Section 2-4 - Resistance & Kirchhoff's Laws

Current
Current in Metals and Other Materials
Resistance and Ohm's Relationship
Power Dissipated as Thermal Energy
Combinations of Resistors
Batteries
Kirchhoff's Rules

Current

So far, we have been discussing *static electricity*, charges in particular static arrangements which were placed that way by having been moved very slowly. Now, we want to start discussing charges which do move. It would be nice to be able to describe the rate of that motion, a quantity we shall call the *current*, and which we shall naively define as the amount of charge which passes a given point per second:
$I = Q/t$.
The unit of current is the *ampere*, or amp, and 1A = 1coulomb/1second.

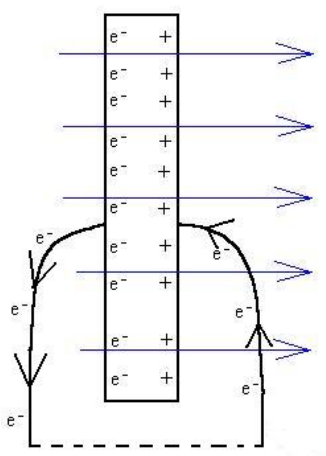
We discussed long ago that Franklin labeled his two charges positive and negative arbitrarily. Unfortunately, we now know that the negatively charged electrons are the charges which move most often, but by the time this was realized, 150 years of convention was ground into the equations we use. So we now speak of the *real current* and more often of the *conventional current*, which is assumed to be moving positive charge. So, for example, if 5 coulombs of electrons pass me in 3 seconds moving to the right, then the current is
$I = Q/t = 5C/3s = 1.67$ amps to the left.

What is the current if, in 2 seconds, 5 coulombs of electrons pass me to the left while 8 coulombs of protons pass me to the right?

Answer

Currents in Metals and Other Materials

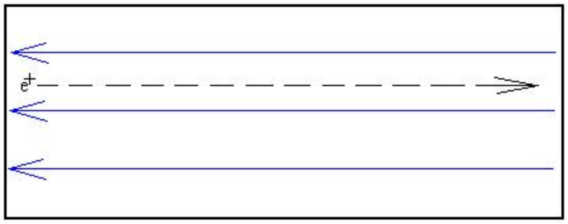
In a previous discussion, we saw how charges re-distribute themselves in a metal (and to a lesser degree in dielectrics) to eliminate the electric field in the metal. In that situation, the charges built up on the surfaces of the metal, but could go no farther. However, we shall now connect the two sides of the metal with wires so that the charge may continue on its way:

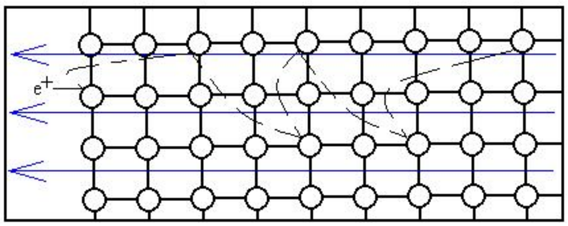


If we were to connect the two wires together, the electrons would eventually find themselves attracted back to the right hand side of the metal. In any event, since the charges are not allowed to build up on the surfaces of the metal, the system never comes to equilibrium, no internal E-field is generated, and the net E-field is not required to go to zero. Certainly, it is not required so to do in a non-metal, anyway.

The ease with which electrons (or other charges) can move through a material is called the *electrical conductivity*, σ. More often, though, we'll refer to its reciprocal, the *electrical resistivity*, ρ, which is a measure of the difficulty with which charge moves through a material. In order of increasing resistivity, the categories of classification are: *super-conductors*, conductors (metals), *semi-*

conductors, *semi-insulators*, and *insulators*.
The model we will use is applicable to metals at very low temperatures, but it gives an idea of what happens in other materials as well. Consider a region which is empty except for electrons. Now turn on an external electric field. Each electron feels a force and accelerates quickly to a very high speed. Now, place a lattice pattern of large atoms in that region. Since metallic atoms occupy a few percent of the volume (still lots of space between the atoms), there is a fairly high probability that the electron will collide with an atom without having had to move very far. This collisions will scatter the electrons in different directions and will result in a loss (actually a conversion) of energy.





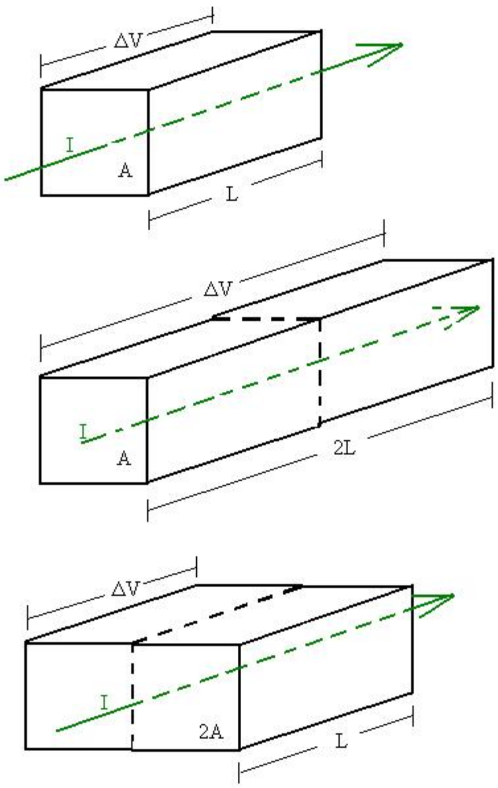
First, this process slows the electrons down to an average velocity of a few millimetres per second. Second, the energy lost by the electrons is gained by the atoms; this shows up as an increase in the vibration of the atoms, which we know from past discussions can be detected macroscopically as an increase in the metal's temperature. We'll discuss how much energy is 'lost' as thermal energy later.

