Section 2-2 - Electric Potential

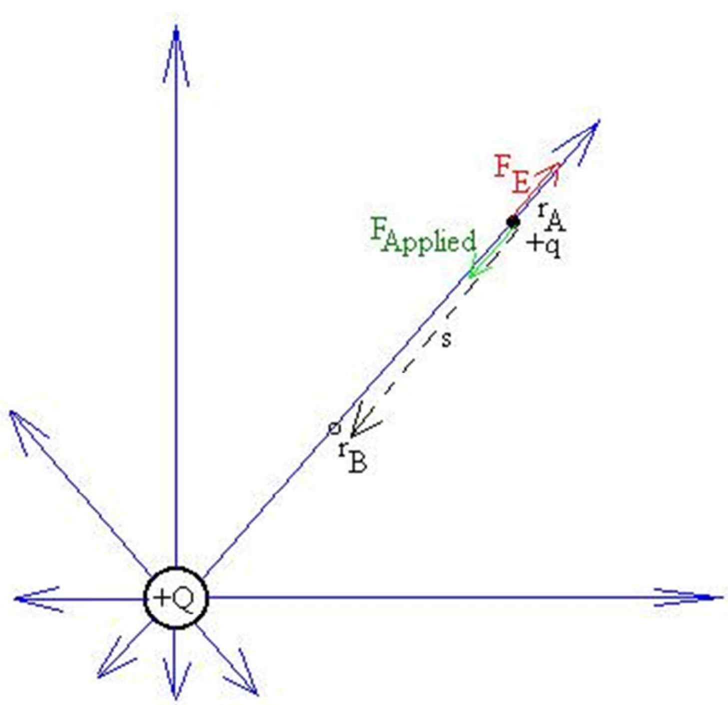
Potential

In much the same way as we used several 'pictures' to study motion (force and acceleration, work and energy), we will do the same this semester. We shall define the *electric potential* (V) at a given point in space to be the *electrical potential energy* (EPE) per unit of charge of a test charge located at that point; in a manner similar to that for the definition of the electric field, we are able to let the test charge diminish in size until it vanishes, leaving the quotient of our definition intact:

$$V \equiv lim_{q_{TEST} \to 0} \frac{EPE}{q_{TEST}}$$

Note that there need not actually be a test charge at the point in order for there to exist a potential at that point, in the same way that there need not be a test charge at some point in order for an electric field to exist.

Let's do a simple example which will give us a useful result and perhaps provide a more concrete picture of the electric potential, V:



Consider a point charge +Q fixed in space. Place a small test charge $+q_{TEST}$ a distance r_A from Q. There will be a repulsive force F_E acting on it of magnitude k_eQq_{TEST}/r^2. Now, suppose that for whatever reason the charge q_{TEST} moves closer to Q, to a distance r_B. As the electric force acts though a distance s, work is done. How much? Can we simply multiply F_E s $\cos\theta_{F,\,s}$?

Answer

$$W_E = \int_{r_A}^{r_B} \boldsymbol{F}_E \bullet d\boldsymbol{r} = \int_{r_A}^{r_B} \frac{k_e Q q_{TEST}}{r^2} dr = -k_e Q q_{TEST}\left(\frac{1}{r_B} - \frac{1}{r_A}\right). \quad *$$

Note that we have found the work done by the electric force only, and have ignored any work done by other forces.

Later, we will show that the electric force is a conservative force; in that case, we can re-write the work done by the electric force as a change in the *electrical potential energy* (EPE) of charge q:

$$\Delta \mathrm{EPE} = -\mathrm{W_E} = +k_e Q q_{TEST}\left(\frac{1}{r_B} - \frac{1}{r_A}\right). \quad *$$

Now, we previously defined V as EPE/q_{TEST}, so it's safe to say that

$$\Delta \mathrm{V} = \mathrm{V_B} - \mathrm{V_A} = \frac{\Delta \mathrm{EPE}}{\mathrm{q_{TEST}}} = +k_e Q\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = \frac{k_e Q}{r_B} - \frac{k_e Q}{r_A}. \quad *$$

Think back to when you were working with gravitational potential energy (GPE). The derivation we did then was that the change in GPE for an object moving vertically was:

$GPE_f - GPE_i = mgh_f - mgh_i$.

