Section 2-1 - Electric Charges, Forces, & Fields

Interactions between Materials

We started by discussing that static electricity had been known for thousands of years, the name deriving from the Greek for a type of amber which, when rubbed, produced sparks. We investigated the interactions of several types of materials: wool, silk, glass, and bakelite. W e saw that certain combinations attract one another and other combinations repel. We summarized the interactions in a chart:

	bakelite	wool	glass	silk
bakelite	R	A	A	R
wool	A	R	R	A
glass	A	R	R	A
silk	R	A	A	R

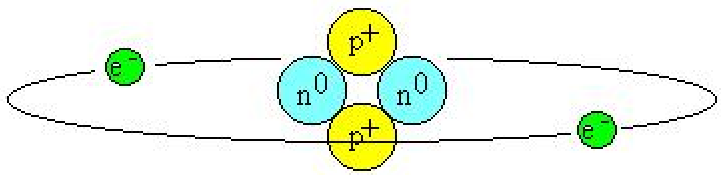
where 'A' means that an attraction was seen, and 'R' means that the two materials repelled one another. We discussed that we should make a table which list every possible material, and check its interaction with every other type of material, and that this would be a valid model for the effect. We define the property of the material which governs the interaction *charge*, and we postulate that there may well be a different type of charge for each type of material, *e.g.*, silk-type charge or glass-type charge, which in this case happen to attract one another, and a set of $(N^2 + N)/2$ rules, where N is the number of materials in the universe. However, from grade school conditioning, students remember that there is an alternate model in which there are only two types of charge, which are arbitrarily named *positive* and *negative*, that obey these two rules: like charges repel; unlike charges attract. This is often referred to as Franklin's 'One Fluid Model.' How could we develope this model from the information given, and can this model account for the behaviors seen in the experiments we performed? Let's arbitrarily choose the bakelite to be negative; since we would expect both pieces of bakelite to possess the same charge, let's postulate that negative charge repels negative charge. Then the wool would have to be positive to account for the attraction, from which we might conclude tentatively that opposite charges attract.

	bakelite (-)	wool (+)	glass	silk
bakelite (-)	R	A	A	R
wool (+)	A	R	R	A
glass	A	R	R	A
silk	R	A	A	R

Then, we see that the glass would have to be positive to attract the negative bakelite. We might then predict that the (+) glass would then repel the (+) wool (that is , that positive charge repels positive charge, or more generally that like charges repel), and this is in fact consistent with our observations. We predict that the silk is negative, since it was attracted to the (+) glass, and we finish off by predicting the rest of the interactions:
bakelite (-) should repel silk (-) - correct
wool (+) should attract silk (-) - correct.
We need to verify this model for many more combinations of materials, but we may well have assured ourselves that it is a valid working model.
Now we have two working models: one which requires two types of charges and two rules, and another which has an infinite number of types of charges (silk-type, wool- type, *et c*.) and an infinite number of rules. Which should we accept as correct? We employ *Occam's Razor* to assume that the simpler of two equally valid explanations is the correct one.

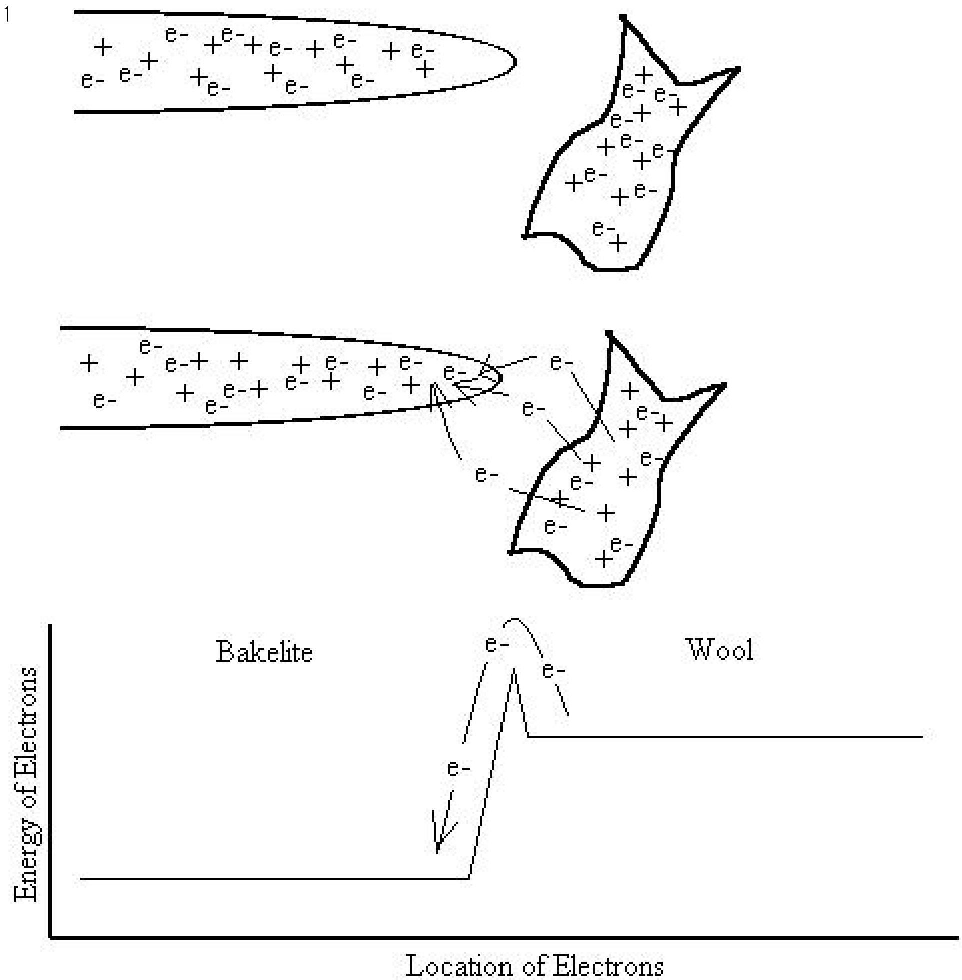
A Twentieth Century Explanation

Every school child knows what an atom looks like:



This model, referred to as *Bohr's planetary model,* is actually the result of about 40 years of intense experimental research and hard thinking by a lot of very smart people. Since it is not our topic for today, we'll accept it as correct without question (at least for today). The *protons* carry what we now call positive charge and are located in the *nucleus*, along with the neutral *neutrons*. The negatively charged *electrons* orbit the nucleus, and are about 1/2000 the mass of either the protons or the neutrons and carry a charge equal to that of the proton, though negative. In general, we expect there to be equal numbers of protons and electrons in each atom. We will see eventually that life would have been much easier for us if Franklin had assigned his negative and positive labels in reverse, but since the electron was discovered a century and a half later, I guess we should forgive him.

In the demos we did in the previous section, we always charged the objects by rubbing them together. This took charge from one object and deposited it on the other; which way the charge goes depends entirely on energy considerations. We see that the electrons are almost always what moves, since they are one thousand to ten million times easier to remove from the outer edges of an atom than are the protons deep in the nucleus. Once removed, they are 2000 times easier to move around, due to their smaller mass. For example, when we rubbed the wool on the bakelite, electrons were transferred from the wool to the bakelite, causing an excess of electrons on the bakelite, making it negatively charged, and a deficit of electrons on the wool, rendering it positively charged. This occurred because the electrons have a slightly lower energy when on



The small amount of work we do by rubbing provides the energy necessary for the electrons to make it over the necessary activation 'hump.'

Properties of Charge

We say that charge is *conserved*, that is that the net amount of charge in a closed system remains constant. In the demonstrations above, the absolute value of the excess negative charge on the bakelite was exactly the same as that of the excess positive charge on the wool. We didn't create charge on the bakelite and the wool, nor did we create some on the bakelite and destroy some on the wool; we transferred it from one object to the other, so that the excess of electrons on the bakelite equals the deficit of electrons on the wool.

Here is an example to consider: *anti-matter* does exist outside of *Star Trek*®. For each sub-atomic particle of matter, there is an anti-particle which is identical in every way except that it has the opposite charge. When anti-pairs contact one another, both are destroyed (*annihilated*) and the energy is carried away in electro-magnetic radiation, usually *gamma rays*. Consider this reaction: $p^+ + p^- = 2\,\gamma^o$. Is charge conserved in this reaction?

We also say that charge is *quantised*, by which we mean that there is a smallest non-zero amount of charge possible (given the symbol *e*), and that all charges are integer multiples of that fundamental charge (*i.e.*, 0, ± e, ± 2e, ± 3e, *et c*.). This is shown by the famous *Millikan oil drop experiment* (Yes, O.K., earlier work by Faraday suggested quantisation). We find that the fundamental charge amount corresponds to the charge on a proton, which is equal although opposite to the charge on the electron; since the electron was only discovered in 1899 and the proton after that, this is a recent discovery. Before the discovery of this fundamental quantity, charges were measured in larger, more convenient, but completely arbitrary units called *coulombs*; $1e = 1.6\text{x}10^{-19}$ C, or $1\text{C} = 6\text{x}10^{+18}\ e$.

As an aside, there is a theory that there are particles called *quarks* that have charges less than *e* (± $^1/_3e$ or ± $^2/_3e$). It is thought that combinations of quarks make up protons, neutrons, and many other particles (but not electrons). This model has been very successful in predicting the results of various reactions between particles, however, individual quarks have never been seen. For the purposes of this class, they do not exist.

Coulomb's Law

Let's quantify the interactions between charges. Coulomb performed an experiment consisting in hanging two conductors from a metal wire, for which the torque necessary to twist the wire through a given angle is known. The conductors are charged, then brought in proximity to two other spherical, charged objects, which then either repel or attract the hanging conductors. By measuring the amount of rotation of the hanging conductors, Coulomb could measure the forces of attraction or repulsion, and plotted these against the distance between the sets of conductors. He found that the forces of both attraction and repulsion follow a $1/r^2$ law similar to gravity's. Further investigation showed that the force is proportional to the product of the charges as well; again, very similar to gravity:

$$F_e = \frac{k_e q_1 q_2}{r^2}\ .$$

This result is valid for the interaction between point charges, or uniform spherical charges (we'll see why the latter, later). The value of the proportionality constant is about $9\text{x}10^9\ \text{Nm}^2/\text{C}^2$.

How should we approach situations in which there are more than two charges? We asserted in Section 4 of the first semestre that forces can be super-imposed, that is, added as vectors. So we find the force exerted on any given charge by each of the other charged, then add.

Example:
Consider two identical positive charges q which are separated by a distance R; a third, negative charge of the same magnitude is located exactly between the other two. Find the net force
a) on the left hand charge.
b) on the centre charge.

