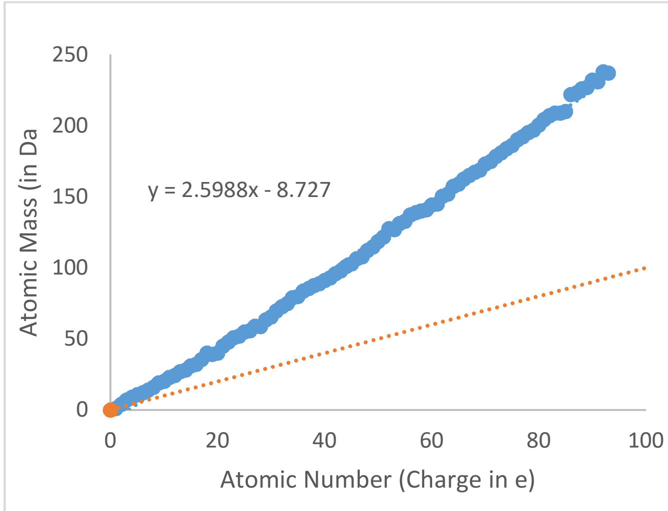
Section 14 - The Nucleus

"If, as I have reason to believe, I have disintegrated the nucleus of the atom, this is of greater significance than the war."

Ernest Rutherford



Since the nucleus of the hydrogen atom can be readily identified as a proton with charge e, and since we can suss the charge on the nucleus of each of the various elements, we presume that the charge will be Ze, with Z the number of protons. We also should be able to predict the masses of any nucleus to be Z m_P ,[1] that is, an integer times the mass of a proton. However, as seen in the graph at right, the masses are higher than expected by roughly a factor of 2.5! What's more, the mass of each element is generally not actually an integer multiple of m_P.

Isotopes

One experiment that will help us solve this puzzle is the discovery of *isotopes*, sets of nuclei with the same electric charge, but different masses. Although there was some early work by Thomson (neon has isotopes of 20 and 22 times the proton mass[2]), we'll concentrate on the results of F.W. Aston.[3] Aston made use of the *mass spectrometer*, a device we examined briefly in Semester Two. Material is heated to the gas phase, ionized (*i.e.*, some number of electrons are removed), sent through a velocity selector, then turned in a large magnetic field. The radius of the resulting curves path is proportional to the mass to charge ratio. Since e is a well-known value, Aston could determine the masses to within 0.1%.

One of the questions asked above was, why are the masses of atoms of a particular element not integer multiples of the proton mass? For example, neon has an atomic mass of approximately 20.2 m_P. Aston confirmed the existence of two types of neon, one (Ne-20) with a mass of almost exactly 20 m_P and the other with a mass of 22 m_P, with Ne-22 being about 10% of the total.

EXAMPLE 14-1

What would be the average mass of neon if 90% were Ne-20 and 10% were Ne-22?

Atomic mass = $0.9\times20 + 0.1\times22 =$ 20.2 m_P, as expected.

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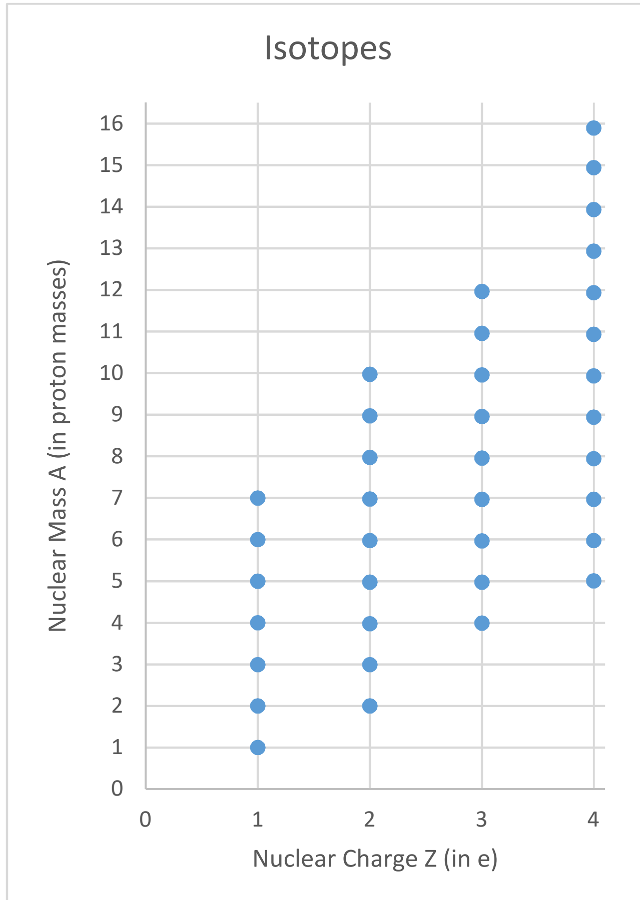
[1] For now, we'll use the mass of the proton as the unit of mass. You may be familiar with the atomic mass unit (a.m.u.), which is 1/16th the mass of an oxygen atom and which is now considered an antiquated unit. The currently correct unit is the Dalton, which is 1/12 the mass of a carbon-12 atom. We'll be more specific later.

[2] These numbers <u>are</u> almost exactly integers.

[3] Francis W. Aston, 'Mass Spectra and Isotopes,' Nobel Lecture, December 12th, 1922.

HOMEWORK 14-1

> Aston measured two isotopes of chlorine to have masses of 35 and 37 m_P, respectively. The average mass of chlorine is 35.46 m_P. What is the ratio of Cl-35 to Cl-37?



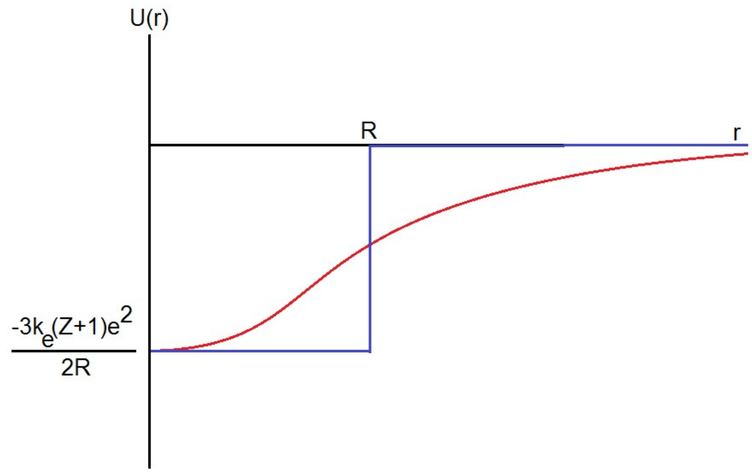
Aston went on to measure the masses of 84 isotopes of 34 different elements; each mass came out to be an integer number of proton masses, to within 0.1%. Here are some modern data for the light end of the periodic table; none of these masses is more than 0.7% away from an integer number of proton masses. It's difficult to believe that the nucleus is composed of anything but protons!

The Aston Model

Here is the model put forth by Aston. As usual, Z is the net charge of the nucleus. A will be the atomic mass in proton masses, which equals the number of protons. Since A>Z, we need to neutralize some of the charge by adding A-Z electrons to the nucleus. Since the charge of the electron is exactly opposite that of the proton, one electron cancels one proton. In terms of mass, the electrons are 0.05% of a proton mass, and will have little effect on the mass measurements.

