Section 8 - The Action and Some Possibly Interesting Examples

Let's look at a period of transition between the classical picture we know and where we're going. The topics discussed in this section are often referred to as the 'old' quantum mechanics and are a blend of classical ideas with some of the new quantum notions.

The Wilson-Sommerfeld Quantization Rule

For systems that move through a cycle, the *action* is *quantized*. We haven't discussed the action, but it comes from classical mechanics; you'll probably see it next year in your intermediate physics course. Being quantized means that only certain values of a quantity are allowed. For now:

$$\oint \mathrm{p_x}\, d\mathrm{x} = \mathrm{nh}, \quad \text{n a positive integer,}$$

is an example.

EXAMPLE 8-1

Consider a particle of mass m in a one dimensional box from x = 0 to x = a.[1] The walls are very hard, so the particle (assumed to be non-relativistic) bounces back and forth elastically with energy, E, and momentum, $\vec{p}$. These remain constant (except for $\vec{p}$'s direction) because there is no potential energy for $0 \le x \le a$; the energy is then all kinetic (E = K). The momentum's magnitude can be written as

$$K = \frac{p_x^2}{2m} \quad \rightarrow \quad p_x = \sqrt{2mK} = \sqrt{2mE}\,.$$

The integral then becomes

$$\int_0^{\mathrm{a}} \sqrt{2\mathrm{mE}}\, d\mathrm{x} + \int_{\mathrm{a}}^0 -\sqrt{2\mathrm{mE}}\, d\mathrm{x} = \mathrm{nh}.$$

$$2\int_0^{\mathrm{a}} \sqrt{2\mathrm{mE}}\, d\mathrm{x} = 2\mathrm{a}\sqrt{2\mathrm{mE}} = \mathrm{nh}.$$

Solving for the energy, E,

$$\mathrm{E_n} = \mathrm{n}^2 \frac{\mathrm{h}^2}{8\mathrm{ma}^2}\,, \quad \mathrm{n} = 1, 2, 3, \ldots.$$

We see that only certain energies are allowed to the particle.

We can also see that these energies correspond to those allowed by requiring the De Broglie wave for this particle to form a standing wave with a node at each turning point. We already know that, for such a transverse wave on a string of length a,

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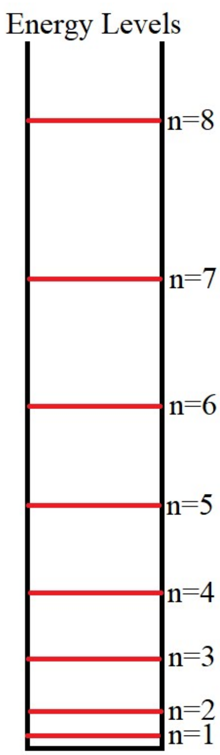
[1] Here, x = 0 and x = a are the *turning points* for the particle, *i.e.* where it reverses direction.



$$a = \frac{n\lambda}{2},$$

and

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{h}{\sqrt{2mE}},$$

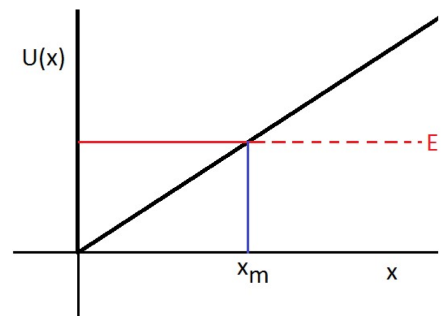
and, so,

$$a = \frac{nh}{2\sqrt{2mE}} \quad \rightarrow \quad E_n = n^2 \frac{h^2}{8ma^2} .$$

We'll do this problem several more times and see that this is indeed the correct relationship.

EXAMPLE 8-2

Let's keep the hard wall at x = 0, but tilt the floor upward for x > 0 so that the particle has potential energy U(x) = Fx (F > 0) in that region. This corresponds to a constant force, F, in the negative x-direction, perhaps the particle's weight. Now, we have that E = K + U, or



$$K = E - U = E - Fx \quad \rightarrow \quad p = \sqrt{2m(E - Fx)}.$$

Notice that, as expected, the momentum is not constant during the trip. The particle bounces off the hard wall at x = 0, but the maximum position x_m in the +x direction depends on its energy. The right-hand turning point is found by setting K = 0, E = F x_m, so x_m = E/F and our integral becomes

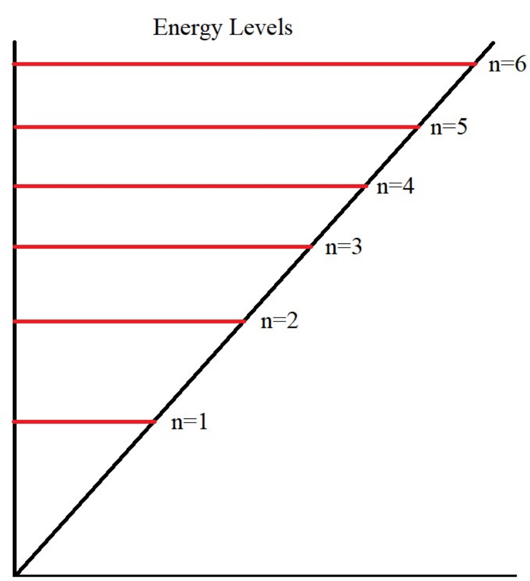
$$\int_0^{E/F} \sqrt{2m(E - Fx)}\, dx + \int_{E/F}^{0} -\sqrt{2m(E - Fx)}\, dx = nh$$

$$2\int_0^{E/F} \sqrt{2m(E - Fx)}\, dx = nh$$

By letting u = 2m(E – Fx) and du = -2mF dx,

$$-\frac{1}{mF}\int_{\sqrt{2mE}}^{0} u^{\frac{1}{2}}\, du = nh \quad ,$$

we obtain



$$\frac{2\,(2\mathrm{mE})^{3/2}}{3\mathrm{mF}} = \mathrm{nh} \quad \rightarrow \quad \boxed{E_n = n^{2/3}\left(\frac{9h^2F^2}{32m}\right)^{1/3}}\,.$$

We will revisit this problem later as well.