First, find the magnitude of each force, then worry about its direction by adding the appropriate signs before each term, then add components.

a) Let the positive direction be to the right: $F_{\text{on left charge}} = + \left|k_e q\,|-q|/(R/2)^2\right| - \left|k_e q\,|q|/(R)^2\right| = +3k_e q^2/R^2$

b) Let the positive direction be to the right: $F_{\text{on centre charge}} = - \left|k_e q\,|-q|/(R/2)^2\right| + \left|k_e q\,|-q|/(R/2)^2\right| = 0$.

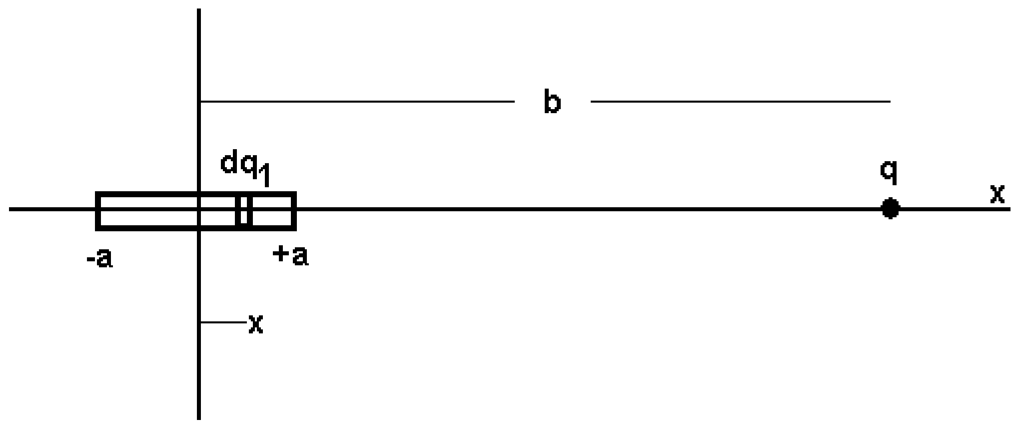
Example:
Three charges are arranged as described below:

	Charge	x co-ordinate	y co-ordinate
Q_1	-2 nC	0	-0.1 m
Q_2	+5 nC	0	0
Q_3	+6 nC	+0.3 m	0

Find the magnitude and direction of the electro-static force acting on Q_2.

$F_{12} = 9\text{x}10^9(2\text{x}10^{-9})(5\text{x}10^{-9})/0.1^2 = 9\text{x}10^{-6}$ N in the -y direction (attractive force).
$F_{32} = 9\text{x}10^9(6\text{x}10^{-9})(5\text{x}10^{-9})/0.3^2 = 3\text{x}10^{-6}$ N in the -x direction (repulsive force).
In this case, these form the x and y components of the net force, so
$F_{NET} = [(-3\text{x}10^{-6})^2 + (-9\text{x}10^{-6})^2]^{1/2} = 9.5\text{x}10^{-6}$ N at an angle of $\arctan[-9\text{x}10^{-6}/-3\text{x}10^{-6}] = 252^{\circ}$.

Now let's do a more complicated example. Instead of two point charges, we have a rod of length 2a centred on the origin and a point charge q located at x=b (b>a). The rod has its charge q distributed evenly along its length with linear charge density $\lambda = q/2a$.



We start the problem by making it look like one we do know how to do, forces between point charges. Break the rod into small charges dq of length dx such that $dq = \lambda dx = (q/2a)dx$. The position of dq_1 is at x and the position of q_2 is at x= b. The distance between the two point charges is then r = b – x. The force of repulsion between the two charges is then given by

$$dF = \frac{k_e dq_1 q_2}{(b - x)^2} = \frac{k_e q^2 dx}{2a(b - x)^2}$$

Now, since every dq_1 on the left rod repels q_2 and does so along the x-axis, we can simply add up the force contributions for each (*i.e.,* the magnitude of the sum of the vectors is the sum of the magnitudes).

$$F = \frac{k_e q^2}{2a} \int_{-a}^{+a} \frac{dx}{(b - x)^2}$$

where the limits of integration are chosen so as to include all of the 'point charges' on the rod.

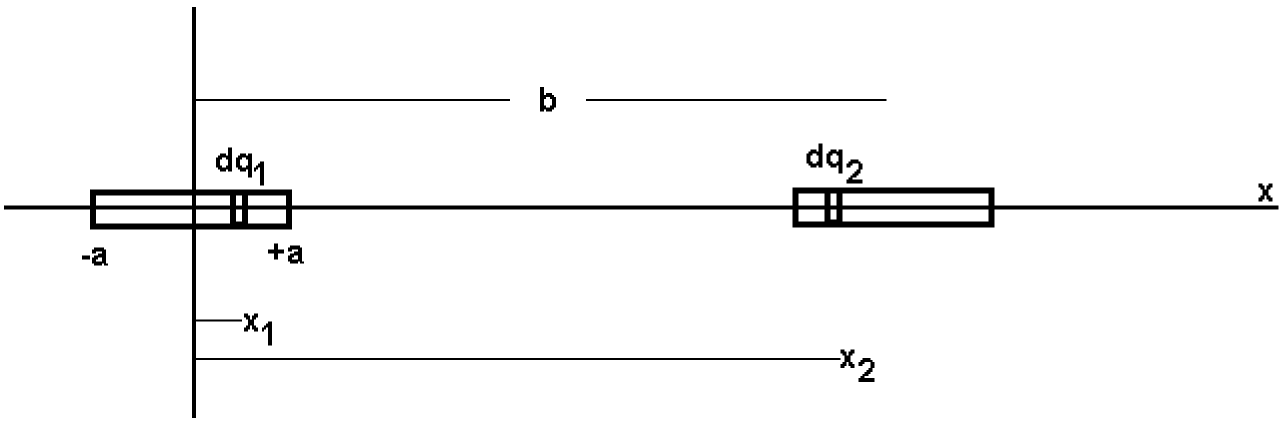
Making use of a u substitution (u = b – x and so $du = - dx$) results in

$$F = \frac{k_e q^2}{2a} \left[\frac{1}{b - a} - \frac{1}{b + a}\right] = \frac{k_e q^2}{2a} \frac{2a}{b^2 - a^2} = \frac{k_e q^2}{b^2 - a^2} .$$

It's always a good idea to try to check the result. If, for example, the length of the rod were to be reduced to zero, the result above should in turn reduce to that for two identical point charges separated by distance b. This is clearly the case:

$$\lim_{a \to 0} \frac{k_e q^2}{b^2 - a^2} = \frac{k_e q^2}{b^2} .$$

Now let's do an even more complicated example. Instead of two point charges, we have two rods of length 2a whose centres are separated by distance b>2a. Each has charge q distributed evenly along its length with linear charge density λ = q/2a.



We start the problem by making it look like one we do know how to do, forces between point charges. Break each rod into small charges *d*q of length *d*x such that *d*q = λ*d*x = (q/2a)*d*x. The position of dq_1 is at x_1 and the position of dq_2 is at x_2. The distance between the two point charges is then $r = x_2 - x_1$. The force of repulsion between the two charges is then given by

$$dF = \frac{k_e dq_1 dq_2}{(x_2 - x_1)^2} = \frac{k_e q^2 dx_1 dx_2}{4a^2(x_2 - x_1)^2}$$

Now, since every dq_1 on the left rod repels every dq_2 on the right rod, and does so along the x-axis, we can simply add up the force contributions for each pair (*i.e.,* the magnitude of the sum of the vectors is the sum of the magnitudes).

$$F = \frac{k_e q^2}{4a^2} \int_{b-a}^{b+a} \int_{-a}^{+a} \frac{dx_1 dx_2}{(x_2 - x_1)^2}$$

where x_1 and x_2 are independent variables.

Integrating over x_1 results in

$$F = \frac{k_e q^2}{4a^2} \int_{b-a}^{b+a} \frac{2a\, dx_2}{x_2^2 - a^2}$$

Integrating over x_2 ultimately results in

$$F = \frac{k_e q^2}{4a^2}\, ln\left(\frac{b^2}{b^2 - 4a^2}\right)$$

We should see that, if a were to be reduced to zero, the force of two point charges q separated by distance b should result. As $a \to 0$, the argument of the log term above approaches 1. Let's make an expansion in the natural log term near that point, using $\ln(z) \approx z - 1$ when $z \approx 1$.[1]

$$F = \frac{k_e q^2}{4a^2} \ln\left(\frac{b^2}{b^2 - 4a^2}\right) \approx \frac{k_e q^2}{4a^2}\left(\frac{b^2}{b^2 - 4a^2} - 1\right) = \frac{k_e q^2}{4a^2}\left(\frac{4a^2}{b^2 - 4a^2}\right) = \frac{k_e q^2}{b^2 - 4a^2} \to \frac{k_e q^2}{b^2}.$$

The Electric Field

Let's review gravity a bit. We usually describe $\mathbf{a}_g$ as the acceleration due to gravity, which numerically is 9.8 m/s^2 near the surface of the earth. The weight of a mass m is given by **W** and is initially found conceptually by dropping an object and looking at the magnitude of the force that is necessary to produce that acceleration. If I cut the mass of an object in half, the weight also decreases by half, and a_g is still 9.8 m/s^2. Let me continue to reduce the mass, and by proportion the weight, until I have no mass at all left. What then is a_g if there is no mass to accelerate? Let's define **g** as the force *per* unit mass exerted by the gravity of the earth: **g** = **W**/m = 9.8 N/kg, downward. This definition works whether there is a mass at any particular spot or not, that is, it is the force the earth would exert on each kg of mass if there were indeed an object at that spot, or if you prefer,

$$\boldsymbol{g} = \lim_{m \to 0} \frac{W}{m}.$$

Note also that different units are used, although the dimension of **g** must remain the same as for $\mathbf{a}_g$. We will now refer to **g** as the *gravitational field strength*.

Newton worked out, based on Kepler's Laws of Planetary Motion, that two masses M_1 and M_2, separated by distance r, will attract each other with (mutual) forces whose (equal) magnitudes are given by

$$F = \frac{GM_1M_2}{r^2}.$$

Here, G is a constant used just to make the units work out and is equal to 6.67×10^{-11} Nm^2/kg^2. This relationship is Newton's *Law of Universal Gravitation*.
Let's check it out. We already know that the weight of an object near the earth's surface is W = *g*m, and that this same force is described by

$$W = \frac{GmM_{Earth}}{R_{Earth}^2}.$$

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[1] F(z) near $z = z_o$ is given by $F(z)|_{zo} + F'(z)|_{zo}(z - z_o) + \frac{1}{2}F''(z)|_{zo}(z - z_o)^2 + \ldots \approx \ln z|_{z=1} + 1/z|_{z=1}(z-1) = z-1$.