Element number = charge of nucleus	Z
Atomic mass	A
Protons in the nucleus	A
Electrons in the nucleus	A - Z
Electrons in orbits around the nucleus	Z

Now, let's see why this model won't work. Consider one of the 'nuclear electrons.' It is essentially trapped in the nucleus, presumably due to coulomb forces. Assume we have a fairly massive nucleus, so that A is a large number. We'll assume that the A protons and the other A-Z-1 nuclear electrons are uniformly distributed in a sphere of radius R, the net charge of which would then be (Z+1)e. Ignoring the orbiting electrons (we're well inside a gaussian surface that excludes them), we can find the potential energy of our nuclear electron as a function of distance from the center of the nucleus. You did this as an example in Semester Two:



$$U(r) = -\frac{k_e Qq}{r} \rightarrow -\frac{k_e(Z+1)e^2}{r} \quad for\ r > R,$$

$$U(r) = \frac{k_e Qq}{2R^3}r^2 - \frac{3k_e Qq}{2R} \rightarrow \frac{k_e(Z+1)e^2}{2}\left(\frac{r^2}{R^3} - \frac{3}{R}\right) \quad for\ 0 < r < R.$$

Let's approximate this potential well as a cube of edge length 2R. If the walls were infinitely high (which they are not), the lowest possible energy for such an electron would be

$$\mathrm{E}_{1,1,1} = (1^2 + 1^2 + 1^2)\frac{\mathrm{h}^2}{8\mathrm{mL}^2} \cong 3\frac{\mathrm{h}^2}{8\mathrm{m}(2\mathrm{R})^2} = \frac{3}{32}\frac{\mathrm{h}^2}{\mathrm{mR}^2} \text{ above the bottom of the well.}$$

EXAMPLE 14-2

Calculate the depth of the Coulomb well based on the diagram above. An order of magnitude calculation is fine. Assume the radius of the nucleus to be about 10^{-14} m and Z to be 79 (gold).

Calculate the lowest possible energy state for an electron in an infinite cubic well of edge 2R. An order of magnitude calculation is fine.

Alternatively, calculate the energy of an electron confined to the nucleus using the De Broglie wavelength, *i.e.*, $\lambda_{dB} \simeq 2R$.

What do these results indicate?

$$\mathrm{U}(0) = \frac{3\mathrm{k_e}(\mathrm{Z}+1)\mathrm{e}^2}{2\mathrm{R}} = \frac{3(9\times10^9)(80)(1.6\times10^{-19})^2}{2(10^{-14})} \approx 3\times10^{-12}\mathrm{J} \cong 20\ \mathrm{MeV}.$$

$$\mathrm{E}_{1,1,1} = \frac{3}{32}\frac{\mathrm{h}^2}{\mathrm{mR}^2} = \frac{3}{32}\frac{(6.63\times10^{-34})^2}{(9.11\times10^{-31})(10^{-14})^2} \cong 5\times10^{-10}\ \mathrm{J} \cong 300\ \mathrm{MeV}.$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda_{dB}^2} = \frac{h^2}{8mR^2} = \frac{(6.63 \times 10^{-34})^2}{8(9.11 \times 10^{-31})(10^{-14})^2} \cong 6 \times 10^{-10}\ \mathrm{J} \cong 300\ \mathrm{MeV}.$$

So, this suggests that an electron restricted to the nucleus would have an energy much higher than the depth of the potential well. So, it seems unlikely that there are any electrons in the nucleus.[4] Any electrons there would quickly scoot out.

So, it appears Aston's model is a no go.

Enter the Neutron

Like many discoveries of modern physics, that of the neutron was accidental. Streams of neutrons were initially mis-identified as a type of gamma ray; subsequent experiments indicated that this unknown radiation was a neutral particle with a mass roughly equal to that of a proton. So, we now have a candidate to explain the unexpected extra mass of the nucleus and the existence of isotopes; the nucleus contains Z protons that determine the X-ray spectra and, indirectly, the chemical properties of the atom, and A-Z neutrons that account for the 'extra' mass of the atom.

Wrong ideas often linger for a while after discoveries are made. Consider the possibility that a neutron is just a 'paired' proton and electron, that is, that Aston's model is essentially correct but that the electrons, for some reason, can't escape the nucleus. Chadwick and Goldhaber[5] irradiated the nucleus of 'heavy hydrogen' (*deuterium*) with gamma radiation. The assumption was that the nucleus comprises a proton and a neutron, and that the energy of the gamma-ray will split the nucleus. The energy released will appear as the kinetic energies of the proton and neutron. The energy of the proton is fairly easy to measure (it's charged) and the actual mass of the neutron can then be calculated.

EXAMPLE 14-3

The mass of a proton by definition is 1 proton mass. The mass of deuterium is 1.9990 proton masses. A gamma ray of energy 2.62 MeV strikes a stationary deuteron and splits it. As an approximation, we'll assume that the kinetic energies of the resulting particles are equally distributed between them (See Note One). The kinetic energy of the proton K_P is measured to be 0.25 MeV. Then, making use of the relativistic rest energies of the particles:

$$m_D c^2 + hf = m_P c^2 + m_N c^2 + 2K_P$$

$$m_N = m_D - m_P + \frac{hf}{c^2} - \frac{2K_P}{c^2}$$

[4] This is complicated in that we do occasionally see electrons ejected from the nucleus (beta decay). More on this later.

[5] Chadwick, J., and M. Goldhaber, 'A Nuclear Photo-effect: Disintegration of the Diplon by γ-Rays,' *Nature* **134** (1934) p 237.

$$m_N = 1.9990 \times 1.672622 \times 10^{-27} - 1.672622 \times 10^{-27} + \frac{2.62 \times 10^{6} \times 1.602 \times 10^{-19}}{(3 \times 10^{8})^2}$$
$$- \frac{2 \times 0.25 \times 10^{6} \times 1.602 \times 10^{-19}}{(3 \times 10^{8})^2} = 1.674736 \times 10^{-27}\ \text{kg}$$
$$= 1.0013\ \text{proton masses.}$$

The sum of the masses of a proton and separate electron is 1.0005 proton masses, while the mass of a neutron, we've just calculated, is 1.0013 proton masses. This indicates that the neutron is not a bound pair; from special relativity, we know that a stable bound state should have a mass less than the sum of the masses of the individual parts.[6]

It appears that the neutron is a new, distinct particle. So, let's replace all those extra protons in the Aston model with N neutrons. Now, a nucleus of charge Z and mass A contains Z protons and N = A - Z neutrons.

What holds the nucleus together?

In any nucleus more complicated than hydrogen, we have a number of positively charged protons repelling each other. Why doesn't the nucleus fly apart? There must be some other force attracting them together. What about gravity?

EXAMPLE 14-4

Compare the electric potential energy of two protons separated by 10^{-15} m to their gravitational potential energy.

$$U_E = \frac{k_e q^2}{r} = \frac{(9 \times 10^{9})(1.6 \times 10^{-1}\ )^2}{10^{-15}} \approx 10^{-13}\ \text{J}$$
$$U_g = -\frac{Gm^2}{r} = -\frac{(6.67 \times 10^{-11})(1.67 \times 10^{-27})^2}{10^{-15}} \approx -10^{-49}\ \text{J}.$$

Well, it's not gravity.

So, we need a new force. What characteristics should it possess?

1) The force must be fairly strong at short distances so that protons (and we assume neutrons) attract but negligible at distances greater than ~1.5×10^{-14} m (the Rutherford experiment indicates that the Coulomb force is dominant beyond this distance).
2) The force is neither gravitational nor electrical.
3) The force doesn't act on electrons.

