5-5)

The current density is the current passing through a cross-sectional area divided by the area, so that the current is

$$dI = J\,dA = \frac{2I_o}{\pi R^4}\,r^2\,dA \quad .$$

Since the density is symmetric about the line r = 0, we can consider a thin ring of area of radius r and width *d*r where the current density is the same all round. The area of such a ring is 2πr *d*r.

$$dI = \left(\frac{2I_o}{\pi R^4}\,r^2\right)\,(2\pi\, r\, dr) \;=\; \frac{4I_o}{R^4}\,r^3\,dr \;.$$

We'll use this in the next two parts of the problem.

A) To find the total current, let's add up all the *d*Is with r varying from 0 to R:

$$I = \int_0^R \frac{4I_o}{R^4}\,r^3\,dr = \frac{4I_o}{R^4}\int_0^R r^3\,dr = \frac{4I_o}{R^4}\frac{r^4}{4}\bigg|_0^R = \frac{4I_o}{R^4}\frac{R^4}{4} = I_o \quad .$$

B) To find the current within distance r of the axis, use the same integral but stop at r < R:

$$I = \int_0^r \frac{4I_o}{R^4}\,r^3\,dr = \frac{4I_o}{R^4}\int_0^r r^3\,dr = \frac{4I_o}{R^4}\frac{r^4}{4}\bigg|_0^r = \frac{4I_o}{R^4}\frac{r^4}{4} = \frac{I_o}{R^4}\,r^4 \quad .$$

Then, use Ampère's law:

$$\oint B_{||}\,dl = \mu_o I_{Enclosed} = \mu_o \frac{I_o}{R^4}\,r^4$$

$$B\,2\pi\,r = \mu_o \frac{I_o}{R^4}\,r^4$$

$$B = \frac{\mu_o \frac{I_o}{R^4}\,r^4}{2\pi\,r} = \frac{\mu_o\,I_o}{2\pi R^4}\,r^3$$

C) Now, when r>R, the entirety of I_o is enclosed:

$$\oint B_{||}\,dl = \mu_o I_{Enclosed}$$

$$B\,2\pi\,r = \mu_o I_o$$

$$B = \frac{\mu_o I_o}{2\pi\, r} \quad \text{(same as for a thin wire)}\ .$$

Note that these two expressions agree when r = R.