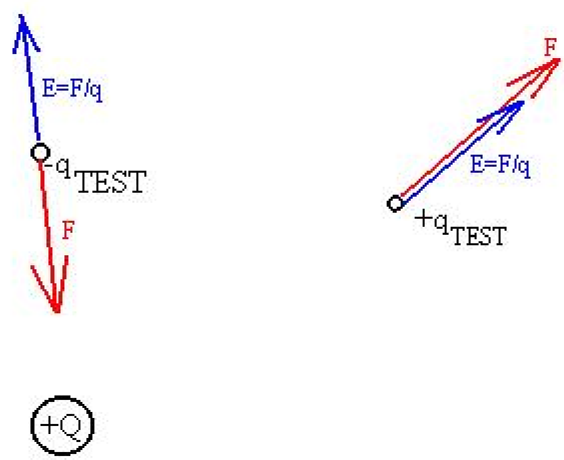
So,

$$g = \frac{W}{m} = \frac{GM_{Earth}}{R_{Earth}^2} = \frac{6.67\text{x}10^{-11} \times 6\text{x}10^{24}}{(6.4\text{x}10^6)^2} = 9.77N/kg,$$

or, about what we expected.

Let's consider a fixed charge +Q in space. If we bring a small test charge +q near to +Q, it will be repelled by a force given by:

$$F_e = \frac{k_e Q q}{r^2}.$$



Let's define the *electric field* of charge +Q at some location as the force that +Q would exert *per* unit charge on a test charge q_{TEST} put in that location:

$$\boldsymbol{E} = \lim_{q \to 0} \frac{F_e}{q_{TEST}}.$$

We note that since **F** is a vector, so is **E**. The units of electric field are newtons *per* coulomb (N/C). So, if we place $+q_{TEST}$ at various locations around +Q and find the direction of **E** at each, we see that the field from +Q points radially outward from +Q. Can we find the magnitude? The force between the charges is

$$F_e = \frac{k_e Q q}{r^2},$$

so the field is:

$$E = \frac{F}{q} = \frac{k_e Q q_{TEST}}{r^2} \Big/ q_{TEST} = \frac{k_e Q}{r^2}$$

radially outward.

NOTE: This relationship is valid only for the field produced by a *point charge*, since the force is based on the interaction between two point charges. The fields produced by other shapes may well have very different dependences.

Let's look at the field produced by a negative charge, -Q. The force on a positive test charge will be attractive, toward -Q. Using the same definition as above, we see that the field of a negative charge is again

$$E = \frac{k_e Q}{r^2}$$

but radially inward.

To be complete, we should see if the results are the same if the test charge were negative. In the first case, the force on the test charge would be radially inward (attractive), but then we would divide that force by a negative test charge, which once again makes the direction of the field radially outward. A similar argument works for -Q.

How do we find the field from more complicated distributions of charge? If the distribution is made up of a collection of point charges, then we find the field at some point P by adding the individual fields (as vectors, remember) due to the individual charges. We did several examples in class.

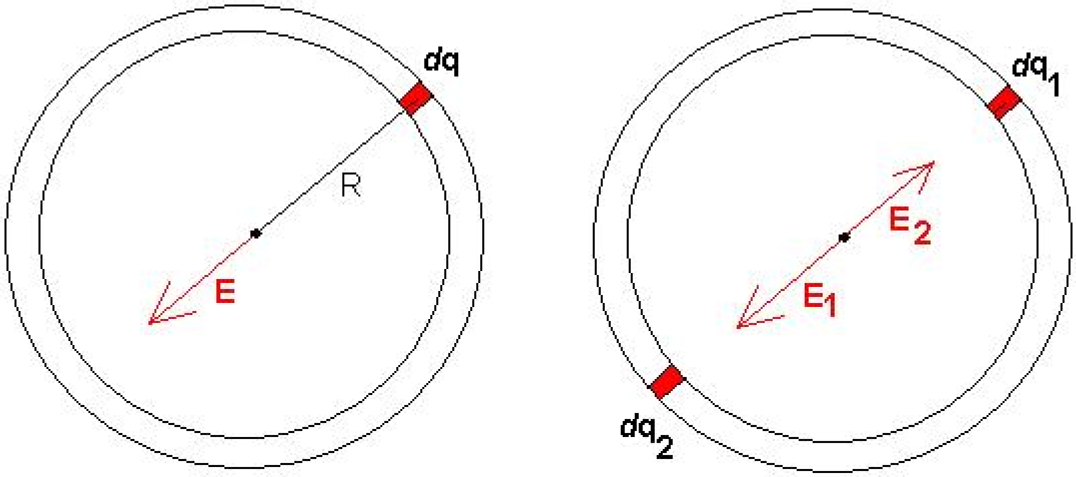
What if the distribution of charge is not composed of discreet points, but rather is continuous. In principle, the method is the same as in the last paragraph, but the execution of that method usually requires calculus. However, there are a few special shapes for which calculus is not required.

Example:
Consider a circular ring of radius R that carries a uniformly distributed charge Q. What is the electric field at the centre of the ring?

Let the charge be broken up into many very small charges, *d*q, that then resemble point charges. The electric field $\mathbf{E}_1$ at the center of the ring for one of them will have magnitude

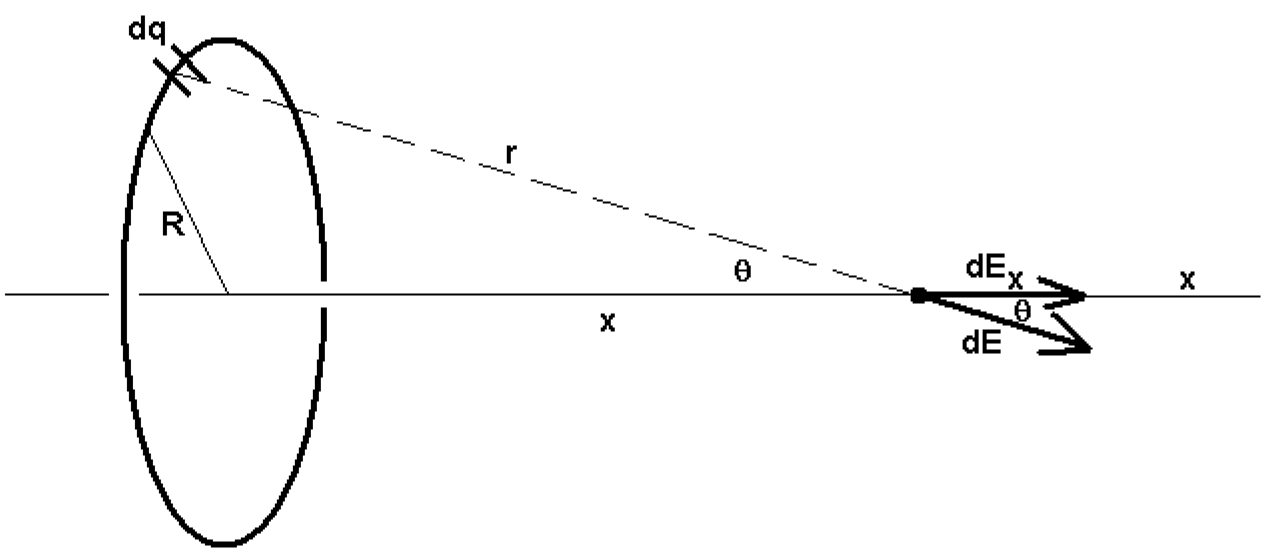
$$E_1 = \frac{k_e dq_1}{r_1^2} = \frac{k_e dq_1}{R^2}$$

and (if the charge is positive) will point away from dq_1 (see left figure below).



However, there is an identical charge dq_2on the opposite side of the ring that produces the same magnitude field at the centre, but in the opposite direction. These two fields will cancel, as indeed will all such fields, in pairs. So, the net field at the centre is zero.

Example:
E-Field along the axis of a hoop of radius R and with uniformly distributed charge Q:



Again, break the ring up into point charges dq, for which the field at a point distance x from the ring will be

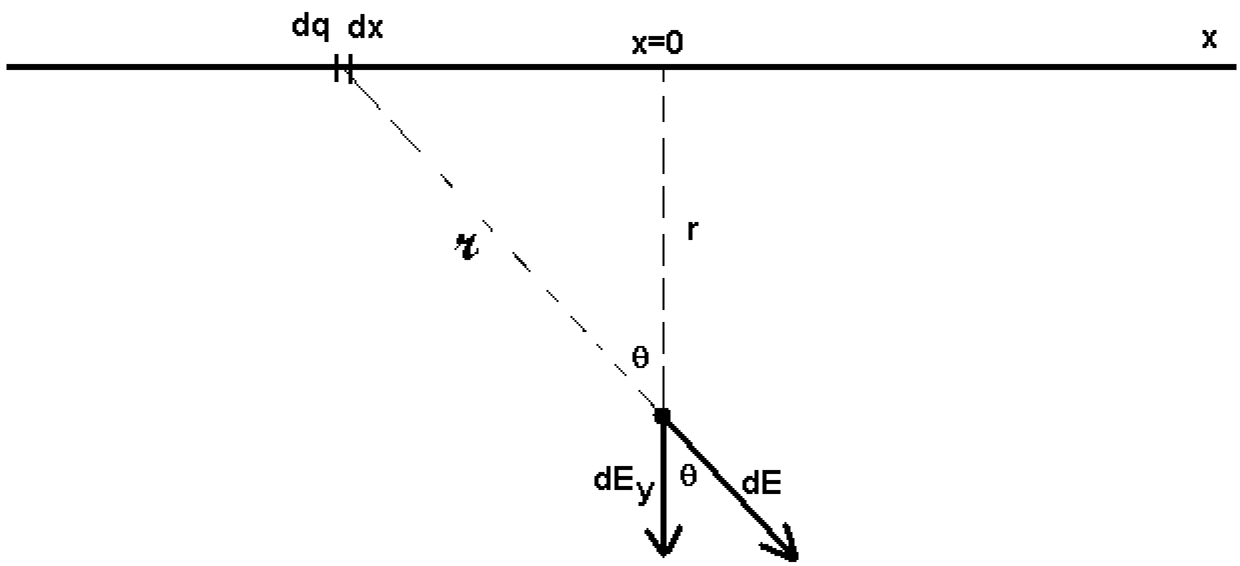
$$dE = \frac{k_e dq}{r^2}$$

The x-component of this field will be $dE_x = dE \cos\theta$. Now consider the component of *d*E that is perpendicular to the x axis. For every *d*q on the ring, there is another charge diametric to it that produces a field contribution whose perpendicular component will exactly cancel the component due to the original *d*q so that only the x components will survive. In that case, the magnitude of the sum of the contributions will be the sum of the magnitudes of the x-contributions. We'll substitute x/r for the cosine and $(R^2+x^2)^{1/2}$ for r:

$$E = E_x = \int dE_x = \int dE\, cos\theta = \int \frac{k_e dq}{r^2}\frac{x}{r} = \frac{k_e x}{(R^2+x^2)^{3/2}}\int dq = \frac{k_e x}{(R^2+x^2)^{3/2}}Q$$

Now, it might be fun to see if this result agrees with some examples we already have seen. If x goes to zero, *i.e.,* we look at the centre of the ring, we obtain E = 0, as expected from the example above. If we let x get very large (or equivalently, let R get very small), we obtain the result for a point charge Q with x replacing the original r.