This model also helps to explain an interesting effect. As the temperature of a metal increases, the electrical resistivity increases. We naively explain that, as the temperature increases, the amount of vibration of the atoms increases, so that they occupy a larger effective volume, thereby making collisions with passing electrons even more likely. Often, we can approximate that behaviour with the empirical relationship:
$\rho(T) = \rho_o (1+\alpha_{To} (T - T_o))$,
where alpha is the *temperature co-efficient for resistivity* for a particular material at a particular reference temperature.

Now that we understand this model, let's correct some of the faulty assumptions it uses. The collisions generally are not with the atoms, but with *defects* in the lattice structure, such as a spot where an atom is missing, or where there is an extra atom, or where a different type of atom (an *impurity*) has been substituted for a correct type. Even then, the model is only good for metals at low temperatures; at higher temperatures, the electrons mostly scatter off each other.

Resistance and Ohm's Relationship

Every piece of some material with the same composition, at the same temperature, *et c*., will have the same conductivity or resistivity (it is an intrinsic or characteristic property). However, it is often convenient to use a quantity which is sample specific, the *conductance* (symbol S, unit the Siemens). The conductance is the proportionality constant between the potential difference applied across an object and the resulting current:
$I = S\,\Delta V$.
Much more often, the quantity used instead is the *resistance* (symbol R, unit the *ohm* Ω), which we shall define as the ratio of the potential difference applied across a sample to the current that that potential difference causes:
$R = \Delta V/I$.
Clearly, the resistance and the conducatnce are reciprcal values.
How is resistance related to the resistivity? Consider a block of material as shown with a potential difference ΔV across the ends and a current I flowing through the end face as shown in the top figure.



Certainly, the resistance will depend on the characteristics of the material; if the resistivity is higher, the resistance of any given piece of material should also be higher by the same factor, so we can say that
$R \sim \rho$.

In the second figure, we double the length L of the piece of material. If the same potential difference ΔV is applied across the ends, the electric field inside the material will be half as great, since
$E = V/d = \Delta V/2L$,
(compared to $\Delta V/L$ in the original piece), in which case the force on each electron is half as great and the acceleration is on average half as great and the average velocity of the electrons is half as great as before (see note below). As a result, it takes twice the time as before for a given amount of charge to pass through the sample, and correspondingly, the current is half as big, and so R is twice as big. Our conclusion is that the resistance R is proportional to the length L of the sample:
$R \sim L$.

In the third figure, the length is returned to its former value and the cross-sectional area is doubled. It seems clear that the total current will be the sum of the currents through each of the original-sized sections, that is, double the original current. So, we see that the current is doubled, and correspondingly the resistance is halved. More generally, the resistance is inversely proportional to the cross-sectional area:
$R \sim 1/A$.

Combining these relationships results in:
$R = \rho L/A$,
for a rectangularly shaped sample. For more complicated shapes, we'd have to consider each part and combine the results (see below).

Note that, if the object above were rotated so that the potential difference were applied across a different pair of ends, the resistance may well have a different value:
$R_X = \rho L_X/L_Y L_Z$ *vs* $R_Y = \rho L_Y/L_X L_Z$ for example.
So R depends not only on the dimensions of the sample, but on its orientation as well.

For many objects, the resistance is a constant (or approximately so), and the current is proportional to the applied potential difference:
$\Delta V = R\,I$.
Such objects are called *ohmic* or are said to obey *Ohm's 'Law.'* I have placed the word 'law' in quotation marks because this relationship, unlike real laws in physics, is not always true. Better to refer to *Ohm's Relationship*. In fact, most of the interesting applications of this principle are in materials and structures that do not obey Ohm's Relationship.

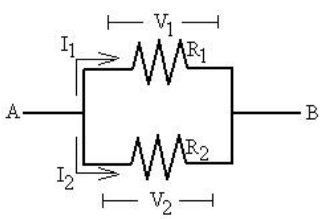
Power

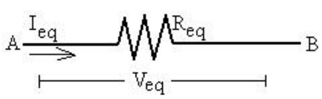
We mentioned that the electrical potential energy of these charges is converted into thermal energy. How much energy is so produced? If charge Q crosses a potential difference V, it loses QV of potential energy ($\Delta EPE = - QV$) and its overall dirft-related kinetic energy component remains constant (at about zero). The power generated as thermal energy is then
$P = W/t = -\Delta EPE/t = +QV/t = [Q/t]V = IV$.
This relationship is always correct. If the material is ohmic, though, we can also write that
$P = IV = V^2/R = I^2R$.

Combinations of Resistors

We consider two basic arrangements of combinations of resistors: series and parallel, somewhat similar to the combinations of capacitors we examined. We would like to see what value resistance R_{eq} will do the same job as the combinations, *i.e.,* for a given potential difference, allow the same current though.