For now, we'll refer to this force as the *nuclear force*.[7]

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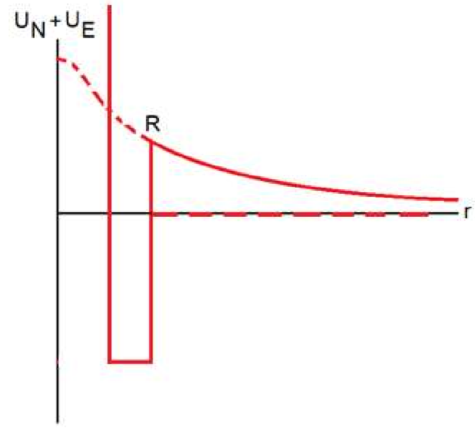
[6] On the other hand, as we'll see in the next Section, a neutron can dissociate into a proton and an electron, plus a 'new' particle known as an *anti-neutrino*, through a process known as *beta decay*. However, the current upper limit on the mass of the anti-neutrino is about 10^{-8} that of a proton, and so this doesn't affect the binding energy argument given above.

[7] There are actually two. We won't deal directly with the other.

There are a number of models that have been used for the force's potential, U_N. These have been refined over the years, based on more detailed experimental results. We will use a very simple function. Combining the electric and nuclear potentials results in a net potential as shown in the figure. Beyond a given radius, R, the nuclear force is essentially zero and only the Coulomb repulsion acts. Within R, the nuclear force must be much stronger than the electric force; we will assume that the two forces create a square well. In addition, since we think that the radius of the nucleus is proportional to the cube root of the number of nucleons (well, for larger nuclei, anyway), there must be some smallest distance they can get to each other. Of course, for proton-neutron or neutron-neutron pairs, the Coulomb potential would be omitted. In the next section, we'll see that the depth of this well is approximately 3×10^{-13} J.



A 'Modern' Unit for Mass

So far, we've been discussing masses either in terms of the proton mass or, with much less accuracy, with the atomic mass number, the sum of the protons and neutrons in the nucleus. From this point on, we'll have to be much more careful about these masses.

I'm old enough to remember the *atomic mass unit* (amu) as the unit of mass; it was defined as one-sixteenth the mass of an oxygen-16 nucleus. Sixty years ago, the switch was made to the *Dalton* (symbol, Da), one-twelfth the mass of a carbon-12 nucleus. This corresponds to neither of our particles' masses.

Mass of proton	1.007276 Da
Mass of neutron	1.008664 Da
Mass of electron	0.00054858 Da

Many of the forthcoming calculations concern the small difference between large numbers; it will be important to carry as many significant figures as possible.

Binding Energy

The energy associated with the binding together of particles (the *binding energy*) can be found using special relativity: the difference between the nucleus mass and the mass of the separated constituent parts, times c^2:

$$E_B = (Zm_P + Nm_N - m_{nucleus})c^2 .$$

Since we're going to be doing this kind of calculation many times, let's make it easier. The rest energy ($m_o c^2$) of one Dalton is about 931.5 MeV. This changes the expression above to a perhaps more convenient:

$$E_B = (Z \times 1.007276 + N \times 1.008664 - m_{nucleus}) \times 931.5\ \text{MeV}.$$

Note that with this formulation, a higher binding energy means that the parts are more tightly bound together.

EXAMPLE 14-5

Calculate the binding energy of the deuteron, given that $m_{Deuteron}$ = 2.013553 Da.

$$E_B = (m_P + m_N - m_D)c^2 = (1.007276 + 1.008664 - 2.013553)\ 931.5\ \text{MeV}$$
$$= 2.223\ \text{MeV}.$$

This is the energy required to separate the deuteron into a proton and a neutron. In Chadwick's experiment, a 2.62 MeV gamma ray split the deuteron, but when the experiment was repeated using gamma rays of energy 1.8 MeV, no dissociation was observed, as would be expected.

HOMEWORK 14-2

Calculate the binding energy of the *triteron* (one proton and two neutrons), given that $m_{Triteron}$ = 3.015501 Da.

Estimating the Binding Energy of a Nucleus

We will discuss aspects of three models, then see if we can combine them into a relationship that will predict the binding energies of nuclei.[8] The first is the *liquid drop model*, which seems self-explanatory. We will find that our final result will be valid only for larger nuclei (A ≥ 20), where the nuclear density is fairly constant. If the density is constant, then the volume of the nucleus should be proportional the A, the number of particles there.

Let's think about some energy considerations.

1) Our first assumption is that all nucleons are attracted to one another with a constant force. The range of this force is very short, and so only *nearest neighbors* interact. Then assuming that each nucleon interacts with the same number of nearest neighbors, the corresponding potential energy term should be proportional to the number of nucleons:

$$\Delta E_1 = C_1 A.$$

2) The second term involves those nucleons at the surface of the drop. Those particles are not completely surrounded with neighbors as the interior ones are, have fewer interactions, and so less binding energy. The number of such nucleons should be proportional to the surface area of the drop. The surface area is proportional to the radius squared (R^2) and the atomic mass

[8] Von Weizsäcker, C. F. 'Zur Theorie der Kernmassen,' *Z. für Physik* **96** pp 431–458.

number A is proportional to the volume, which is in turn proportional to R^3, so this term is proportional $A^{2/3}$.

$$\Delta E_2 = -C_2 A^{\frac{2}{3}}.$$

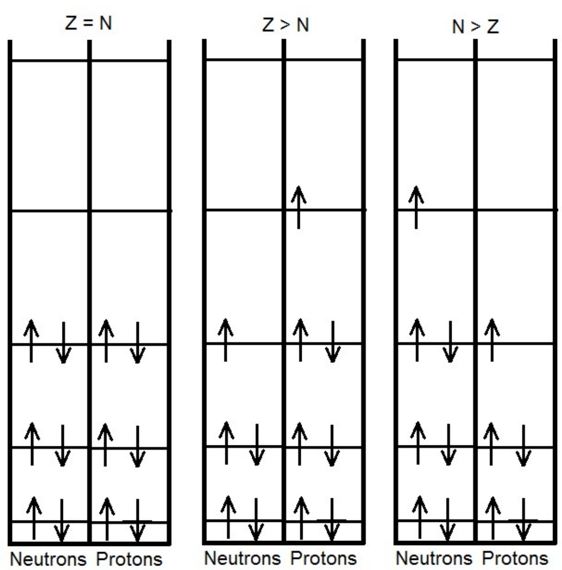
3) The next term accounts for the Coulomb repulsion among the protons. As we showed in Semester Two, the potential energy of a uniformly charged sphere (Q, R) is

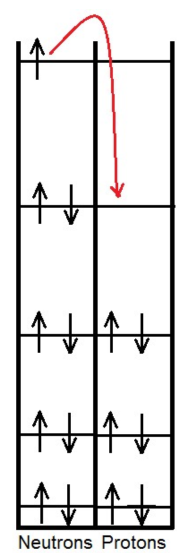
$$U_E = \frac{3k_e Q^2}{5R}.$$

However, we have discrete charges, so the Q^2 terms should be replaced with $Z(Z-1)e^2/2$ to reflect the Z(Z-1)/2 pairs of protons We also realize that R is proportional to $A^{1/3}$. So,

$$\Delta E_3 = -C_3\, Z(Z-1) A^{-\frac{1}{3}}.$$

4) Consider the *Fermi gas model*, in which each type of nucleon fills up energy levels much as the electrons do.[9] Since protons and neutrons are distinct particles, it is possible to have one of each in a particular state without violating the Pauli exclusion principle. Let's take a look at an abstract example. In the figure, we see energy levels for a set of three *isobars*, nuclei with the same atomic mass number, A. On the left, we see the situation with Z = N. Note that the energy levels become more widely spaced as we move up through them. Changing either a proton to a neutron or a neutron to a proton would increase the energy of the system, rendering it therefor less stable.