What we did next was to assign a value of zero to the GPE at some particular level, usually ground level where h = 0 as well. In that way, we simplified this relationship to obtain $GPE_f = mgh_f$, or more generally, GPE = mgh. We're about to do the same thing here. Assign a value of zero potential to locations infinitely far from Q (that is, let $V_A = 0$ at r_A = infinity), so that the relation becomes:

$$\mathrm{V_B} - 0 = \frac{k_e Q}{r_B} - 0,$$

or more generally,

$$\mathrm{V(r)} = \frac{k_e Q}{r}.$$

Note that this result is valid for a point charge only. This actually gives us an alternate notion of the definition of the potential: the potential at a point P is the work done by some agent necessary per unit charge to bring that charge very slowly from infinity to P. The 'very slowly' is so that we need not consider any kinetic energy, and of course only the agent and the electric force can act on the charge.

Note two things: the potential does not depend on the existence of the test charge q_{TEST}, and that this particular relationship is only valid for a point charge (or for regions <u>outside</u> of spherically symmetric charges which produce the same configuration of electric field), although similar relations can be valid for other shapes of charge.

The unit for electric potential is the *volt*. A potential of one volt at some location means that each coulomb of charge placed there will possess one joule of EPE.

Let's return to the starred equations above:

$$EPE\ (\boldsymbol{r}) = -W_E = -\int_{\infty}^{r} \boldsymbol{F}_E \bullet d\boldsymbol{r}\ .$$

Dividing each end by q_{TEST} results in

$$V(\boldsymbol{r}) = \frac{EPE}{q_{TEST}} = -\frac{1}{q_{TEST}}\int_{\infty}^{r} \boldsymbol{F}_E \bullet d\boldsymbol{r} = -\int_{\infty}^{r} \frac{\boldsymbol{F}_{\mathrm{E}}}{q_{\mathrm{TEST}}} \bullet d\boldsymbol{r}$$

$$V(\boldsymbol{r}) = -\int_{\infty}^{r} \boldsymbol{E} \bullet d\boldsymbol{r}\ .$$

So, if we know the electric field, we should be able to find the potential field.

Since potential is related to work and energy, it is a scalar quantity. The total potential at any given point P is the algebraic sum of the potentials at that spot due to the various charges:

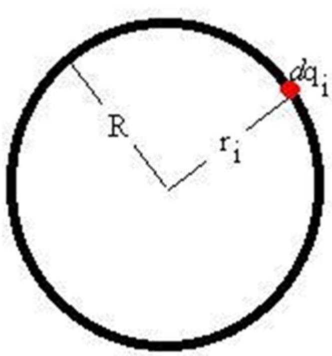
$$V_{TOTAL}(\boldsymbol{r}) = -\int_{\infty}^{r} \boldsymbol{E_{TOTAL}} \bullet d\boldsymbol{r} = -\int_{\infty}^{r} \Sigma_{\mathrm{i}}\, \boldsymbol{E_i} \bullet d\boldsymbol{r} = -\Sigma_{\mathrm{i}} \int_{\infty}^{r} \boldsymbol{E_i} \bullet d\boldsymbol{r} = -\Sigma_{\mathrm{i}} V_i(\boldsymbol{r})\ .$$

So, if we want to find the potential caused by a collection of charges, or by a continuous distribution of charge, in principle, we need only find the potential due to each tiny bit of charge and add.

Also, note that the potential at some point P caused by a charge Q depends on the sign of the charge. For example, if we were to allow a positive test charge to approach a negative charge, we would expect its potential energy, and therefor the potential, to decrease.

Examples:
Consider a ring of radius R with a charge Q_o distributed evenly along its circumference. Find the potential at the centre of the ring.

V = kQo/R



Consider one small charge dq_i, so small it looks like a point charge. We already worked out that the potential due to a point charge is $k_e q/r$, so the contribution to the total potential from one of these little charges is $k_e\, dq_i/r_i$. The total value of the potential at the centre will be the sum of all the contributions from each of the individual charges:

$$V = \int \frac{k_e dq}{r}.$$

In this case, all of the r's are equal R, so

$$V = \int \frac{k_e dq}{r} = \frac{k_e}{R}\int dq = \frac{k_e Q_o}{R}.$$

What is the potential a distance x out along the axis of the ring, through the centre and perpendicular to the plane of the ring?