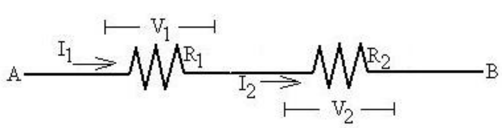
Consider the parallel arrangement shown below:

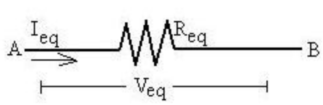




We see a current I_{eq} flowing into the combination, with I_1 passing through R_1 and I_2 passing through R_2. By conservation of charge, we know that
$I_{eq} = I_1 + I_2$,
since we can't create or destroy charge, and there's nowhere for it to be stored. Also, we know that the potential difference across the pair V_{tot} is the same as the difference across each individually (V_1 and V_2),
$V_{eq} = V_1 = V_2$,
since the electric field is a conservative field (and therefore the work done taking a charge slowly from A to B is the same regardless of path). Lastly, for each we can write Ohm's Relationship:
$V_{eq} = I_{eq}R_{eq}$; $V_1 = I_1R_1$; $V_2= I_2R_2$
So, let's start:
$I_{eq} = I_1 + I_2$,
$V_{eq}/R_{eq}= V_1/R_1 + V_2/R_2$
All Vs equal, so they cancel:
$1/R_{eq}= 1/R_1 + 1/R_2$.
This shows that the equivalent resistence of a parallel pair is less that either resistance by itself. This should make sense, because the opening up of a new path for the current will allow more to pass through the circuit, which is the same as saying that the resistance is less. If there are more resistors in parallel, we just keep adding reciprocals of the resistances.

Consider the series arrangement shown below:





We see a current I_{eq} flowing into the combination, with I_1 passing through R_1 and I_2 passing through R_2. By conservation of charge, we know that
$I_{eq}= I_1 = I_2$.
In other words, not only are I_{eq}, I_1 and I_2 equal currents, they are the <u>same</u> current.

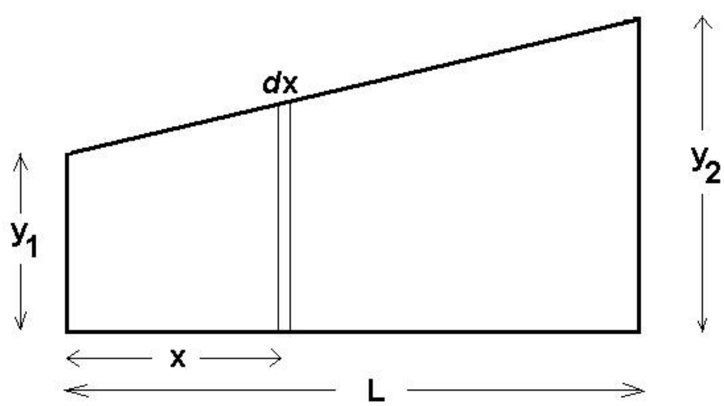
Also, we know that the potential difference across the pair V_{eq} is the same as the sum of the individual differences across each (V_1 and V_2),
$V_{eq} = V_1 + V_2$,
since the electric field is a conservative field.
Lastly, for each we can write Ohm's Relationship:
$V_{eq} = I_{eq}R_{eq}$; $V_1 = I_1R_1$; $V_2= I_2R_2$
So, let's start:
$V_{eq} = V_1 + V_2$,
$I_{eq}R_{eq} = I_1R_1 + I_2R_2$.
All Is are equal, so they cancel:
$R_{eq}= R_1 + R_2$
This shows that the equivalent resistence of a series pair is greater that either resistance by itself. This should make sense, because the combination has more obstacles for the charges to collide with.

Here is an extremely lame analogy: Consider a theatre letting out. The charge is represented by the patrons and the current is the rate at which they leave the theatre. The potential is their desire to get home. In the way is a set of turnstiles, which limits the current. By opening up more turnstiles (adding parallel resistances) the rate of exiting increases, and by placing a second row of turnstiles behind the first, the rate is decreased. Of course, increasing the desire of patrons (potential difference) to leave by yelling 'fire' will also increase the 'current.'

How does the concept of resistance fit into our water analogy? If a wire is like a pipe, then a resistance is like a partial blockage in the pipe; it acts to decrease the flow rate (current) and causes a pressure difference (potential difference) to develop across itself.

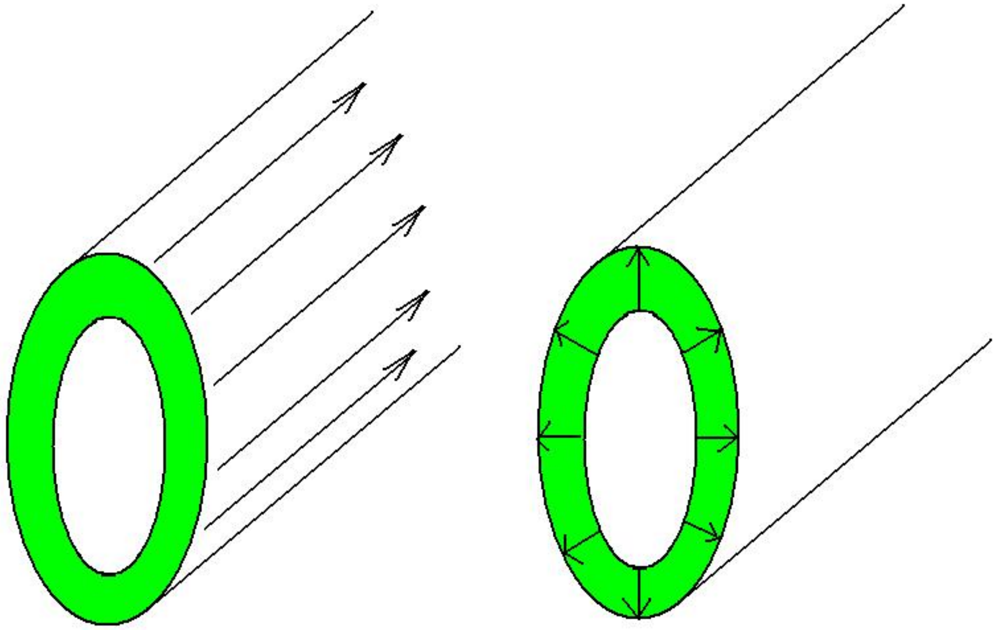
Here is an additional note: Notice that the relationships for parallel and series combinations of capacitors is the reverse from the relationship for combinations of resistors. There is no deep reason for this; each of R and C is a ratio of a charge-related quantity (either Q or I) and potential difference, but they are defined inversely:
R = (potential difference)/(charge-related quantity)   C = (charge-related quantity)/(potential difference).
We could have arranged to define the *conductance* (S = 1/R, unit is the S*iemens*) instead and arrived at similar relationships for S and C, but since few people worry about conductance except transistor engineers, we don't either.