It's easy to see that the closer N and Z are, the lower the system energy will be for a given value of A. So we need a term related to |N – Z|. Let's see if we can work out an expression for this correction. As usual, we will be making a few approximations.

Let u = N – Z. Then N = ½(A + u) and Z = ½(A – u). We'll assume that the difference in energy between the most energetic neutrons and the most energetic protons is not too big and

[9] The Fermi gas modification to the liquid drop picture was actually necessary to prevent a number of flaws.

that A remains constant. The energies of those particles are given by their respective *fermi energies*, which in turn are each proportional to the two-thirds power of the numbers of neutrons and protons, respectively (see NOTE). The combined energies of these outlying particles is then

$$E = N\varepsilon_{fN} + Z\varepsilon_{fP} = D_1\left(N^{5/3} + Z^{5/3}\right) = D_2\left((A+u)^{5/3} + (A-u)^{5/3}\right),$$

with the Ds being constants whose values we do not care about. Let's expand this expression about u = 0, or N = Z:

$$E(u) \approx \left[D_2\left((A+u)^{\frac{5}{3}} + (A-u)^{\frac{5}{3}}\right)\right]_{u=0} + \left[D_3\left((A+u)^{\frac{2}{3}} - (A-u)^{\frac{2}{3}}\right)\right]_{u=0} u$$
$$+ \left[D_4\left((A+u)^{-\frac{1}{3}} + (A-u)^{-\frac{1}{3}}\right)\right]_{u=0} u^2 + \cdots$$

Since the linear term is zero, this becomes

$$E(u) \approx D_5A^{5/3} + D_6A^{-1/3}\,u^2 + \cdots$$

Now, another approximation. If N = Z (our expansion point, u = 0), then the fermi energies of the most energetic neutrons and of the most energetic protons should be just about the same, or just ε_f.

$$A = N + Z = (\varepsilon_f D_1^{-1})^{3/2} + (\varepsilon_f D_1^{-1})^{3/2} = 2(\varepsilon_f D_1^{-1})^{3/2} \quad \rightarrow \quad \frac{\varepsilon_f}{A^{2/3}} = D_7\,.$$

Inserting this into the result above,

$$E(u) \approx \frac{D_5}{D_7}\frac{\varepsilon_f}{A^{\frac{2}{3}}}A^{\frac{5}{3}} + \frac{D_6}{D_7}\frac{\varepsilon_f}{A^{\frac{2}{3}}}A^{-\frac{1}{3}}u^2 + \cdots = D_8A + D_9\frac{(N-Z)^2}{A} + \cdots\,.$$

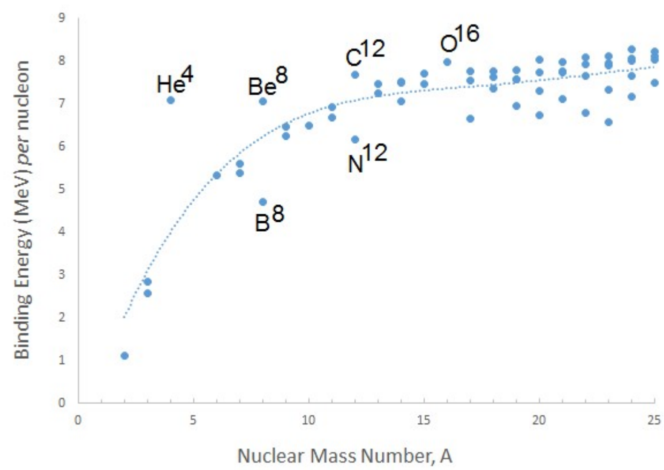
The first term is proportional to A, and so we'll include it in the bulk term above (1), which leaves us with

$$\Delta E_4 = -\,C_4\frac{(N-Z)^2}{A}\quad.$$

5) Out last term estimates the energy differences due to paired or unpaired spins. If both N and Z are even, all spins are paired up, the system is more stable, and the binding energy increases. One of the two odd makes the nucleus less stable, and both odd even more so. Here is an example:

Nucleotide	Z-N	Binding Energy
Zn-34	Even-Even	8.736 Mev/nucleon
Zn-35	Even-Odd	8.724 MeV/nucleon
Ga-34	Odd-Even	8.662 MeV/nucleon
Ga-35	Odd-Odd	8.540 MeV/nucleon

The examples are complicated by the fact that we can not easily eliminate the other effects from the binding energy differences. However, looking at all of the stable nuclei on the periodic table reveals that 58.5% are even-even, 38.7% are either even-odd or odd-even, while only 2.8% are odd-odd. Here's a look at some small nuclei. Note that He-4 (2,2), Be-8 (4,4), C-12 (6, 6), and O-16 (8,8) are noticeably more stable than their immediate neighbors.



We'll write a simple function for this:

$$\Delta E_5 = \eta_{N,Z} C_5 A^{-4/3} ,$$

with $\eta_{N,Z}$ = +1 if Z and N are both even, -1 if both are odd, and 0 if they are mixed.

So, in the end, we have that

$$E_B \approx C_1 A - C_2 A^{\frac{2}{3}} - C_3\, Z(Z-1) A^{-\frac{1}{3}} - C_4 (N-Z)^2 A^{-1} + \eta_{N,Z} C_5 A^{-\frac{4}{3}} .$$

If we fit this function to data from a large number of nuclei, we obtain values for the coëfficients:

C_1 = 15.75 MeV
C_2 = 17.80 MeV
C_3 = 0.710 MeV
C_4 = 23.69 MeV
C_5 = 39 MeV

EXAMPLE 14-6

Calculate the binding energy of tellurium-126 using each method described above. Te-126 has Z = 52 and A = 126, so N = 74. The mass of the Te-126 nucleus is 125.8748 Da.[10]

Mass difference:

$$E_B = (Z \times 1.007276 + N \times 1.008664 - m_{nucleus}) \times 931.5 \text{ MeV}.$$

$$E_B = (52 \times 1.007276 + 74 \times 1.008664 - 125.90331) \times 931.5 \text{ MeV} = 1066.10 \text{ MeV}.$$

Von Weizsäcker:

Since Z and N are both even, $\eta_{N,Z} = +1$.

$$E_B \approx 15.75(126) - 17.80(126)^{\frac{2}{3}} - 0.710\,(52)(52-1)(126)^{-\frac{1}{3}} - 23.69(74-52)^2(126)^{-1} + (+1)39(126)^{-\frac{4}{3}} = 1070.6 \text{ MeV}.$$

That's less than 0.4% off.

HOMEWORK 14-3

Calculate the binding energy (in MeV) of cesium-133 both ways, as in the example above. The atomic mass of Ce-133 is 132.905452 Da.

Fun Fact 14-1

In a pinch, for nuclei larger than about neon, the binding energy *per* nucleon is between 8 and 9 MeV. More specifically, there is a peak of about 9 MeV in the region of iron (A = 56) and a nearly linear taper to 8 MeV to uranium.