V = kQo/[sqrt(R^2 + x^2)]

The calculation is similar to the one above, except that each charge is now a distance

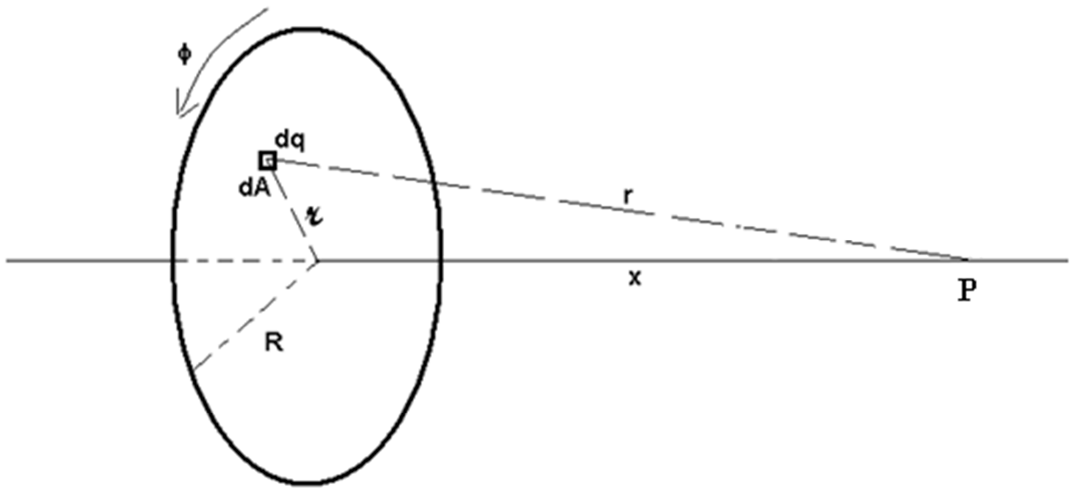
$$r_i = [R^2 + x^2]^{1/2}$$

from the point:

$$V = \int \frac{k_e dq}{r} = \frac{k_e}{(R^2 + x^2)^{1/2}}\int dq = \frac{k_e Q_o}{(R^2 + x^2)^{1/2}}.$$

Note that, as x >> R, this reduces to the result for a point charge, $k_e Q_o/x$.

Now consider a disc of charge Q_o with radius R. Find the potential a distance x from the center of the disc along the axis through the center perpendicular to the plane of the disc.



Again, we consider a small point charge dq with area dA, whose distance from point P is

$r = (x^2 + \mathfrak{r}^2)^{1/2}$.

If the disc is uniformly charged, then we can expect that

$$\frac{dq}{dA} = \frac{Q}{\pi R^2} \quad \rightarrow \quad dq = \frac{Q}{\pi R^2}\mathfrak{r}\, d\mathfrak{r}\, d\phi$$

So,

$$V = \int \frac{k_e dq}{r} = \int \frac{k_e \frac{Q}{\pi R^2}\mathfrak{r}\, d\mathfrak{r}\, d\phi}{\left(x^2 + \mathfrak{r}^2\right)^{1/2}} = \frac{k_e Q}{\pi R^2}\int_0^{2\pi} d\phi \int_0^R \frac{\mathfrak{r}\, d\mathfrak{r}\, d\phi}{\left(x^2 + \mathfrak{r}^2\right)^{1/2}}.$$

Through the use of a u-substitution, $u = x^2 + \mathfrak{r}^2$; $du = 2\,\mathfrak{r}\, d\mathfrak{r}$, this becomes

$$V(x) = \frac{2k_e Q}{R^2}\left(x^2 + \mathfrak{r}^2\right)^{1/2}\Big|_0^R = \frac{2k_e Q}{R^2}\left((x^2 + R^2)^{1/2} - x\right)$$

We should see that as R → 0, V should approach the function of a point charge. This time, we'll use l'Hôpital's Rule[1] to find the limit.

$$\lim_{R\to 0}\frac{2k_e Q}{R^2}\left((x^2 + R^2)^{\frac{1}{2}} - x\right) = \lim_{R\to 0} 2k_e Q\,\frac{\frac{1}{2}(x^2 + R^2)^{-\frac{1}{2}}\,2R}{2R}$$

[1] $\lim_{z\to 0}\frac{f(z)}{g(z)} = \lim_{z\to 0}\frac{f'(z)}{g'(z)}\ if \lim_{z\to 0} f(z) = 0\ and \lim_{z\to 0} g(z) = 0.$

$$= \lim_{R\to 0} \frac{k_e Q}{(x^2 + R^2)^{1/2}} = \frac{k_e Q}{x},$$

as expected.