EXAMPLE:
E-field a distance r from an infinitely long, straight, uniformly charged wire
Since the wire is infinitely long, it necessarily has either zero charge or infinite charge. So, we will again make use of the linear charge density,
$\lambda = Q/L = dq/dx$.



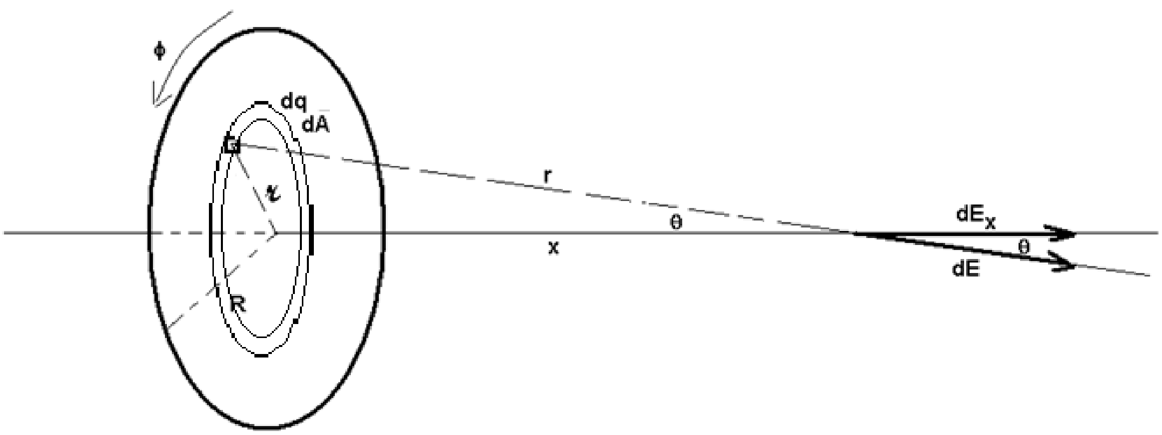
Consider a point charge *d*q located at position x on the wire. The contribution to the electric field at our point of interest is *d*E. For each such charge, there is another charge the same distance of the origin, but on the other side that produces a contribution to the field at the point and whose x component will cancel the x component of the field due to the original charge, so we need only worry about the radial components of the field, labeled dE_y in the figure. In that case, the magnitude of the sum of the contributions will be the sum of the magnitudes of the contributions.

We'll substitute $r/\mathscr{r}$ for the cosine, $(r^2+x^2)^{1/2}$ for $\mathscr{r}$, and λdx for dq:

$$E = E_y = \int dE_y = \int dE\ cos\theta = \int \frac{k_e dq}{\mathscr{r}^2}\frac{r}{\mathscr{r}} = \int_{-\infty}^{+\infty} \frac{k_e\,\lambda\, r\, dx}{(r^2+x^2)^{3/2}} = \frac{2k_e\lambda}{r}$$

Keep this result in mind; we will solve this problem a different way later.

Example:
E-field on the axis of a uniformly charged disc of radius R and charge Q:



The disc has an *areal charge density* $\sigma = Q/\pi R^2$, so $dq = \sigma dA$. Let's break the disc up into a series of concentric rings, each of radius $\mathscr{r}$ and thickness $d\mathscr{r}$**,** charge dq, and area dA. We can find an expression for dA (of the ring) in this way: A disc has area $\pi\ \mathscr{r}^2$. If we increase the radius by $d\ \mathscr{r}$, the area becomes larger by amount $dA = 2\pi\ \mathscr{r}\ d\mathscr{r}$. This is exactly the area between the edges of the original ring and the slightly enlarged ring. We already know (see above) that the field on the axis of a ring is along the axis, so the field from the disc, the sum of the rings' fields, will also be along the axis.

The field along the axis of a ring was found to be

$$E_x = \frac{k_e x}{(R^2+x^2)^{3/2}} Q\ ,$$

but, we have to translate this for our new thin ring, so, $Q \rightarrow dq$ and $R \rightarrow \mathscr{r}$:

$$dE_x = \frac{k_e x\ dq}{(\mathscr{r}^2+x^2)^{3/2}} = \frac{k_e x\ \sigma\ dA}{(\mathscr{r}^2+x^2)^{3/2}} == \frac{k_e x\ \sigma\ 2\pi\ \mathscr{r}\ d\mathscr{r}}{(\mathscr{r}^2+x^2)^{3/2}}$$

To cover the whole disc, we'll integrate from $\mathscr{r} = 0$ to R:

$$E = E_x = \int dE_x = \int_0^R \frac{2\pi\, k_e \sigma\, x\, \mathrm{r}\, d\mathrm{r}}{\left(\mathrm{r}^2 + x^2\right)^{3/2}} = 2\pi k_e \sigma \left[1 - \frac{x}{(x^2 + R^2)^{1/2}}\right]$$

Let's check the limiting cases again. Let the radius of the disc go to infinity, forming a flat infinite sheet of charge. Then the term in brackets clearly becomes 1 and the electric field becomes $2\pi k_e \sigma$. We will do this problem a different way later to verify this result. We should also see that reducing R to zero results in the expression for a point charge (or equivalently, letting $x \to \infty$):

Let the expression in the brackets be the function F(R) for which we want to find an expansion about R = 0.

$$F(R) \approx F(R)|_0 + F'(R)|_0 R + \tfrac{1}{2} F''(R)|_0 R^2 + \cdots$$

$$F|_{R=0} = 0$$

$$F'|_{R=0} = \frac{Rx}{(x^2 + R^2)^{3/2}}\, |_{R=0} = 0$$

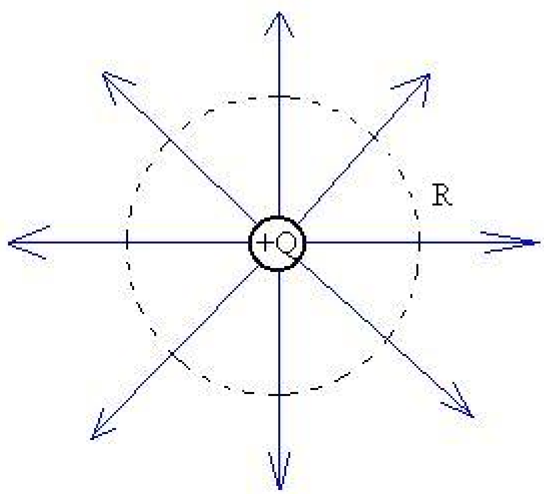
$$F''|_{R=0} = \frac{x^3 - 2xR^2}{(x^2 + R^2)^{5/2}}\, |_{R=0} = \frac{x^3}{x^5} = \frac{1}{x^2}\, .$$

And, so,

$$E \approx 2\pi k_e \sigma \left[\frac{1}{2}\frac{1}{x^2} R^2\right] = k_e(\sigma \pi R^2)\left[\frac{1}{x^2}\right] = \frac{k_e Q}{x^2}\, .$$

Gauss's Law

Let's consider a point charge +Q, whose field we know has magnitude $k_e Q/r^2$. Let's draw an imaginary sphere of radius R around it, so that the center is at the charge.

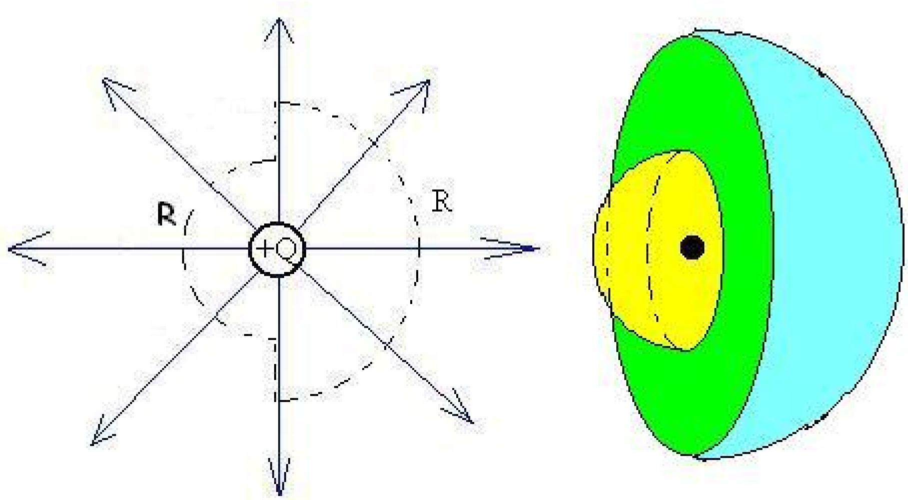


We happen to notice that the product of the field strength at distance R from the charge and the area of the sphere is independent of R:

$$\frac{k_e Q}{R^2}\,4\pi R^2 = 4\pi k_e Q$$

Now, any time we find something which remains constant, we wonder if it may be useful for solving problems. For example, we found that the total of the quantity m**v** in a closed system remains constant before and after collisions, and it was so useful in predicting the outcomes of these collisions that we gave m**v** its own name: *momentum*. Eventually, we shall call this quantity EA the *electric flux*, ϕ_E

Now let's look at a different shape to see if this relationship holds. If we imagine that the sphere is cut into two hemispheres, the quantity EA will be $2\pi k_e Q$ for each half. Let's shrink one half of the sphere to radius **R**.



Since EA is independent of the radius, each 'half' sphere will still have flux $2\pi k_e Q$. However, there is now a third surface to worry about, the flat annular surface which connects the edges of the hemispheres (shown in green). Here, the area $A = \pi[R^2 - \mathbf{R}^2]$ and the field E is certainly not zero (and not even constant!), and so the total flux (as initially defined) is greater than $4\pi k_e Q$. Rather than give up, can we somehow save our notion?