Before moving on, let's do some calculations for the resistances of more complex shapes.
Consider a trapezoidal block of material with uniform resistivity, length L, width W (out of the page), but a height that varies linearly from y_1 to $y_2>y_1$. Let the prospective current flow from left to right.



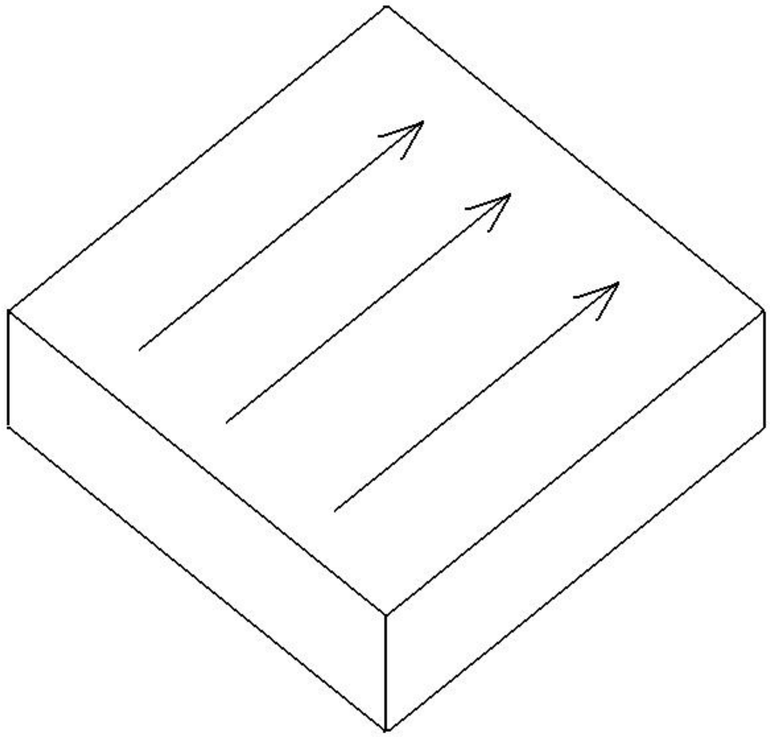
Let's break the object up into thin slices of thickness dx. The resistance of each such slice will be
$dR = \rho dx/A = \rho dx/Wy = \rho dx/W[y_1+(y_2-y_1)x/L]$.
The total resisitance in that orientation is then the sum of all the slices (they are in series):
$R = {}_0\int^L \rho dx/W[y_1+(y_2-y_1)x/L]$
thsi is amenable to u substitution with $u = y_1+(y_2-y_1)x/L$ and $du = (y_2-y_1)x/L$.
$R = \rho L/[W(y_2-y_1)] \int u^{-1}\, du = \rho L/[W(y_2-y_1)] \ln u = \rho L/[W(y_2-y_1)] \ln [y_1+(y_2-y_1)x/L]\, {}_0|^L = \rho L/[W(y_2-y_1)] \ln [y_2/y_1]$

Consider a pair of coaxial surfaces or radiuses r_A and $r_B>r_A$ and length L. The space betweeen them is filled with a material of uniform resistivity, ρ.



First, find the resistance of the material for a current directly along the length.
Well, this is fairly easy because the cross sectional area is constant along the length of the object. So,
$R = \rho L/A = \rho L/[\pi r_B^2 - \pi r_A^2] = \rho L/\pi[r_B^2 - r_A^2]$.

Now find the resistance if the current were to run radially.
Break the material up into thin cylinders. Each has cross-sectional area $2\pi rL$ and thickness dr. The resistance of such a sheet is then $dR = \rho dr/2\pi rL$.
The slices are in series in this case, so we simply add them up:
$R = \int_{rA}^{rB} \rho dr/2\pi rL = [\rho/2\pi L] \int_{rA}^{rB} dr/r = [\rho/2\pi L] \ln(r) |_{rA}^{rB} = [\rho/2\pi L] \ln(r_B/r_A)$.
Consider an LxLxd rectangular slab of uniform resistivity. Find the resistance from one small face to the opposite face.



Again, the cross sectional area is constant along the length, so we have that
$R = \rho L/A = \rho L/Ld = \rho/d$, independent of L!
This is sometimes called the *sheet resistance* and has units of Ohms per square (Ω/[]).

Another example: Consider a cylinder of radius R and length L for which the resistivity is given by $\rho(x) = \rho_o(1 + 3x)$, where x indicates the position along the lenght of the cylinder (we'll assume that the numbers in brackets have the correct dimensions). Let the potential difference be applied across the ends of the cylinder.