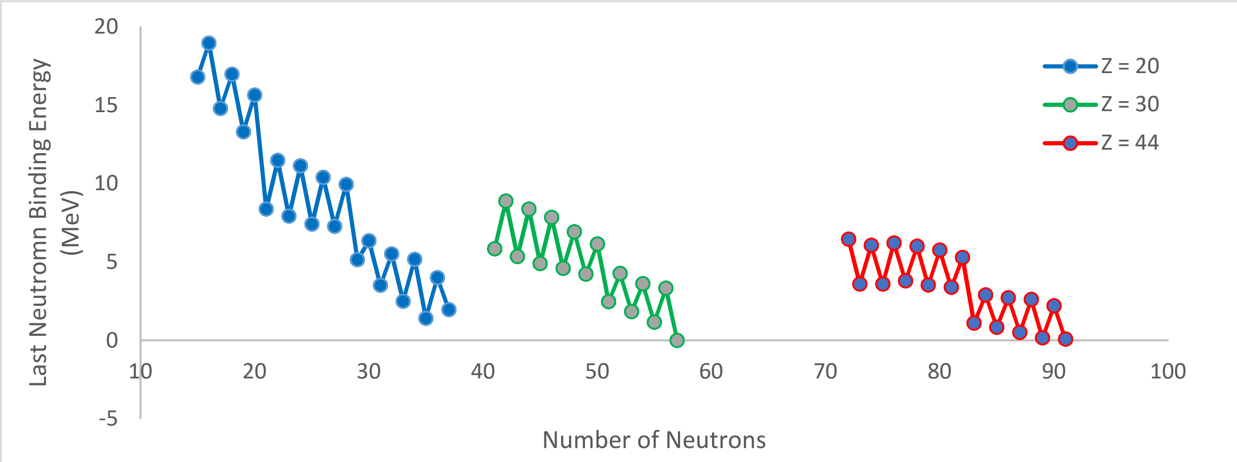
The Shell Model

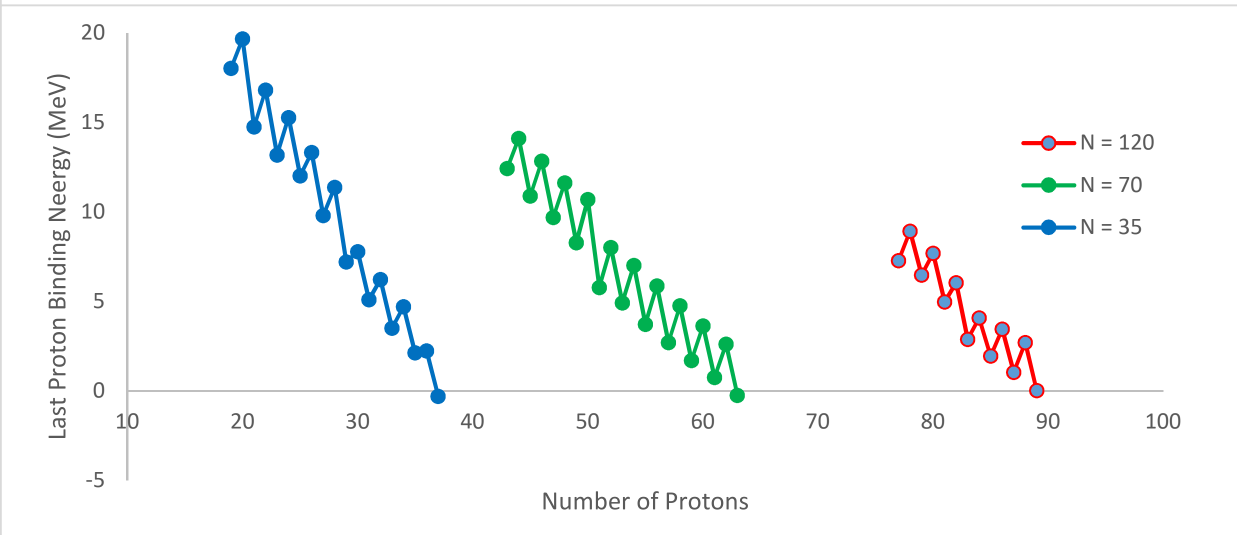
There is some evidence to suggest that the nucleons form shells, in much the same way as do electrons. You may remember that atoms are most chemically stable when the number of electrons is one of the following numbers, when the outer shells are filled: 2, 10, 18, 36, 54, 86, and, presumably, 118. Similarly, there is evidence that nuclei are more stable when either the number of protons or neutrons (or better yet, both) are: 2, 8, 20, 28, 50, 82, and 126. These are referred to as *magic numbers*. Here are some data. In the first graph, the binding energy of the last neutron of several elements is plotted against the number of neutrons.

$$E_{B\ Last\ neutron\ Z,N} = E_{B\ Z,N} - E_{B\ Z,N-1}.$$

[10] You need to be careful about looking up the masses of nuclei. Often, you will find the *atomic mass* that includes the electrons. These should be subtracted off to obtain the nuclear mass. For example, the atomic mass of tellurium 126 is given as125.90331, but that includes 52 electrons. 125.90331 – 52(0.00055) = 125.8748 Da for just the nucleus.

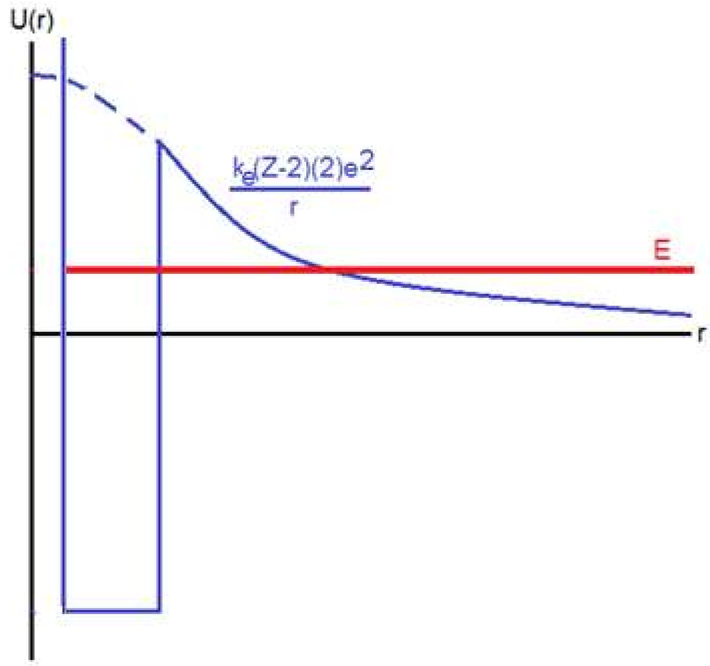
For N = 20, 28, 50, and 82, the binding energy is noticeably larger (more stable) than the nucleus with one more neutron.





Although less obvious, the same holds for the last proton at the magic numbers.

Alpha Decay



Let's talk about nuclei that are particularly unstable, specifically those that undergo *alpha decay*. We've already discussed that an alpha is identical to a He-4 nucleus, two protons and two neutrons bound tightly together (two is one of the magic numbers for stability). First, we'll discuss what is seen from the outside of the nucleus, then see if we can explain it with quantum mechanics. We'll treat the nucleus of atomic number Z as an alpha particle and a ball of charge (Z-2)e. As noted above, the combined nuclear and electric forces give us a potential well U(r) in which the alpha particle is bound, but also a barrier through which it could tunnel. The alpha particle, being in a bound state within the nucleus, continually bounces against the barrier. We can say that the probability P of decay (or tunneling) of any one nucleus in a given time interval *d*t is proportional to the length of that interval, or

$$P = \eta \, dt.$$

If there are some number N of nuclei at the beginning of a particular time interval *d*t, then the number of nuclei that are expected to decay by the end of the interval, -*d*N, is

$$-dN = N\,P = N\,\eta\,dt.$$

Re-arranging this relationship leads to the familiar differential equation you encountered in Semester Two:

$$\frac{dN}{dt} = -\eta\,N.$$

Often, it is stated in textbooks that the rate of nuclear decay in a sample is proportional to the number of atoms that are present. Although this statement is true, it sounds a bit like magic and obfuscates the true reason for this behavior, developed from the equations above. The solution of this equation is well-known:

$$N(t) = N_o e^{-\eta t}\,.$$

The *decay parameter* η is often described in terms of a quantity called the *half-life*, $t_{1/2}$. The half-life is the amount of time after which only half of the original nuclei are expected to remain.