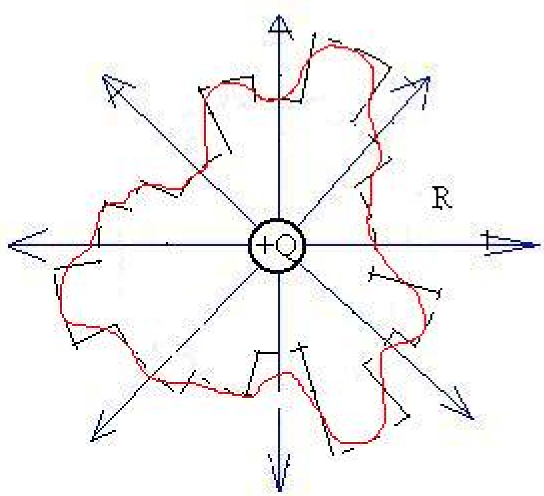
Notice that the electric field on the hemispherical portions is perpendicular to the surfaces, but that the field along the annulus is parallel to the surface. If we were to change our definition of flux to $E_{perp}A$, the contribution from the annulus would be zero and the total would be

$$\phi_E = (E_{perp})_1 A_1 + (E_{perp})_2 A_2 + (E_{perp})_3 A_3 =$$

$$[k_e Q/\mathbf{R}^2]*2\pi\mathbf{R}^2 + [k_e Q/R^2]*2\pi R^2 + [0]*\pi[R^2 - \mathbf{R}^2] =$$

$$2\pi k_e Q + 2\pi k_e Q + 0 = 4\pi k_e Q \text{ once again.}$$

Now, what if the surface surrounding +Q is of any arbitrary shape? We can always approximate the surface's shape to an arbitrary degree of accuracy with a series of spherical sections with their centers at +Q.



This gives us two types of individual surfaces dA_i to work with: the curved sections for which E is still perpendicular and for which the flux is $(E_{perp})_i dA_i$, and the sides of the sections for which E runs along the surfaces and for which the flux contribution is zero ($E_{perp} = 0$). The total flux through such a surface should be the sum of the fluxes through each small surface, or

$$\phi_E = \Sigma_i E_{perp} dA_i \rightarrow \oiint E_{perp} dA,$$

where the integral becomes appropriate as the small surfaces become very small and infinite in number.

We can then push and pull the spherical sections until they form a sphere again, remembering that the flux through each individual section does not change as the section is moved, since those that move closer to +Q get smaller areas but have E-field strengths which increase by the same factor that the areas decrease. In the end, we have a sphere with flux $4\pi k_e Q$ again. So, we can safely assert that the flux from +Q through any closed surface surrounding it must be $4\pi k_e Q$, and that the total flux through that surface can be found from the sums of the fluxes through the component parts of the surface.

What if there is more than one charge? The total field at any point is the vector sum of the fields due to the individual charges Q_1 and Q_2:

$$\mathbf{E}_{total} = \mathbf{E}_1 + \mathbf{E}_2.$$

As a result, we can also say that the perpendicular component of the total field is the sum of the perpendicular components of the fields due to each charge:

$$(E_{total})_{perp} = E_{1perp} + E_{2perp}.$$

We can write that

$$\phi_{total} = \Sigma_i\, [(E_{total})_{perp}]_i \delta A_i = \Sigma\, [[E_{1perp}]_i + [E_{2perp}]_i]\delta A_i =$$

$$\Sigma\, [E_{1perp}]_i \delta A_i + \Sigma\, [E_{2perp}]_i \delta A_i = 4\pi k_e Q_1 + 4\pi k_e Q_2 = 4\pi k_e (Q_1 + Q_2),$$

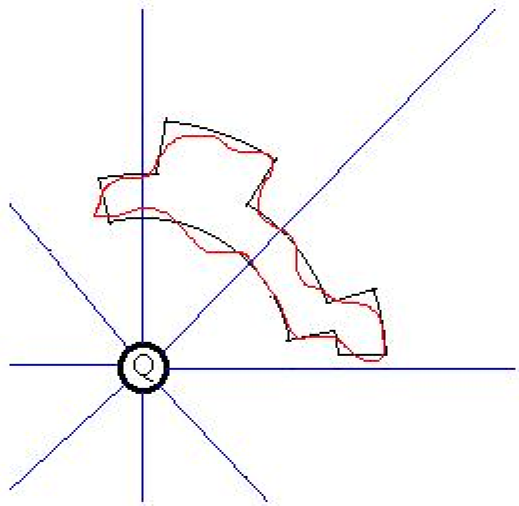
which we can generalize to as many charges as we like.

What if the charge inside is negative? Hey, good question, shows you're thinking. Suppose that I put a positive charge +Q and a smaller negative charge (-q) inside a spherical surface. It should be pretty clear that the total electric field at any point on the surface generally will be reduced from the value it would have had when only the positive charge was there, since the outward pointing field from +Q will be partially if not completely canceled by the inward pointing field from (-q). This reduction in the magnitude of E_{total} will correspondingly reduce the total flux

$$\phi_{total} = \Sigma_i\, [(E_{total})_{perp}]_I\, dA_i.$$

From the last section, we showed that the total flux should be the sum of the flux due to +Q alone and the flux due to (-q) alone. Since adding the flux from (-q) <u>reduces</u> the total flux, that number must be negative. So we define the flux due to a negative charge to be negative. Now the field at the surface doesn't know or care what causes it, and the flux calculation will be the same regardless, so we generalize the result to say that flux due to <u>any</u> E-line going from the outside of the surface to the inside is counted as negative flux.

What if the charge is outside of the surface? In that case, the total flux through that surface due to that charge is zero. Once again, any shape closed surface can be approximated to arbitrary accuracy with the spherical sections centered on the charge. As above, I can then push and pull the sections to get a nice double spherical surface shape. Since the quantity EA has already been shown to be independent of the distance from the charge, and since we have just defined the flux from E-lines entering a surface to be negative while the flux from those leaving a surface are positive, and since the fluxes through those side portions of the sections are zero since $E_{perp} = 0$, we see that the total flux through this surface due to an exterior charge is zero.



At this point, we have arrived at something useful.

Gauss's Law for Electricity: The net electric flux through a closed surface is proportional to the net charge inside the surface.

The flux is defined as

$$\phi_E = \Sigma_i E_{perp} dA_i \;\; or \;\; \oint\!\!\oint E_{perp} dA,$$

and it is calculated by looking at each little piece of area, δA_i, multiplying by the component of E_i that is perpendicular to the area, assigning a sign (positive if E points from the inside of the surface to the outside, negative if the reverse is so), and adding up all the contributions from each area. When this is done, the total should equal $4\pi k_e Q_{net\ enclosed}$.

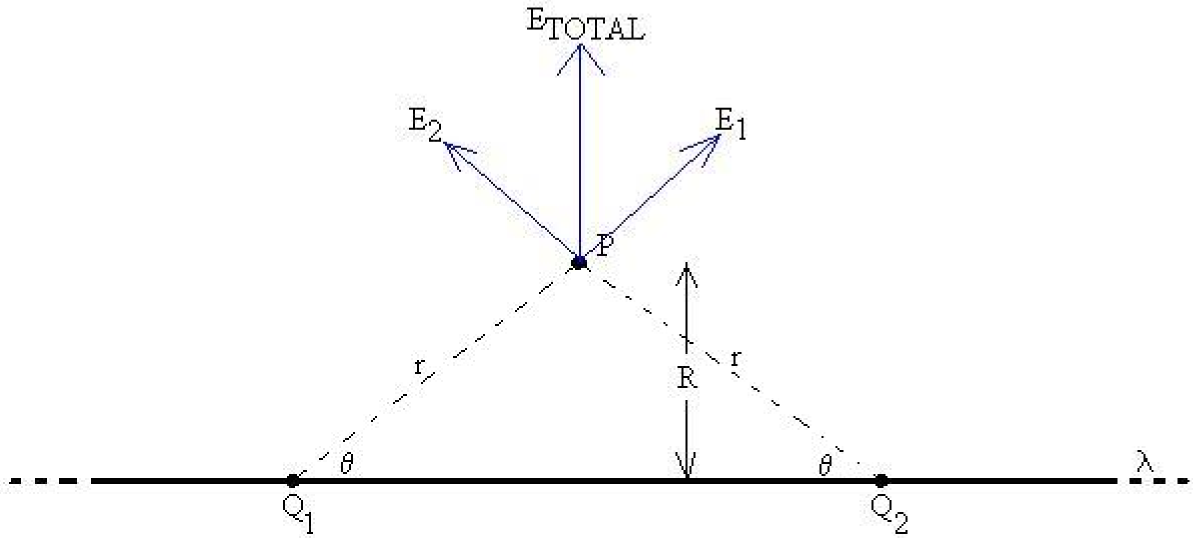
Here, we introduce a new and sometimes more convenient constant, ε_o, the *permittivity of free space*. The value of ε_o is $1/[4\pi k_e]$. So, finally, we have that

$$\phi_E = \Sigma_i E_{perp} dA_i \;\; or \;\; \oint\!\!\oint E_{perp} dA = \frac{Q_{net\ enclosed}}{\varepsilon_o}.$$

Let's now use Gauss's Law to find the electric field due to some very special, highly symmetric distributions of charge. The symmetry is important since, although Gauss's law is always true, it is not always possible to work it backwards to find **E**.

Uniform Infinite Straight Line Charge:

Consider a straight, infinitely long wire which carries a uniform *linear charge density* $\lambda = Q/L$. Let's get an idea what the E-field looks like at some point P a distance R from the line charge



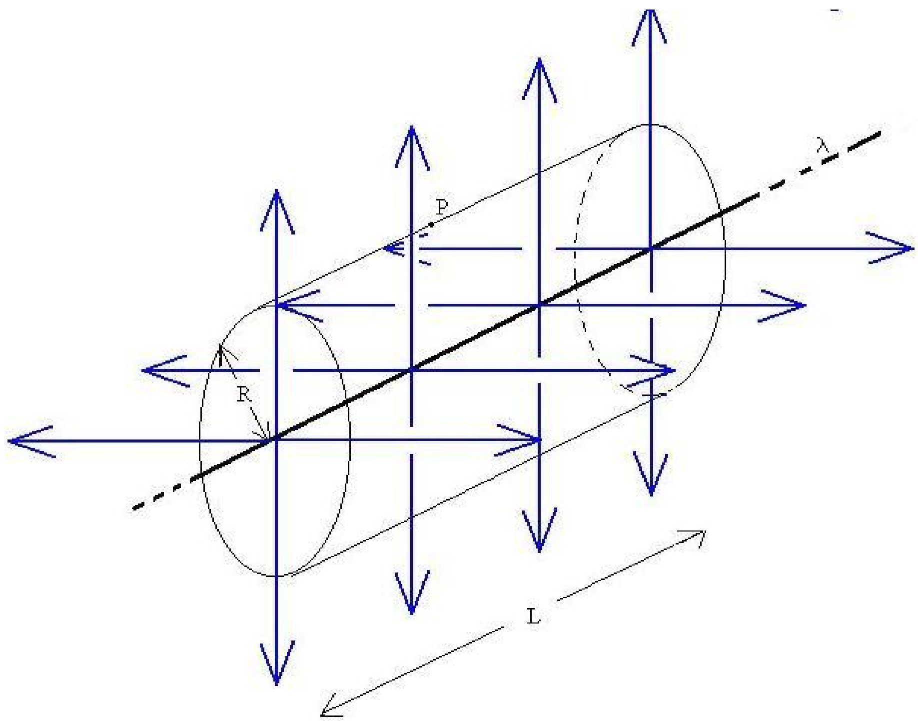
Consider some small bit of charge Q_1, which will cause a field E_1 at P. For every one of these charges, there is another charge Q_2 the same distance away, but on the other side of P. The fields E_1 and E_2 will have the same magnitudes, since P is the same distance r from each charge. In addition, the horizontal components of the two fields will be the same (although in opposite directions) because the angles θ are the same. The resultant E-field from Q_1 and Q_2 is directly away from the line charge. This is true for any pair of charges, and also for the single charge directly under Point P, so the total E-field at Point P must be radially outward perpendicular to the wire. What's more, the magnitude of E is the same for any point distance R from the wire, since the argument can be repeated at any spot along the wire.