Slice the cylinder up into discs; for each, the resistance throughits face is
$dR = \rho(x)dx/A = \rho_o(1 + 3x)dx/\pi R^2$
Since the slices are arranging in series, the total resistance is then

$$R = {}_0\int^L \rho_o(1 + 3x)dx/\pi R^2 = \rho_o(x + {}^3/_2x^2)/\pi R^2\,{}_0|^L = \rho_o(L + {}^3/_2L^2)/\pi R^2$$

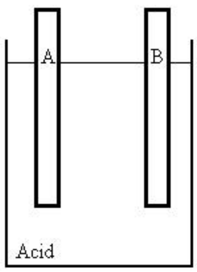
Consider two hemispheres (r_A and r_B) with a resistive material between them. The resistivity is given by:
$\rho(x) = \rho_o(r + a)$
(we'll assume that the numbers in brackets have the correct dimensions). Let the potential difference be applied across the hemispherical surfaces.
Break the material up into thin hemispheres of thickness dr and area $2\pi r^2$. The resistance of such an object with the current normal to its curved surface will be:
$dR = \rho(x)dr/A = \rho_o(r + a)dr/2\pi r^2$.
Since all of these layers are arranged in series, we just add the individual resistances:
$R = {}_{rA}\int^{rB} \rho_o(r + a)dr/2\pi r^2 = {}_{rA}\int^{rB} [\rho_o/2\pi]\, r^{-1}dr + {}_{rA}\int^{rB} [a\rho_o/2\pi]\, r^{-2}\, dr$ = (skipping a few steps) $[\rho_o/2\pi] \ln(r_B/r_A) + [a\rho_o/2\pi]\,[1/r_A - 1/r_B]$

One last example: Consider a cone of material of uniform resistivity. The cone has length L and the side is described by $r = ax$, where x is the distance along the central axis measured from the apex. Let eh current run fronm teh apex to the base. Again, we slice the cone into discs of thickness dx and area $\pi r^2 = \pi a^2x^2$.
The resistance of each slice is then
$dR = \rho dx/ \pi a^2x^2$
These slices are in series, so once again,
$R = {}_0\int^L \rho dx/ \pi a^2x^2 = {}_0\int^L [\rho/\pi a^2]\, x^{-2}\, dx = -[\rho/\pi a^2]\, x^{-1}\,{}_0|^L = -[\rho/\pi a^2]\,[L^{-1} - 0^{-1}]$ = infinite.

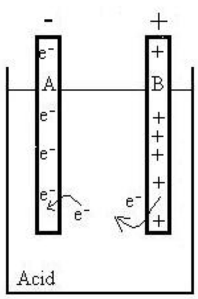
Does this make sense? What happens at the apex?

Batteries

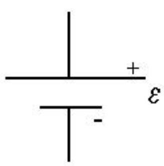
We have been rather incautiously applying potential differences to capacitors and resistors. Let's discuss one device which can do this, the *battery*. The classic battery is a container of acid into which are placed two electrodes composed of <u>different</u> metals (here, A and B).



The materials are chosen so that two chemical reactions take place. One has the result of removing electrons from Metal B and placing them into the acid solution, the other has the effect of placing electrons onto Metal A from the solution. As a result, electrode A becomes negatively charged while B becomes positively charged.



The symbol for a battery is:



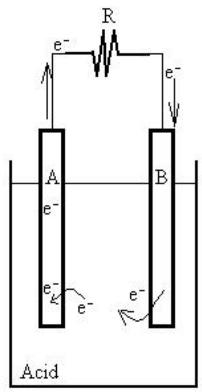
The longer line represents the *positive terminal* and the shorter line the *negative terminal* (makes sense, since we need twice as much line to make a cross as a dash); here the vertical lines represent wires connected to the battery terminals.

Why do the metals have to be different?

Answer

When does this reaction stop? There is a *chemical potential difference* which drives the electrons over to electrode A, but as A is charged up, it gets harder and harder to place the next electron over there (electrostatic repulsion), and so there is a reverse electrical potential difference forming as well. When these two cancel each other, the reaction stops. The electrical potential difference of the battery in this state is called the *emf* (ε, unit is volts). The letters of 'emf' do stand for words, but they are misleading and do not accurately describe the nature of emf; better to think of emf as a new word meaning the potential difference between the terminals of a battery when this static equilibrium has been reached.

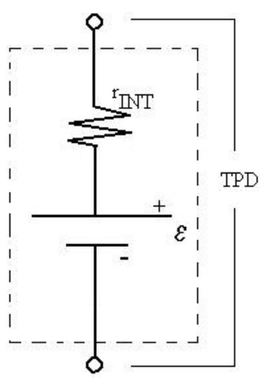
Now, connect a wire (with some small resistance) from the *terminal* of electrode A to that of B. Now, the electrons have a way of returning to B without having to fight against the chemical potential. As charge is transferred on an outside path from one terminal to the other, the electrical potantial difference between the electrodes will decrease and the chemical reation will restart, resupplying charges to the terminals. So, unlike in a capacitor, the plates or terminals are continually resupplied with charge.



As the charge flows, a new dynamic equilibrium will be achieved such that the charge transfer due to the chemical reactions and the current in the outside circuit are equal; this will occur however at a potential difference between the terminals (TPD) which is less than the emf. The larger the current through the battery, the lower this equilibrium TPD will be.