$$N(t_{1/2}) = \tfrac{1}{2}N_o = N_o e^{-\eta t_{1/2}}$$

$$\frac{1}{2} = e^{-\eta t_{1/2}}$$

$$t_{1/2} = \frac{\ln 2}{\eta} = \frac{0.693}{\eta}$$

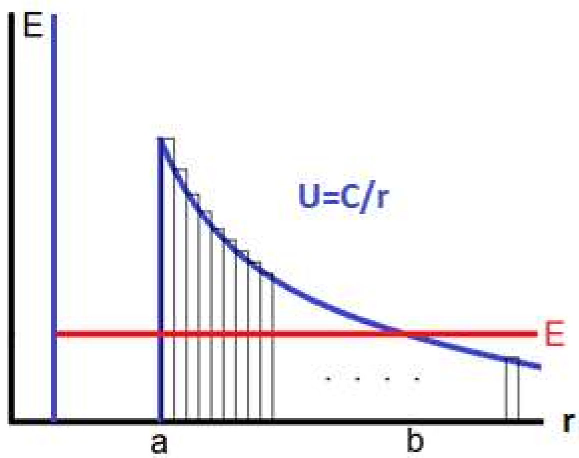
After an additional half-life, only half of the half, or one quarter, of the original number will remain un-decayed. After each subsequent half-life has passed, half of the remaining nuclei will have decayed.

The *activity* A of a sample is the rate at which nuclei decay:

$$A = -\frac{dN}{dt} = \eta N_o e^{-\eta t}.$$

Generally, this is the quantity that is actually measured.

We'll use the same technique we used in the last section on tunneling to verify that this is indeed the mechanism that explains alpha decay. We'll model the barrier as a series of thin rectangular barriers whose heights follow the Coulomb potential. For convenience, we'll let C represent the constants in the numerator of Coulomb's formula ($k_e(Z\text{-}2)2e^2$). Remembering that the transmission coefficient for all the barriers combined is the product of the individual coefficients, and that the product of exponentials is the exponential of the sum, we obtain[11]



$$T \approx e^{-\int \frac{\sqrt{8m(U(r)-E)}}{\hbar} dr} = e^{-\sqrt{\frac{8m}{\hbar^2}}\int \sqrt{\frac{C}{r} - E}\, dx} = e^{-\sqrt{\frac{8mC}{\hbar^2}}\int \sqrt{\frac{1}{r} - \frac{E}{C}}\, dr}.$$

Let's concentrate on the integral. The limits should be from a to b, but b is a function of E. If we set

[11] Gamow, G., 'Zur Quantentheorie des Atomskernes,' *Z. für Physik* **51**, pp 201-212.

$$U(b) = \frac{C}{b} = E \;,$$

we see that b = C/E, and

$$\int_a^{C/E} \sqrt{\frac{1}{r} - \frac{E}{C}}\, dr \;.$$

This integral isn't too bad, although a bit tedious. Let r = b $\cos^2\theta$; this may seem strange, but remember that r is always less than or equal to b. Correspondingly, $dr = -2b \sin\theta \cos\theta\, d\theta$. The new limits of integration are $\theta_a = \arccos((a/b)^{1/2})$ and $\theta_b = 0$.

$$-2\int_{\arccos\sqrt{\frac{a}{b}}}^{0} \sqrt{\left(\frac{1}{b\cos^2\theta} - \frac{1}{b}\right)}\; b \sin\theta \cos\theta\, d\theta$$

$$= -2\sqrt{b}\int \sqrt{\left(\frac{1}{\cos^2\theta} - 1\right)}\; \sin\theta \cos\theta\, d\theta$$

$$= -2\sqrt{b}\int \sqrt{\left(\frac{1 - \cos^2\theta}{\cos^2\theta}\right)}\; \sin\theta \cos\theta\, d\theta \;.$$

$$= -2\sqrt{b}\int_{\arccos\sqrt{\frac{a}{b}}}^{0} \sin^2\theta\, d\theta$$

$$= -2\sqrt{b}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\, \Big|_{\arccos\sqrt{\frac{a}{b}}}^{0}\right]$$

$$= +2\sqrt{b}\left[\frac{\arccos\sqrt{\frac{a}{b}}}{2} - \frac{\sin\left(2\arccos\sqrt{\frac{a}{b}}\right)}{4}\right]$$

At this point, we'll make use of a trig identity and two approximations, specifically:

$$\sin(2\arccos(x)) = 2\,x\sqrt{1 - x^2} \approx 2x \text{ and } \arccos(x) \approx \frac{\pi}{2} - x, \text{ if x is small.}$$

$$2\sqrt{b}\left[\frac{\left(\frac{\pi}{2} - \sqrt{\frac{a}{b}}\right)}{2} - \frac{2\sqrt{\frac{a}{b}}}{4}\right] = \sqrt{b}\,\left(\frac{\pi}{2} - 2\sqrt{\frac{a}{b}}\right)$$

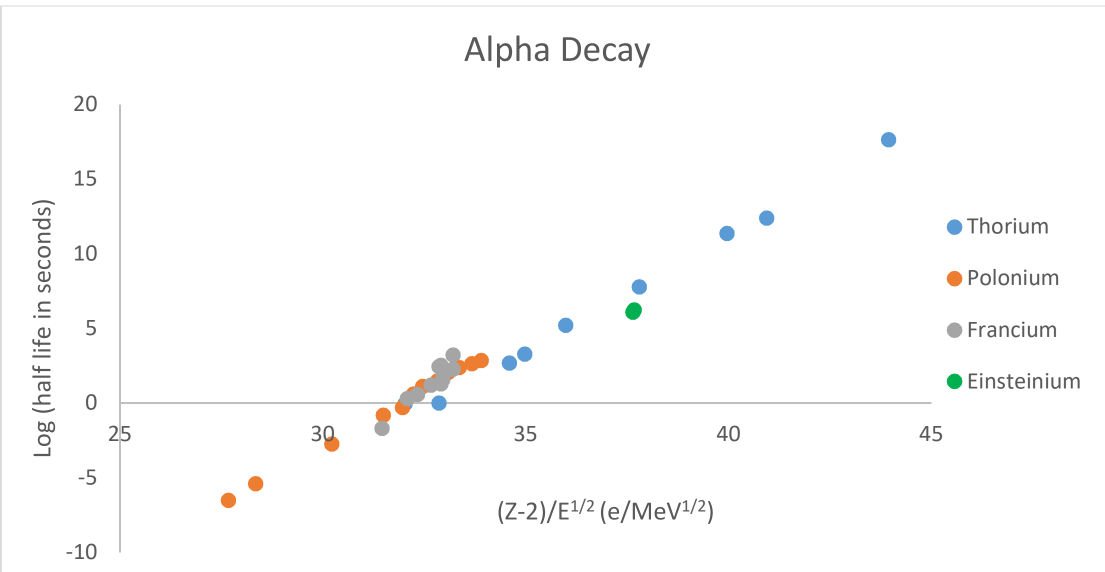
Since a<<b, we might even drop the very last term. Let's put what's left back into the expression we started with, with b = C/E:

$$T \approx exp\left(-\sqrt{\frac{8mCb}{\hbar^2}}\left(\frac{\pi}{2}\right)\right) = exp\left(-\sqrt{\frac{2\pi^2 mC^2}{\hbar^2 E}}\right) = exp\left(-\sqrt{\frac{2\pi^2 m}{\hbar^2}}\frac{(Z-2)}{\sqrt{E_{Alpha}}}\right)$$

Since we assumed that the half-life of the decay is roughly inversely related to the transmittance factor, we would expect

$$\log(t_{1/2}) \sim \frac{(Z-2)}{\sqrt{E_{Alpha}}} \ .$$

In the graph below, I've plotted a few alpha decay half-lifes over <u>many</u> orders of magnitude; it is quite clear from the linear shape of the relationship that our rough estimate agrees fairly well with reality.