Now, in order to use Gauss's law, we need a *gaussian surface* over which to find the flux. Since the surface is imaginary, we can pick one which will minimize the difficulty of the calculations we must do while still delivering the result we want. In general, we should pick a surface such that either:

1) the electric field runs along the surface (or part of a surface) so that the perpendicular component of E is zero and the contribution to the flux is zero, or

2) the electric field along a surface (or part of a surface) is constant in magnitude and, if possible, perpendicular to the surface, so that $\Sigma\, E_{perp}\, \delta A = EA$.

Choose the gaussian surface to be a cylinder of variable radius R and length L, co-axial with the wire. The surface can be considered to comprise three individual surfaces, the circular end caps and the curved section connecting them.



Gauss's Law: for a closed surface,

$$\phi_E = \Sigma_i [E_{perp}]_i \delta A_i = 4\pi k_e Q_{enclosed}$$

The total flux is the sum of the fluxes through each part of the surface:

$$\phi_E = \Sigma_{left\ cap} [E_{perp}]_i \delta A_i + \Sigma_{right\ cap} [E_{perp}]_i \delta A_i + \Sigma_{curved\ part} [E_{perp}]_i \delta A_i$$

In that case, the first two terms are zero, since the field runs <u>along</u> the surfaces, not perpendicular to them. On the curved surface, however, E is perpendicular to the gaussian surface everywhere, so $E_{perp} = E$ and

$$\Sigma_{curved\ part} [E_{perp}]_i \delta A_i = \Sigma\ E_i\ \delta A_i$$

We said above that E is constant in magnitude for any given radius from the centre, so this is true everywhere on the curved surface ($E_i = E$), so

$$\Sigma_{curved\ part}\ E_i\ \delta A_i = E\ \Sigma_{curved\ part} \delta A_i = EA = E\ 2\pi RL = 4\pi k_e Q_{enclosed}$$

Solve for E:

$$E = 2k_e Q_{encl}/LR = 2k_e(Q_{encl}/L)/R = 2k_e(\lambda)/R$$

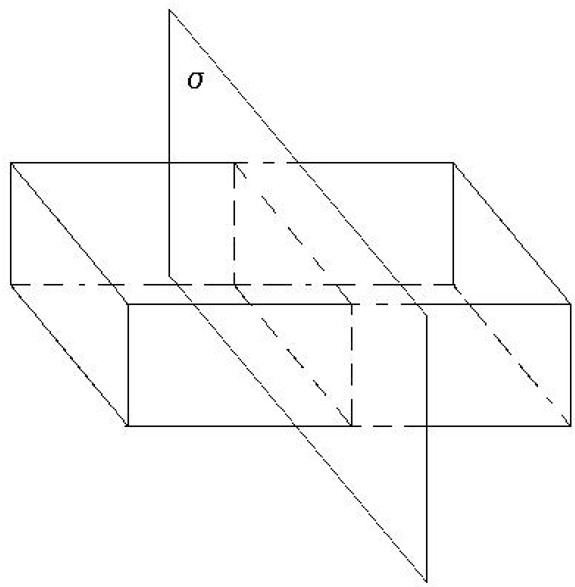
(radially outward if λ is positive, radially inward if λ is negative.)

Note that L dropped out of the result. This is very important. Why?

R was the radius of the gaussian surface. Why didn't R drop out of the answer?

Uniformly Charged Infinite Flat Sheet

Exactly what it sounds like: a flat sheet that extends infinitely in four directions. We define σ as the amount of charge *per* unit area:
$\sigma = Q/A$
We need to get some idea what the E-field looks like before we start. Use an argument similar to the one outlined above, or use a symmetry argument (done in class) to see that the field must be straight away from the sheet (or towards it, if σ is negative). Also, the magnitude must be the same at any given distance from the sheet, on either side. Pick a gaussian surface in the shape of a rectangular box (with six sides, top, bottom, left, right, front, back), positioned symmetrically on the sheet, as shown in the figure.



Start with Gauss's Law:

$\phi_E = \Sigma_i [E_{perp}]_i \delta A_i = 4\pi k_e Q_{enclosed}$ for a closed surface. Then,

$$\phi_E = \Sigma_{top} E_{perp} \delta A_{top} + \Sigma_{bot} E_{perp} \delta A_{bot} + \Sigma_{front} E_{perp} \delta A_{front} + \Sigma_{back} E_{perp} \delta A_{back} + \Sigma_{left} E_{perp} \delta A_{left} + \Sigma_{right} E_{perp} \delta A_{right}.$$

Since **E** is perpendicular to the sheet of charge, it is along the top, bottom, front, and back surfaces, so that there, $E_{perpendicular} = 0$.

Then, we're left with

$$\phi_E = \Sigma_{left}\, E_{perp}\, \delta A_{left} + \Sigma_{right}\, E_{perp}\, \delta A_{right}$$

For the same reason, on the left and right surfaces, $E = E_{perpendicular}$, so that

$$\phi_E = \Sigma_{left}\, E\, \delta A + \Sigma_{right}\, E\, \delta A.$$

Also, E is a constant along each of the right and left hand surfaces and equal in magnitude along both, and the right and left hand sides are the same area as the area of the sheet enclosed by the gaussian surface, so

$$\phi_E = E\, \Sigma_{left}\delta A + E\, \Sigma_{right}\delta A = 2\, EA$$

By Gauss's law, $\phi_E = 4\pi k_e Q_{enclosed} = 4\pi k_e \sigma A$. So,

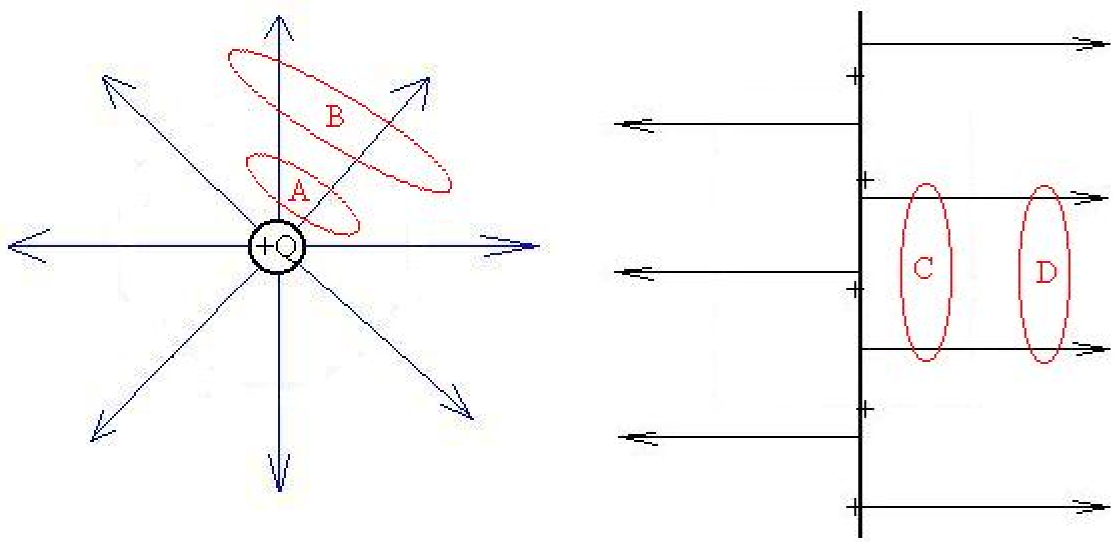
$$2EA = 4\pi k_e \sigma A.$$

The area cancels out, as we know it should, so that we obtain

$$E = 2\pi k_e \sigma = \sigma/2\varepsilon_o.$$

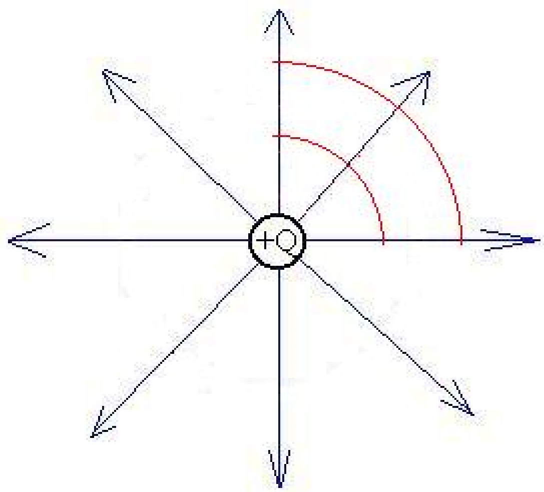
So, the electric field of an infinite, flat, uniformly charged sheet is constant (although he direction reverses on each side).

Maxwell developed the notion of the *electric field line*. Let's see if we can make some connections to what we've done previously to see if we can suss out any information from their appearance. Consider the point charge with a radially outward field of magnitude $E = k_e Q/r^2$ and the infinite flat sheet with a field of parallel lines with constant magnitude:



We note that in Region A, the lines are close together and the field is strong, while in Region B the lines are far apart and the field is weak. This correlation seems fine and is consistent with the other example: lines which stay evenly separated (Regions C & D) represent constant field strength.

Following through on this, we can also say that the electric flux through a surface is proportional to the number of field lines passing through it. For example, we argued that the flux through a spherical section due to a point charge remained constant regardless of r, since the field strength and the area both have r^2 terms which then cancel in their product, while we can easily see that the number of lines through each surface is also the same.