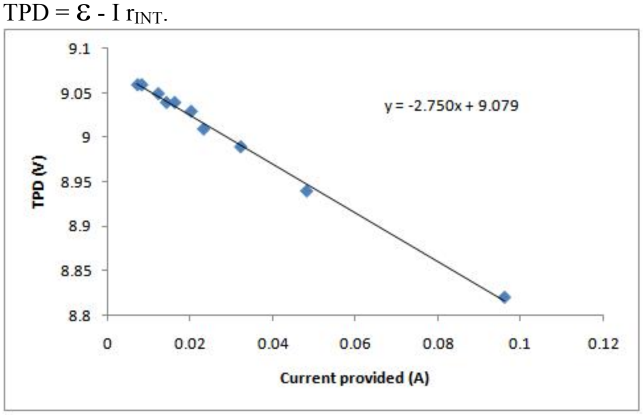
Let's use the water analogy, in which water corresponds to charge and water depth corresponds nicely to potential difference. The cistern in your toilet has a valve attached to a lever with a floater; when the water level in the tank lowers, the floater also lowers and the valve is opened, thus refilling the tank. Often, though not always, the rate at which water refills the tank is greater when the level of water in the tank is lower (since the valve is more open). This device represents the chemical reaction, since the rate of the chemical reaction will increase when the TPD is lower.
Now, suppose that you have a leak (current) in the tank; the water level will sink, opening the valve, until the rate of leakage and the rate of refilling match (current in and out of the battery). This will necessarily occur at a water level (TPD) lower than 'full.' If the leak (current) becomes larger, the equilibrium will be reached when the water level (TPD) is even lower. Stopping the leak allows the water level (TPD) to return to its 'full' value (the emf).

We model this effect by assuming that there is an *internal resistance* r_{INT} inside the battery, across which there is a potential difference ($I^* r_{INT}$) while current is flowing.



In that case, the potential across the terminals of the battery should be given by:

$TPD = \varepsilon - I\, r_{INT}$.



We can calculate r_{INT} by looking at the slope of the line in the graph above (in this case, about 2.75 Ohms).

We find that larger batteries tend to have smaller internal resistances, since the larger plates inside provide more surface area for the reactions to take place more efficiently, and also that the internal resistance of a given battery tends to increase as the reactants are used up.

Remember though that there is not really such a resistor inside batteries; this is just a model.

Here are some questions to consider:

Batteries generally get hot while providing a current. Why?

Answer

Why do batteries 'run out?'

Answer

Why do batteries run down slowly, rather than work fine and suddenly stop working?

Answer

What could one do to resuscitate a dead battery?

Answer

Why are some batteries non-rechargeable?

Answer

Can the TPD ever be greater than the emf?

Answer

Here is an example. Consider your car battery with an emf of about 12.6 V. Turn on your headlights, which draw a current of about 20A. Now, try to start your car. What happens to the headlights?

Answer

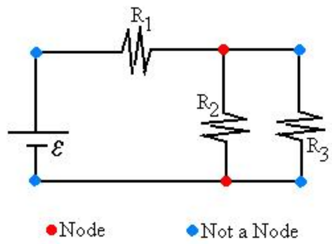
The starter of a typical car draws about 400A from the battery. Because of this huge drain, the TPD of the battery is reduced to about 6V, and correspondingly this reduction in the potential difference across the lamps forces less current through them, making them dimmer. Once the starter is disconnected from the battery, the TPD rises back to about 12V and the lights regain their former brilliance.

Let's consider briefly the possibility of placing batteries in series or in parallel; the same rules apply, of course. In series the total emf will be the sum of teh individual emfs, and the current will be the same through each battery. In parallel, the total current out of the combination will be the sum of the individual currents, but we have to be careful in terms of the potential differences across each which must be the same; two different emf batteries in parallel will cause one to discharge and the other to charge. This is why mixing old and new batteries (even of the same nominal emf) in a device is not a good idea.

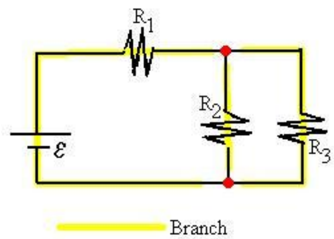
Kirchhoff's Rules

Some arrangements of resistors (and of other components) are not so amenable to the reduction treatment outlined above. For those, we need to use a more powerful, if more tedious method, now known as *Kirchhoff's Rules*. These are restatements of the same two concepts discussed previously: conservation of charge and the fact that the electric field is conservative. As we work through Kirchhoff's Rules, let's consider a simple concrete example, simple enough that we need not even use the rules to find the currents and potential differences.

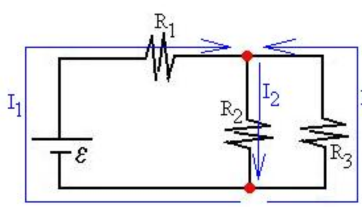
Consider a *node*, a spot where there is a choice of directions to go in a circuit. Some books define a node as a spot where two wires are connected, but some students invariably have problems with distinguishing nodes from bends in a wire.
Mark all of the nodes in the circuit. Here, we see that there are two nodes.



Next, we shall look first for the *branches* of the circuit. A branch is any path which connects two nodes without passing through any other node.



Here, we see that there are three branches. Each branch carries a current, I_i. We have at this time no idea even in which directions the currents are flowing, but the beauty of Kirchhoff's rules is that everything will work out in the end; if we guess a current direction incorrectly, we shall merely obtain a negative answer. Assign currents to each branch (I_1, I_2, *et c.*) along with a direction for each (indicated on circuit diagram with arrows).



Note that the current is I_1 in <u>all</u> of the first branch, that is, before, after, and through the battery.

Conservation of charge indicates that the sum of all currents entering a node must equal the sum of all currents leaving a node, else charge would be either created or destroyed.