Equipotential Surfaces

An equipotential surface is one along which the electric potential is constant, or if you like, along which the electric force does no work on a test charge. Consider again a positive test charge in an electric field. The field will exert a force on q in the direction of **E**. If the test charge moves in the direction of **E**, work is done, and the potential energy of q changes, and since V = EPE/q, the potential changes as well. A similar argument holds if q moves against the field. The only way that the potential will not change is if no work is done on q, which means it must move perpendicularly to the field, *i.e.*, make the displacement perpendicular to the electric force (remember that $W = Fd \cos\theta_{F,d}$).

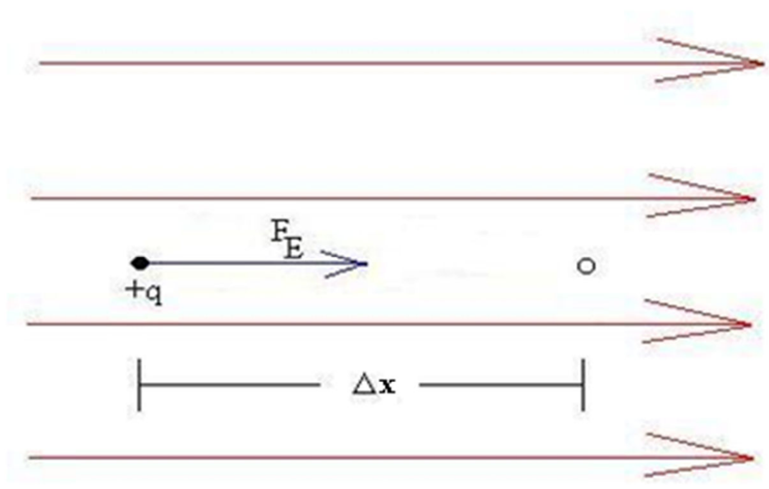
So, we conclude that equipotential surfaces are always perpendicular to the electric field.

Now, since we know that the electric field must always be perpendicular to the surface of a metal, what can we say about the potential on the surface of a piece of metal?

What can you say about the potential inside of a metal?

Relationship between V and E.

There is a relationship between the potential field and the electric field which goes beyond just the direction of **E**; we can get a sense of the magnitude of **E** as well from V. Let's once again consider a very special case which will give some insight:



Suppose that we put a positive test charge q in a uniform electric field, **E**, that points in the x-direction.

There will be a force from the field on q with magnitude F_E = qE acting to the right. Let q move toward the right a distance

Δs. The work done by the electric field will then be

$$W_E = F_E \Delta x = qE\Delta x.$$

We know that the work done can be written as a change in EPE:

$$W_E = -\Delta EPE = -q\Delta V.$$

When we compare these results, we see that

$$qE\Delta x = -q\Delta V \rightarrow E = -\frac{\Delta V}{\Delta x}.$$

This tells us that the magnitude of **E** can be obtained by looking at how quickly the potential changes with distance along a field line. The negative sign indicates that the **E**-field points in the direction in which V is decreasing. Also, this gives us a different set of units for **E**: V/m as well as N/C. Now, this result is only exactly true for the situation from which it was developed, but conceptually, it is O.K. in almost any circumstance. Over very short distances $\Delta x \rightarrow dx$, this becomes

$$E = -\frac{dV}{dx}.$$

What if the electric field had all three components, E_x, E_y, and E_z? Lets place a test charge q_{TEST} at some point P where such an electric field exists. Let's move the test charge a very small distance *d*x in the ***i*** direction. Since the E_y and E_z components of the field are at right angles to the displacement, they do no work on the charge and contribute nothing to the change in the EPE of the test charge, and therefor also nothing to the change in potential along that short path. The change in potential as we move in the x direction is due to the x-component of the electric field, only. Similar arguments can be made for the y and z components, so that

$$E_x = -\frac{\partial V}{\partial x};\ E_y = -\frac{\partial V}{\partial y};\ E_z = -\frac{\partial V}{\partial z}\quad or\quad \vec{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right),$$

which is written most efficiently as

$$\vec{E} = -\vec{\nabla}V = -\boldsymbol{grad\,V}.$$[2]

Let's test this relationship with some of the examples we've already done.