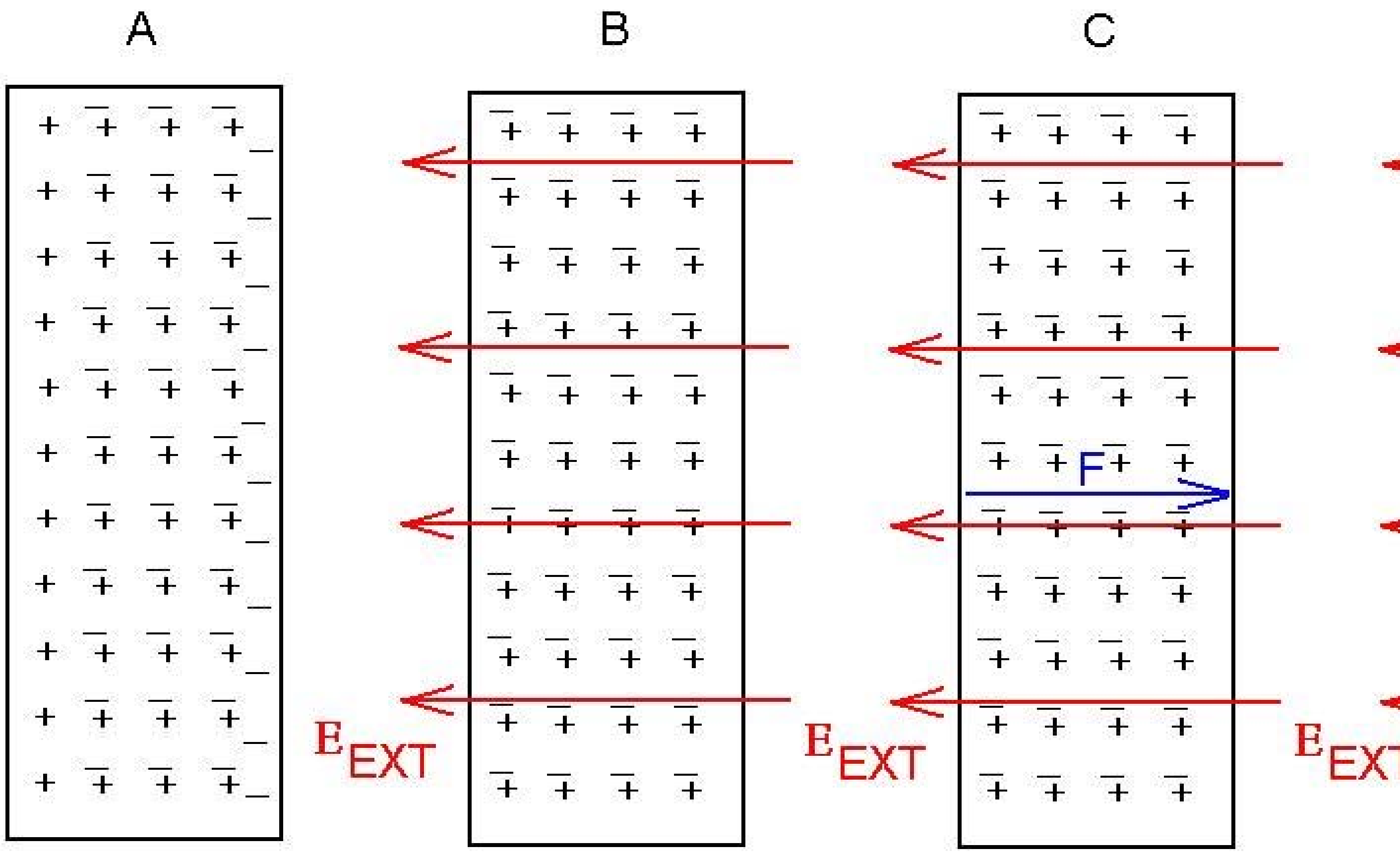


Since arbitrary shapes of charge can be approximated by a collection of point charges, this should be true for any shape of charge.

Metals

One way in which materials can be classified is by how well they allow charge to move about in their interiors. In order of decreasing *mobility*, the categories are:
Superconductors (*e.g.*, Hg or Ni at very low temperature, more recently some complicated ceramics have been discovered to be SCs)
Conductors (typically a metal, *e.g.*, Cu, Ag, Al)
Semi-conductors (*e.g.*, Si, Ge, GaAs)
Semi-insulators (*e.g.*, SnO_2)
Insulators (*e.g.*, glass, paper)
Let's talk about conductors. Metals typically have one, two, or even three electrons per atom which can easily leave the atom and travel fairly uninhibitedly about the interior of the metal.

Consider a block of metal; the distributions of electrons and protons are fairly uniform, even to just above the atomic scale (Figure A).



Let's place this block of metal in an external electric field, $\mathbf{E}_{EXT}$ (Figure B, $\mathbf{E}_{EXT}$ is shown in red).

The electrons will experience an electric force (q**E**) to the left (Figure C, the force is shown in blue), and since they are free to move, they will start to do so toward the rigthhand surface of the metal (they can't go any farther than that). The protons will experience a force to the left, but each is about 2000 times heavier than the electrons, there are 10 to 100 times as many of them as there are free electrons (plus the neutrons too!), each is locked up inside of an atom, and the atoms are linked to each other in a strong lattice structure, so that we might think of them as composing a single large object, which in turn has macroscopic forces acting from, for example, the table on which the block sits; it's safe to say that the net force on each proton is about zero, so they don't move.
As the electrons move rightward, the separation of charge gives rise to an internal electric field, $\mathbf{E}_{INT}$ (Figure D, induced field shown in green), which points from the excess positive charge on the left toward the excess negative charge on the right. The net field inside the metal is the vector sum of $\mathbf{E}_{EXT}$ and $\mathbf{E}_{INT}$, which has a magnitude of (roughly)
‘

$E_{NET} = E_{EXT} - E_{INT}$.

Since there is still a net field in the metal, even more electrons will travel to the left surface, thus increasing the internal fireld and decreasing the net field. When does this stop?

So, we see that the electric field inside a conductor in equilibrium must be zero (Figure E); if it were not, then charges would arrange themselves until it is. Do not be surprised when I qualify this statement in a week or so.

The same argument can be applied to the electric field at the surface of a metal. Consider an external field which has a component perpendicular to the metal's surface and another along the metal's surface; since charge can move along the syrface, they will once again re-arrange themselves and produce their own field parallel to the surface until the total field has no net component in that direction. So, we can also say that the electric field at the surface of a conductor in equilibrium must be perpendicular to that surface.

Now, let's return to Gauss's Law for some more examples.

Example 1:
Consider a metal sphere of radius R which has a charge $3Q_o$. Where does that charge reside?

Why is that so?
Three reasons:

Example 2:
Consider a hollow conducting sphere (inner radius R_1 and outer radius R_2) with a total charge -$3Q_o$. Now, place a point charge $+2Q_o$ at the centre of the hollow sphere. What charge will be on the inner and outer surfaces of the sphere?

We'll label the charge on the inside surface of the sphere Q_{INSIDE} and the charge on the outside of the sphere $Q_{OUTSIDE}$. From the problem, we see that

$$Q_{INSIDE} + Q_{OUTSIDE} = -3Q_o.$$

Consider a gaussian surface inside the metal of the sphere. Again, we know that the field there is zero, so the total charge enclosed by that surface will be zero:

$$+2Q_o + Q_{INSIDE} = 0.$$

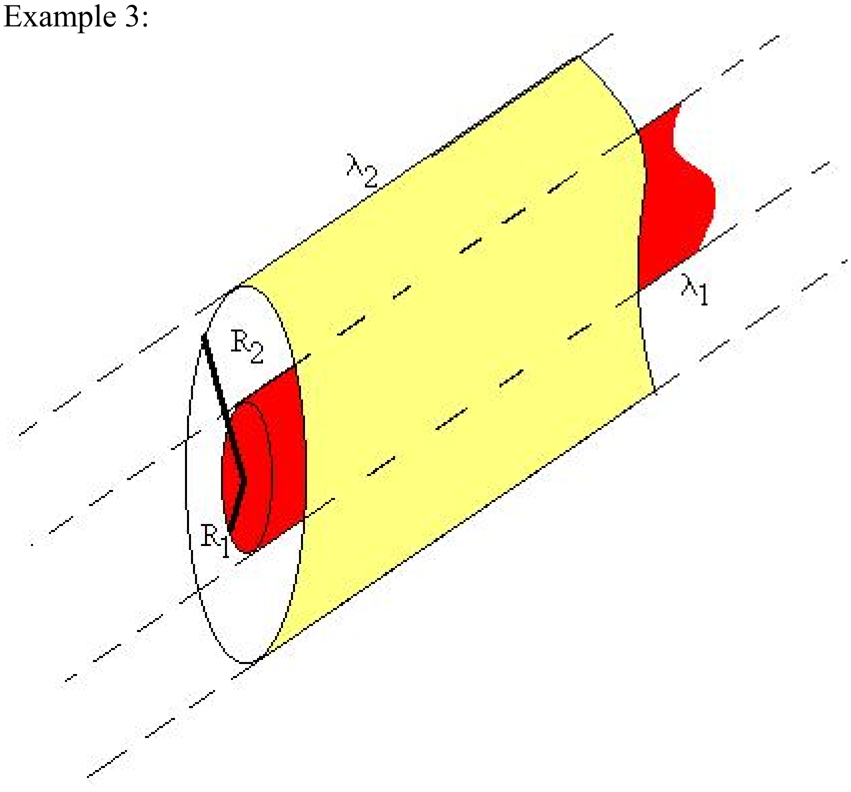
So, there must be a charge $-2Q_o$ on the inside surface of the sphere. This should make some sense, in that we would expect the positive point charge to attract negative charges toward itself. This then means that there is a charge

$$Q_{OUTSIDE} = -3Q_o - Q_{INSIDE} = -3Q_o - -2Q_o = -Q_o$$

on the outside surface.

What does the electric field look like? Again consider a concentric gaussian surface of variable radius r.
For $0 < r < R_1$, $Q_{ENCLOSED} = +2Q_o$, so $E = 2kQ_o/r^2$ outward.
For $R_1 < r < R_2$, $E = 0$ (inside metal).
For $r > R_2$, $Q_{ENCLOSED} = +2Q_o + (-2Q_o) + (-Q_o) = -Q_o$, so $E = kQ_o/r^2$ inward.