$\Sigma_i (I_{in})_i = \Sigma_j (I_{out})_j$

I would prefer to write this differently. Let's say that the sum of all currents entering a node is zero, but make those actually entering be counted as positive, and those currents actually leaving the node be negative ('negative in' = 'out,' right?):

$\Sigma_i (I_{in})_i = 0$

These equations may look very different, but remember that they are simply based on different bookkeeping methods. In this example, we would write:

top node: $I_1 - I_2 + I_3 = 0$,

bottom node: $- I_1 + I_2 - I_3 = 0$.

Notice the redundancy of the two equations; the second one gives us no additional information over the first one, and so we can ignore it. This is discussed in more detail below.

Now let's look at the potential difference we encounter as we trace a *loop* around the circuit; a loop is a closed path around the circuit. Because of the fact that the electric field is conservative, the work we do moving some test charge completely around the loop back to its starting place must be zero. So, then, the sums of all the potential differences we encounter while doing this must add to zero; some will be increases in potential and others must be decreases in potential:

$\Sigma_i \Delta V_i = 0$.

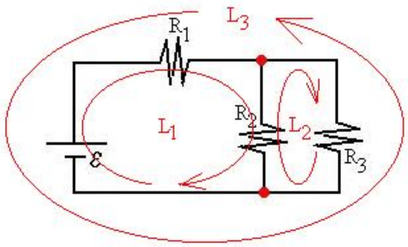
These potential differences will be of two types (for now): the *emfs* of batteries and potential differences across resistors. I prefer to count these two types separately by summing the voltage rises across the batteries (a drop would be counted as a 'negative rise') and the drops across the resistors. Since I will put these two types of terms on opposite sides of my equation, the sign problem will be accounted for:

$\Sigma_i R_i I_i = \Sigma_i \mathcal{E}_i$

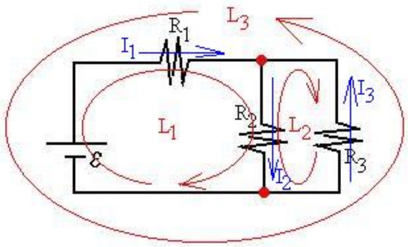
What sign should be assigned to each term? For resistors, current always flows from higher potential to lower potential, so it seems that if I traverse the resistor with the current, I should count that as a positive drop, and if I cross the resistor against the current, that would be an increase in potential or a 'negative drop,' and I should insert a minus sign. If I cross the battery from the negative terminal to the positive terminal, that would be an increase, so I stick a plus sign in front of it, but if I cross it the other way, that's a decrease and I should insert a negative sign. Note that the sign given to a battery emf does not depend on the direction of the current though it.

Let's look at out example. How many loops are there?

Answer



Note that it doesn't matter which way we travel around a loop, since drops will become rises and rises drops, and we'll end up with the same equation in the end. Let's write the loop rule for each of the loops shown in the figure:



L_1: $+ R_1I_1 + R_2I_2 = +\varepsilon$ [We traverse R_1 and R_2 with the respective currents, and we cross the battery from low to high potential.]
L_2: $- R_2I_2 - R_3I_3 = 0$ [We traverse R_2 and R_3 against the respective currents.]
L_3: $-R_1I_1 + R_3I_3 = -\varepsilon$ [We traverse R_1 against the current and R_3 with the current, and we cross the battery from high to low potential.]
We notice that the third equation can be obtained by adding the first and the second then multiplying by negative one, and so it gives us no new information and we therefor ignore it (more on this later).

So, we have three equations which we can use to solve for three unknown currents:

$I_1 + \quad - I_2 + \quad I_3 = 0$

$R_1 I_1 + \quad R_2 I_2 \qquad = +\varepsilon$

$\qquad - R_2 I_2 + - R_3 I_3 = 0$

Generally, at this point the math gets really ugly, even for so simple a circuit.
Multiply the first equation by $R_1$and subtract it from the second equation:

$R_1 I_1 + -R_1 I_2 + R_1 I_3 = 0$

$R_1 I_1 + \quad R_2 I_2 \qquad = +\varepsilon$

to get:

$(R_1 + R_2) I_2 - R_1 I_3 = +\varepsilon$

Now multiply this equation by R_3 and the third of the original equations by R_1,

$(R_1 + R_2)R_3 I_2 + - R_1R_3 I_3 = +R_3\varepsilon$

$- R_1R_2 \qquad I_2 + - R_1R_3 I_3 = 0$

and subtract to get

$(R_3(R_2 + R_1) + R_1R_2) I_2 = +R_3\varepsilon$

or that

$I_2 = R_3\varepsilon/[R_1R_2 + R_2R_3 + R_2R_3]$.
Substituting this back into the third of our original equations gives:

$I_3 = -[R_2/R_3]I_2 = -[R_2/R_3]R_3\varepsilon/[R_1R_2 + R_2R_3 + R_2R_3] = -R_2\varepsilon/[R_1R_2 + R_2R_3 + R_2R_3]$.
The negative sign indicates that we guessed the direction of I_3 incorrectly and that it actually flows in the opposite direction, as we would expect from looking at the circuit.

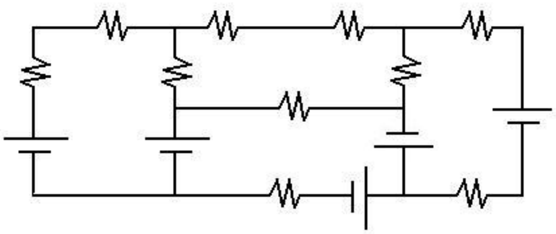
Now, substitute these values for I_2 and I_3 back into a previous equation to find I_1:

$I_1 = I_2 - I_3 = R_3\mathcal{E}/[R_1R_2 + R_2R_3 + R_2R_3] - {}^{-}R_2\mathcal{E}/[R_1R_2 + R_2R_3 + R_2R_3] = [R_2 + R_3]\mathcal{E}/[R_1R_2 + R_2R_3 + R_2R_3]$.