This indicates two things. First, tunneling 'is a thing.' Second, alpha decay can be explained with wave mechanics.

Now, let's back up a bit. Why is it that an alpha particle can often scoot out of a nucleus, but not just a proton or neutron? As is usually the case, the solution is found by considering energy. As a specific example, consider U-232.[12] The chart lists the energies of a number of conceivable decay results relative to U-232, including possible beta decay processes.

Process	Energy (relative to U-232)
U-232 → Th-228 + α^{+2}	-5.42 MeV
U-232 → Th-232 + $2\beta^{+}$	+0.97 MeV
U-232 → Pa-232 + β^{+}	+1.25 MeV
U-232 → Np-232 + β^{-}	+2.54 MeV
U-232 → Pa-232 + $2\beta^{-}$	+3.64 MeV

[12] Enge, H, <u>Introduction to Nuclear Physics</u>, Addison Wesley, Reading (1966) pp275-276.

U-232 → Pa-231 + p^+	+6.13 MeV
U-232 → U-231 + n^0	+7.27 MeV
U-232 → Th-229 + He-3	+9.77 MeV
U-232 → Pa-229 + $3p^+$	+10.09 MeV
U-232 → Pa-230 + $2p^+$	+10.55 MeV

So, we see that only the release of an alpha particle put the system into a lower energy state. While these data correspond to a very specific example, we find that nuclei with masses above approximately 140 Daltons will undergo alpha decay.

Other Types of Decay

Other types of decay are possible, and sometimes energetically preferred, such as electron emission, or the emission of an anti-electron (both referred to as *beta decay*), *proton emission*, and *electron capture*, where an n = 1 orbital electron is captured by the nucleus. Most heavy nuclei decay through a combination of alpha and beta emission. Since the alpha particle has charge +2 and mass 4, there are really only four decay 'chains' on nuclei.

EXAMPLE 14-7

Consider carbon-14, which decays into nitrogen-14 with a half life of 5730 ± 30 years.[13] The fraction of all carbon of C-14 in the earth's atmosphere is 1 in 10^{12}. Since C-14 is chemically identical to C-12, one might presume that the fraction in living organisms is the same. However, once the organism dies, no more C-14 is taken in and the amount decreases due to the decay to nitrogen. Carbon-12, being stable, remains. If a corpse is found to have only 1 in 10^{13} of its carbon atoms to be C-14, when did the person die?

First, convert the half-life to the decay parameter eta:

$$\eta = \frac{0.693}{t_{1/2}} = \frac{0.693}{5730} = 1.209 \times 10^{-4}\ \text{yrs}^{-1}$$

$$N(t) = N_o e^{-\eta t}$$

$$\frac{N}{N_o} = e^{-\eta t}$$

$$-\eta t = \ln\frac{N}{N_o}$$

$$t = -\frac{1}{\eta}\ln\frac{N}{N_o} = -\frac{1}{1.209 \times 10^{-4}}\ln(0.1) = 19{,}039\ \text{years ago}\ .$$

[13] Ahem, this is actually a *beta decay* process, but I wanted to make use of C-14 because it's the one I suspect everyone has heard of.

HOMEWORK 14-4

Eons ago, a rock formed from molten rock. Any gases present at the time escaped the rock, but since then, K-40 has been decaying into Ar-40 with a half-life of 1.248×10^9 years. The argon gas is then trapped in voids within the rock. Assuming that all decayed K-40 turns into Ar-40, what is the ratio of K-40 to Ar-40 after 5 billion years?

HOMEWORK 14-5

An assumption made in the previous problem is in fact incorrect: only 11% of the K-40 atoms decay into Ar-40 while the rest decay into Ca-40. What is the actual ratio of K-40 to Ar-40 after 5 billion years?

Just FYI, the Ca-40 eventually decays to Ar-40 as well, but with a half-life of 10^{21} years, so effectively, never. When the half-lifes in a two step process are close to equal, the problem becomes quite a bit more difficult.

HOMEWORK 14-6

Naturally produced alpha particles have energies from about 2 MeV to about 10 MeV, characteristic of the nucleus from which each comes.[14] That is, for example, the alphas emitted by Ra-226 always have energy 4.871 MeV, while those emitted by At-218 always have energy 6.874 MeV.

Calculate the energy of an alpha particle emitted by Pb-210. The atomic mass of Pb-210 is 209.9841885 Da and the atomic mass of its *daughter*, Hg-206, is 205.977514 Da. Don't forget that some kinetic energy is given to the mercury daughter nucleus. The alpha is non-relativistic and has mass 4.001506 Da.

How big is the nucleus?

Let's return once again to the question of the size of the nucleus. First, some terms to know. We've discussed isotopes, two or more nuclei with the same number of protons but different numbers of neutrons. *Isotones* are two or more nuclei with the same number of neutrons but different numbers of protons. *Isobars* are two or more nuclei that have the same atomic mass numbers but different numbers of protons and neutrons. For example, Th-235 (90 p^+, 145 n^0), Pa-235 (91 p^+, 144 n^0), U-235 (92 p^+, 143 n^0), Np-235(93 p^+, 142 n^0), and Pu-235 (94 p^+, 141 n^0) are isobars. We're going to look at a particular subset of isobars known as *mirror nuclei*; two isobars are mirror nuclei if their numbers of protons and neutrons are reversed. Two sets of examples are tritium H-3 (1 p^+, 2n^0) and He-3 (2p^+, 1n^0), and Mg-26 (12 p^+, 14 n^0) and Si-26 (14 p^+, 12 n^0). We

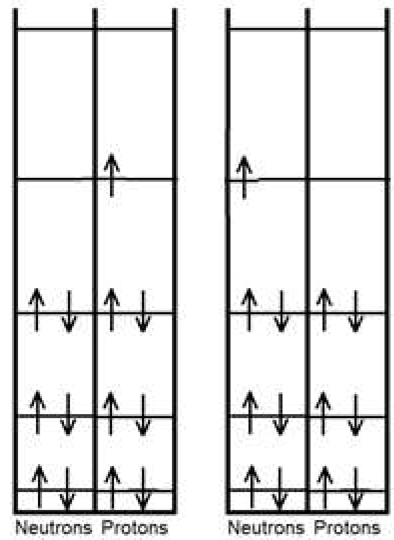
[14] This was an important constraint on the Geiger-Marsden experiment.

will be examining an even more special subset where the numbers of protons and neutrons differ by one.

Consider two nuclei, both of atomic number A, with $Z_1 = ½ A + ½$, $N_1 = ½ A – ½$, $Z_2 = ½ A – ½$, and $N_2 = ½ A + ½$. That is, if we magically change a proton in the first nucleus to a neutron, we obtain the second. First, our assumptions:

a) Any nucleus is spherical with its charge distributed uniformly throughout its volume.
b) The radius of a nucleus depends only on the number of nucleons, A.
c) The nuclear force is independent of the charge of the nucleons.