Example 3:



Consider an infinitely long straight cylinder (radius R_1) of charge with uniform density, that is, the charge is distributed evenly throughout the cylinder. Let the linear charge density be λ_1. Let a thin cylindrical shell of radius R_2 and with linear charge density λ_2 be co-axial with it. Find the electric field everywhere.
Draw a gaussian surface as a cylinder (variable radius r and length L) co-axial with the real cylinders, and make use of the result of the derivation above for the infinitely long straight wire:

$$E = 2k_e\lambda_{ENC}/r.$$

For $r > R_2$, both λ_1 and λ_2 are enclosed, so $E = 2k(\lambda_1 + \lambda_2)/r$.
For $R_1 < r < R_2$, only λ_1 is enclosed, so $E = 2k\lambda_1/r$.
For $0 < r < R_1$, only part of the charge on the central cylinder will be enclosed; we need to figure out how much. If the charge is distributed evenly within the cylinder, then the fraction of charge enclosed by the gaussian surface is the same as the fraction of the volume enclosed:

$$\lambda_{ENC}/\lambda_1 = \pi r^2L/\pi R_1^2L,$$

so that

$$\lambda_{ENC} = \lambda_1 r^2/R_1^2.$$

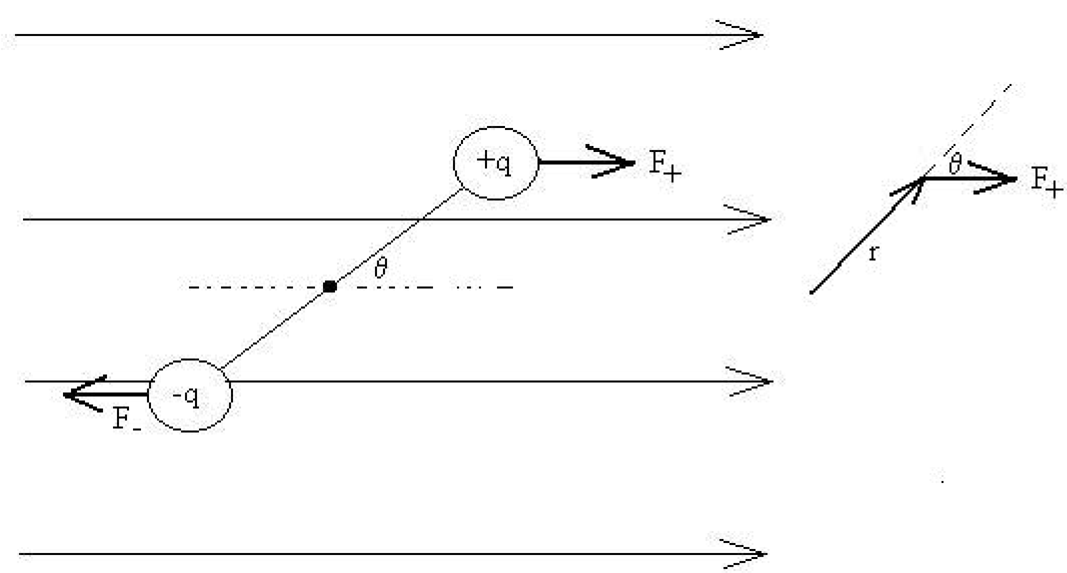
Substituting gives

$$E = 2k\lambda_{ENC}/r = 2k[\lambda_1 r^2/R_1^2]/r = 2k\lambda_1 r/R_1^2.$$

Example 4:
A point charge of $+3Q_o$ is located at the centre of a hollow sphere (inner radius R_1, outer radius R_2) that has a uniform charge density throughout and a total charge of $+Q_o$. Find the electric field everywhere.

Click here for solution.

Dipoles



Consider charges +q and -q, separated by a constant distance L, a bit like a barbell with one weight positive and the other negative. Let's

place this object in a uniform electric field, **E,** tilted at an angle θ as shown. There will be forces acting on each of the charges, $F_+ = qE$ to the right and $F_- = qE$ to the left. Since the field is uniform in both magnitude and in direction, these two forces should cancel, and the net force on the object will be zero. What is not zero is the net torque. Let's calculate it:
We can probably safely assume that the object will tend to rotate about its centre of mass, which is at its physical centre, mid-way between the charges; we'll use that as the pivot point, although in fact it doesn't matter. Remember that torque is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$

or

$$|\tau| = r\ F\ \sin\theta_{r,F}\ \text{(RHR)}.$$

The angles between each r vector and the associated F vector is the same as θ, so,

$$\tau_+ = (L/2)qE\ \sin\theta\ \text{(into the page)}$$

and

$$\tau_- = (L/2)qE\ \sin\theta\ \text{(also into the page)}$$

for a total torque of

$$\tau = 2(L/2)qE\ \sin\theta\ = LqE\ \sin\theta\ \text{(into the page)}.$$

What we do now is to define Lq as the *electric dipole moment*, **p**. We'll let **p** be a vector, whose magnitude we just defined and whose direction points from the negative charge toward the positive charge. Why? Well, why not?

Then, we should be able to write

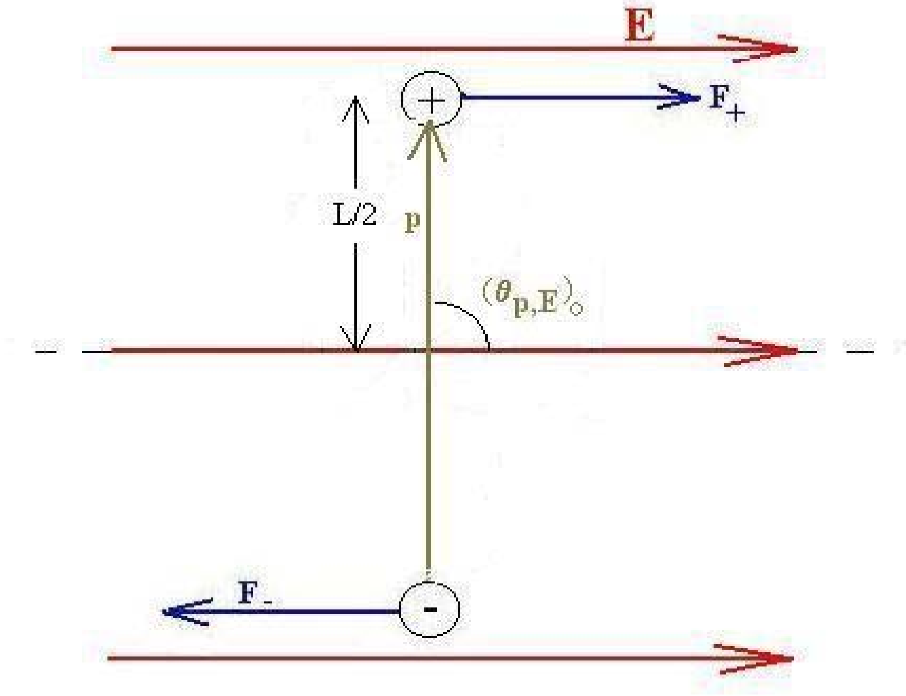
$$\tau = pE\ \sin\theta_{p,E}\ \text{(RHR), or } \boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}.$$

So, what will happen to this object? The torque will twist the dipole in an attempt to align **p** and **E**. Once θ = 0, however, we expect that, due to rotational inertia, the dipole will continue to swing past that point, only to be slowed , stopped, and twisted back toward θ = 0. The behaviour should be identical to that of a pendulum.

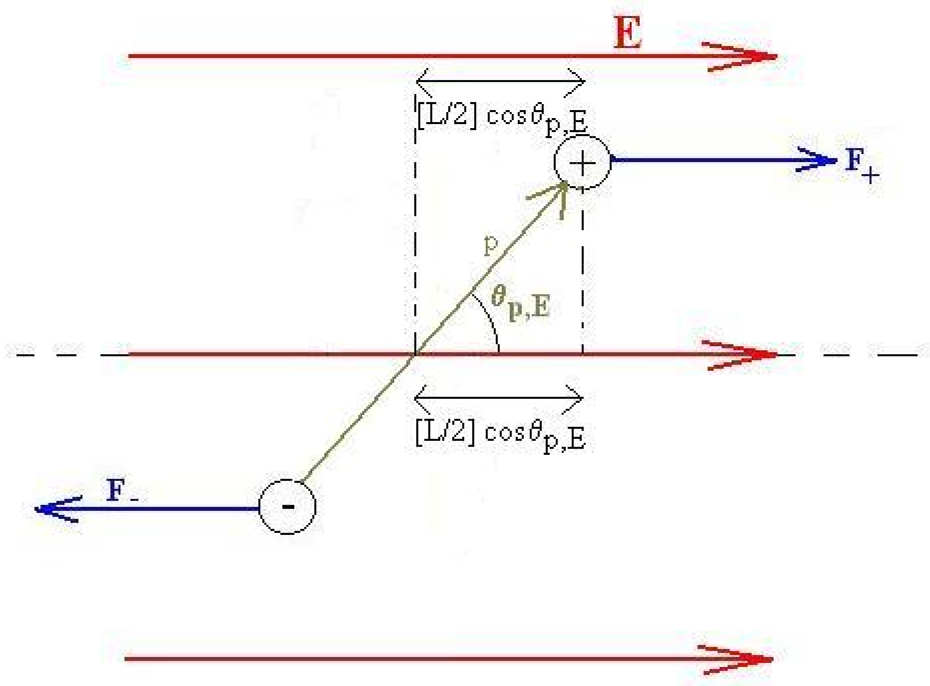
There is some potential energy associated with the dipole, since the kinetic energy of this motion had to come from somewhere (we can think of the source either as PE <u>or</u> as the work done by the electric force, but never as both). That PE is

$$PE_{dipole} = -pE\cos\theta_{\ p,E};$$

let's see if we can derive this result. Start with the dipole arranged such that $(\theta_{p,E})_o$ is 90°.



Now, let the dipole rotate to an arbitrary angle $\theta_{p,E}$.



To calculate the work done by the electric force on, say, the positive charge, we need only consider the displacement in the direction of the force, which will be $[L/2]\cos\theta_{p,E}$. The work done is then

$$[F_+][L/2]\cos\theta_{p,E} = [qE\ L/2]\cos\theta_{p,E} = {}^1\!/_2\ [qL\ E]\ \cos\theta_{p,E} = {}^1\!/_2\ pE\ \cos\theta_{p,E}.$$

A similar amount of work is done on the negative charge, so that the total work is

$$W_{E\text{-field}} = pE\ \cos\theta_{p,E}.$$

But, we defined the change in PE to be

$$\Delta PE_{dipole} = -\ W_{E\text{-Field}} = -\ pE\ \cos\theta_{p,E}.$$

Now, if we set the PE to zero when $\theta_{p,E} = 90^o$ (and why not?), then we have our result, that $PE_{dipole} = -\ pE\ \cos\theta_{p,E}$, = $-\mathbf{p}\cdot\mathbf{E}$.

Mastery Question

Consider two flat, infinite, parallel conducting plates. One plate has overall charge density $+2\sigma_o$ while the other has density $+\sigma_o$. How much charge resides on each side of each plate?

Click here for solution.

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