Find the electric field of a ring of radius R with a uniformly distributed charge Q_o a distance x along the axis perpendicular to the plane and passing through its center.

We already know that

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[2] This function is called the *gradient* of V.

$$V(x) = \frac{k_e Q_o}{(R^2 + x^2)^{1/2}}.$$

Then,

$$E_x = -\frac{dV}{dx} = -\frac{k_e Q_o}{(R^2 + x^2)^{\frac{3}{2}}}\left(-\tfrac{1}{2}\right)(2x) = \frac{k_e Q_o x}{(R^2 + x^2)^{\frac{3}{2}}};$$
$$E_y = -\frac{dV}{dy} = 0;\ E_z = -\frac{dV}{dz} = 0,$$

which agrees with the result in Section One.

Find the electric field of a disc of radius R with a uniformly distributed charge Q_o a distance x along the axis perpendicular to the plane and passing through its center.

We already know that

$$V(x) = \frac{2k_e Q}{R^2}\left((x^2 + R^2)^{1/2} - x\right)$$

Then, once again with $E_y = E_z = 0$,

$$E_x = -\frac{dV}{dx} = -\frac{2k_e Q}{R^2}\left[\frac{\frac{1}{2}(2x)}{(R^2 + x^2)^{1/2}} - 1\right] = \frac{2k_e Q}{R^2}\left[1 - \frac{x}{(R^2 + x^2)^{1/2}}\right]$$

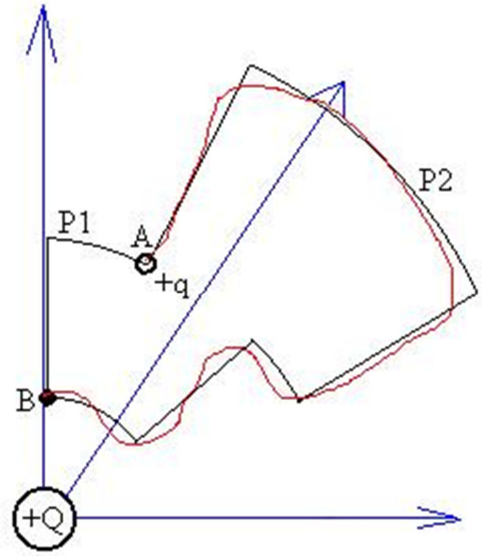
which agrees with the result in Section One.

Find the electric field a distance r from a point charge Q_o. This needs to be done in spherical coördinates, but since there is no dependence on either of the coördinate angles, the gradient reduces to $\mathbf{r}\,{}^{d}/_{dr}$:

$$\vec{E} = -\frac{dV}{dr}\hat{r} = -\frac{d}{dr}\left(\frac{k_e Q_o}{r}\right)\hat{r} = \frac{k_e Q_o}{r^2}\hat{r}.$$

The Conservativeness of the Electric Field

Here, we'll finally make the argument for **E** being a conservative field. Consider a point charge Q. The electric field, and therefor the electric force on a test charge q, goes as $1/r^2$ and is radial (inward or outward, depending on the relative signs of the charges). Take the test charge q from Point A to Point B along Path One:



Along Path 1, no work is done along the curved part, since the electric force is perpendicular to the displacement at all points, and there is some work done along the radial part of the path (which we won't actually calculate). Path 2 (in red) is any other path from Point A to Point B; this path can be approximated to an arbitrary degree of accuracy with a number of radial and circular sections, as shown. Along each circular section, no work is done. Along each radial section, the work must be calculated, but it should be clear that work done during any outward motion from r_1 to r_2 will be cancelled by the work done moving radially from r_2 to r_1. Only the net motion from r_A to r_B will result in a net amount of work done. Since the electric force has the same magnitude and relative orientation to the displacement at all points a distance r from Q, the amount of work done will be independent of the path taken. This last is one way of saying that the force is conservative.

Now, what if the charge distribution had been more complicated than a point charge? We can approximate any shape charge distribution by a collection of point charges; the work done by the E-field while the test charge moves is the sum of the works done by the individual contributions to the field of the individual charges and so it is also conservative.

For more advanced students, I mention without proof that a test for the conservativeness of a field is to take its *curl*; if the curl is zero, the field is conservative.

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