at^2 \quad \rightarrow \quad x_f = 0 + v_o\left(\frac{2v_o}{7\mu_K g}\right) + \tfrac{1}{2}(-\mu_K g)\left(\frac{2v_o}{7\mu_K g}\right)^2 = \frac{10}{49}\frac{v_o^2}{\mu_K g}$$