Now, in this particular case, it would have been easier to use reduction to find the equivalent resistance, but this was a good exercise. Some systems can not be solved with reduction. Also, in this case, we assumed that we knew the emf and the resistances and were looking for the currents, but we could just as easily have known one current and not known one resistor value, *et c*., so long as we have an equal number of independent equations and unknown variables (Note, mathematically, other conditions have to be met, but since we assume that there is only one physical solution for a system like this, that condition suffices).

In general, how many of each type of equation (loop and node) do we need in order to solve a problem? We find that if we have N nodes, we need N-1 node equations, since the last node equation will be some combination of the others and it therefor presents no new information. If there are M small loops in the circuit, then we need M loop equations. Again, any more past that number will be combinations of the others and we shall learn nothing new about the circuit from them. Generally, we know all the Rs and $\mathcal{E}$s, and are asked to find the Is. We can double check our information, since the rule is that we need Z equations to find Z unknowns. You should find that M + N - 1 is equal to the number of branches in the circuit, and therefor also equal to the number of currents we need to find. The rest is just algebra.....

So, consider this generic example:



How many nodes are there?

Answer

How many loops?

Answer

How many independent loops?

Answer

How many branches?

Answer

How many of each equation type should we write to solve this problem?

Answer

There is a Kirchhoff's Rules Solver on the course Excel Workbook. This will determine up to six currents (and other quantities with some imagination). Just write the (M+N-1) equations and place the co-efficients in the correct cells. Below is an example of how the worksheet should be filled in with the co-efficients for the example done above with $R_1 = 20\Omega$, $R_2 = 30\Omega$ and $R_3 = 50\Omega$, and $\mathcal{E}$ = 14V:

$$I_1 + \quad - I_2 + \quad I_3 = 0$$
$$20\,I_1 + \quad 30\,I_2 \quad = +14$$
$$- 30\,I_2 + - 50\,I_3 = 0$$

KIRCHHOFF'S RULES SOLVER

1	I1 +	-1	I2 +	1	I3 +	0	I4 +	0	I5 +	0	I6 =	0
20	I1 +	30	I2 +	0	I3 +	0	I4 +	0	I5 +	0	I6 =	14
0	I1 +	-30	I2 +	-50	I3 +	0	I4 +	0	I5 +	0	I6 =	0
0	I1 +	0	I2 +	0	I3 +	1	I4 +	0	I5 +	0	I6 =	0

0	I1 +	0	I2 +	0	I3 +	0	I4 +	1	I5 +	0	I6 =	0	
0	I1 +	0	I2 +	0	I3 +	0	I4 +	0	I5 +	1	I6 =	0	
I1 =	**0.361**	I2 =	**0.226**	I3 =	**- 0.140**	I4 =	**0**	I5 =	**0**	I6 =	**0**		

A Further Note for Those Who Care

If one were to try to work out the assertions made in this paragraph in detail, one would probably fail. The reason is that the whole story has not been presented. One might say the following:
L is doubled, so for constant ΔV, the E field is halved.
E is halved, so the force $F_E = qE$ is halved and the acceleration $a = F/m$ is halved.
The electrons are assumed to start from rest after each collision and accelerate against the field until the next collision with a scattering site, perhaps a distance d away. Using the kinematic equations from previous discussions, one sees that $v_{AVE} = {}^1/_2\, v_f = {}^1/_2\, [2ad]^{1/2}$, so that the average velocity does not halve when the acceleration halves.

The resolution to this problem is in the details of the motions of the electrons. When the electric field is zero, the electrons are not motionless, but actually moving very quickly. The fact that the current in this situation is zero only means that just as many electrons are flying in one direction as in the opposite direction, so that there is no net movement of charge. In fact, in a metal, the conduction electrons are bouncing around inside the metal in much the same way as gas molecules in a box do. One can even use the results of the *equipartition of energy theorem* to estimate the rms speed:
${}^1/_2 m\, v_{rms}^2 = {}^3/_2 k_B T$; at room temperature, $v_{rms} = 10^5$ m/s, compared to the asserted average *drift speed* of the current, $\sim 10^{-4}$ m/s.
The point here being that the quantity which is constant is the time between collisions, not the distance traveled between collisions. So the relationship one should consider is
$(v_{Drift})_{AVE} = {}^1/_2\, (v_{Drift})_f = at/2$,
so that the average drift speed of the motion against the electric field which is superimposed on the random motion is indeed proportional to the acceleration and so also to the electric field.

Let's back up a little bit: how do we know that the drift speed is so low? Consider a typical metal wire with cross sectional area 1mm^2 carrying 1A of current. In one second, $6\text{x}10^{18}$ electrons will pass a given point. These electrons on average occupy the same volume that their 'host' atoms do; since metal atoms are typically separated by about $3A^o = 3\text{x}10^{-10}$m, the total volume occupied by the electrons will be $6\text{x}10^{18} * [3\text{x}10^{-10}]^3 = 1.6\text{x}10^{-10}\text{m}^3$. Given that the cross-sectional area of the wire is 10^{-6}m^2, the electrons must then move at $0.16\text{x}10^{-3}$ m/s or as stated at about 0.1 mm/s.

Continue on to the Next Section
Return to the Notes Directory