Consider the binding energy of each nucleus. Assumption (c) implies that, because each nucleus has the same number of, and presumably the same arrangement of, nucleons as the other, the first and second terms in the Von Weizsäcker expression will have the same value for both. The fourth term is also the same; we're just removing a proton and substituting a neutron with the same energy (see figure). Term five is also the same, in that both nuclei are odd-even ($\eta_{N,Z} = 0$). The only term that should be different between the two nuclei is the Coulomb term. Since we can write those terms separately,



$$E_B = U_{Nuclear} + U_{Coul}$$

and the nuclear energy terms are the same value for both isobars, we should be able to write that

$$U_{Nuclear} = E_{B1} - U_{Coulomb\,1} = E_{B2} - U_{Coulomb\,2}\,.$$

However, there is a slight twist: 'replacing' a proton with a neutron, positron, and neutrino releases some energy, specifically,

$$m_n c^2 - (m_p + m_{e+} + m_\nu)c^2\ ,$$

so that

$$E_{B1} - U_{Coulomb\,1} + (m_n - m_p - m_{e+} - m_\nu)c^2 = E_{B2} - U_{Coulomb\,2}$$

$$\Delta U_{Coul} = U_{Coul\,1} - U_{Coul\,2} = E_{B1} - E_{B2} + (m_n - m_p - m_{e+} - m_\nu)c^2$$

In Semester Two, we calculated that a sphere of radius R with a charge Q distributed uniformly through its volume has an electrical potential energy of $3k_eQ^2/5R$ (see Note Two). The difference in the electrostatic potential energies of the two nuclei will be

$$\Delta U = \frac{3k_eQ_1^2}{5R} - \frac{3k_eQ_2^2}{5R} = \frac{3k_ee^2Z_1^2}{5R} - \frac{3k_ee^2Z_2^2}{5R} = \frac{3k_ee^2}{5R}\,(Z_1^2 - Z_2^2)$$
$$= \frac{3k_ee^2}{5R}\left[\left(\frac{A}{2} + \frac{1}{2}\right)^2 - \left(\frac{A}{2} - \frac{1}{2}\right)^2\right] = \frac{3k_ee^2}{5R}A\ .$$

Then,

$$R = \frac{3k_e e^2 A}{5\left(\mathrm{E_{B1}} - \mathrm{E_{B2}} + \left(\mathrm{m_n} - \mathrm{m_p} - \mathrm{m_e} - \mathrm{m_\nu}\right)\mathrm{c}^2\right)} \; .$$

Filling in some values,

$$R = \frac{0.864}{\varepsilon_1 - \varepsilon_2 + 0.7867/A} \, ,$$

where ε_1 and ε_2 are the binding energies *per* nucleon in MeVs for the isobars and R is in femtometers.

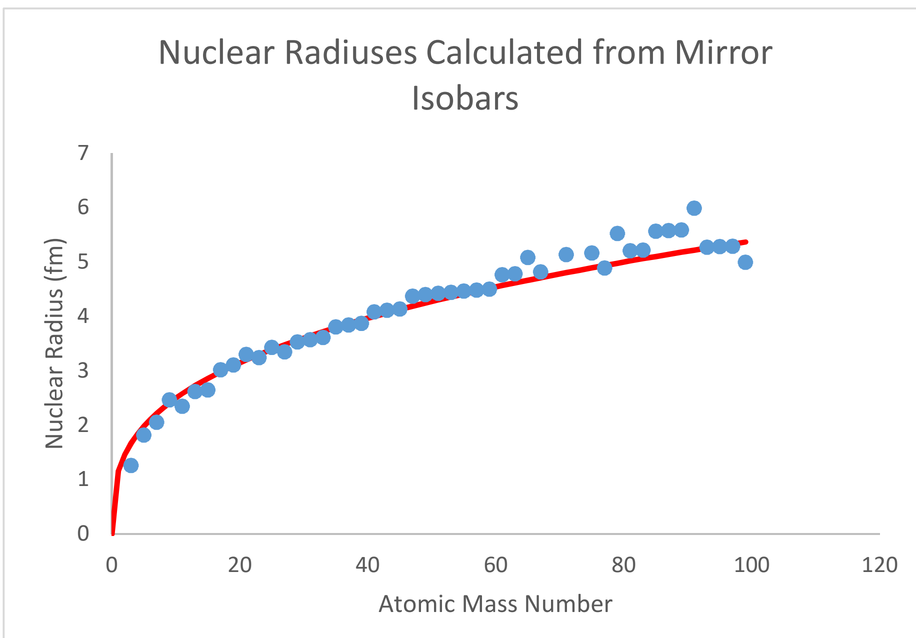
EXAMPLE 14-8

Find the radius of a nucleus of atomic number 39.

Making use of the method above, we'll examine K-39 and Ca-39. The binding energy *per* nucleon for the potassium is given as -8.56 MeV and -8.37 MeV for calcium.

$$R = \frac{0.864}{-8.37 - (-8.56) + 0.7867/39} = 4.11 \text{ fm}.$$

The graph shows the results of this calculation for many such mirror nuclei. The red line corresponds to A being proportional to R^3, or to the volume. The values suggest that the densities of all nuclei are about equal.



HOMEWORK 14-7

If the curve at right can be described by

$$R = 1.16\,A^{1/3}\,,$$

with the radius in femtometers and A the atomic mass number, find the approximate density of a nucleus in kg/m^3.

Note One

First, compare the momentum of the gamma-ray with that of the proton:

$$p_\gamma = \frac{E}{c} = \frac{2.62 \times 10^6 \times 1.6 \times 10^{-19}}{3 \times 10^8} = 1.4 \times 10^{-21}\ kg\frac{m}{s}.$$

$$p_p = \sqrt{2mK} = \sqrt{2 \times 1.67 \times 10^{-2}\ \ \times 0.25 \times 10^6 \times 1.6 \times 10^{-1}} = 1.2 \times 10^{-20}\ kg\frac{m}{s}.$$

Note Two

As a review, say that we wish to construct a sphere of charge Q and radius R with the charge distributed uniformly throughout the volume. We'll do so by bringing in small pieces of charge dq from r = infinity and put them in place. Consider an intermediate step in the process, when we have a sphere of charge q and radius r, such that

$$\frac{\mathrm{q}}{\frac{4\pi}{3}\mathrm{r}^3} = \frac{\mathrm{Q}}{\frac{4\pi}{3}\mathrm{R}^3} \ \rightarrow\ \mathrm{q} = \frac{\mathrm{Q}}{\mathrm{R}^3}\mathrm{r}^3\ .$$

The electric potential at the surface of this sphere will be

$$\mathrm{V} = \frac{\mathrm{k_e q}}{\mathrm{r}}\ .$$

If we bring in an additional charge *dq* from infinity and form a thin layer on the surface of our sphere, the potential energy of dq will be

$$d\mathrm{U} = \mathrm{V}\, d\mathrm{q} = \frac{\mathrm{k_e q}}{\mathrm{r}}\, d\mathrm{q} = \left(\frac{\mathrm{k_e}}{\mathrm{r}}\frac{\mathrm{Q}}{\mathrm{R}^3}\mathrm{r}^3\right)\left(\frac{\mathrm{Q}}{\mathrm{R}^3}3\mathrm{r}^2\, d\mathrm{r}\right) = \frac{3\mathrm{k_e Q}^2}{\mathrm{R}^6}\mathrm{r}^4 d\mathrm{r}\ .$$

To find the final energy of the completed sphere, we'll integrate dU from r = 0 to r = R:

$$\mathrm{U} = \int_0^{\mathrm{R}} \frac{3\mathrm{k_e Q}^2}{\mathrm{R}^6}\mathrm{r}^4 d\mathrm{r} = \frac{3\mathrm{k_e Q}^2}{5\mathrm{R}}\ .$$