EXERCISE 8-1

Do the math to verify the result of Example 8-2.

EXAMPLE 8-3

This is one of the most important systems in physics: the simple harmonic oscillator. We know from Semester One that the potential function in one dimension is $U(x) = {}^1\!/_2 Cx^2$. Then,

$$K = E - U(x) = E - \tfrac{1}{2}Cx^2 \;\text{ and } p = \sqrt{2mK} = \sqrt{2m\left(E - \tfrac{1}{2}Cx^2\right)}\;.$$

The turning points for a particle of energy E (*i.e.*, when K = 0) will be

$$E = \frac{1}{2}Cx_m^2 \quad \rightarrow \quad x_m = \pm\sqrt{\frac{2E}{C}}\;.$$

The action integral is then

$$\int_{-\sqrt{\frac{2E}{C}}}^{+\sqrt{\frac{2E}{C}}} \sqrt{2m\left(E - \tfrac{1}{2}Cx^2\right)}\;dx + \int_{+\sqrt{\frac{2E}{C}}}^{-\sqrt{\frac{2E}{C}}} -\sqrt{2m\left(E - \tfrac{1}{2}Cx^2\right)}\;dx = nh$$

Re-arranging a bit,

$$4\int_0^{+\sqrt{\frac{2E}{C}}} \sqrt{2m\left(E - \tfrac{1}{2}Cx^2\right)}\;dx = nh\,.$$

Some factoring,

$$4\sqrt{2mE}\int_0^{+\sqrt{\frac{2E}{C}}} \sqrt{1 - \frac{Cx^2}{2E}}\;dx = nh\,,$$

and let $\sin\theta = \sqrt{\frac{C}{2E}}\,x$, $dx = \sqrt{\frac{2E}{C}}\cos\theta\; d\theta$ to get

$$8\sqrt{\frac{m}{C}}\,E\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\;\cos\theta\; dx = nh\,,$$

$$8\sqrt{\frac{m}{C}}\,E\int_0^{\frac{\pi}{2}} \cos^2\theta\; dx = nh\,.$$

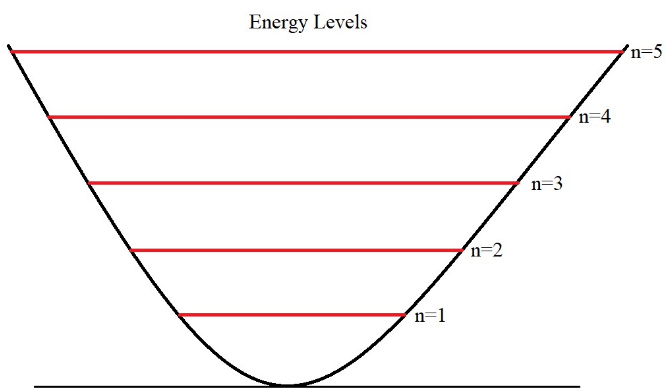
It is ‘well-known’ that the integral equals $\frac{\pi}{4}$ (think about r.m.s. values in Semester Two),[2] and so

$$8\sqrt{\frac{m}{C}}E\left(\frac{\pi}{4}\right) = nh\,,$$

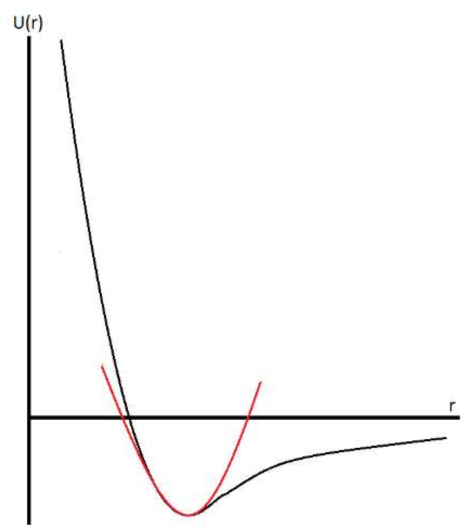
or finally,

$$E_n = \frac{nh}{2\pi}\sqrt{\frac{C}{m}} = n\,\hbar\,\omega_o\,,$$

where ω_o is the classical oscillation frequency of a SHO with mass m and ‘spring constant’ C. Note that the allowed energy levels are even spaced.



Note that this jibes exactly with Planck’s notion that the oscillators on the inside surface of the black body cavity could have energies that are integer multiples of the corresponding wave energies, and matches Einstein’s notion of being able to have any integer number of photons of energy h*f* in any given state.



Now, any potential energy curve with a stable equilibrium point can be to some degree be approximated by a parabolic function, *i.e.* treated as a SHO (see the figure). So, consider a diatomic molecule comprising two atoms of mass M connected with a spring-like bond of constant C. Let’s look at some data for N_2.[3]

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[2] The average of $\cos^2$ over a full 2π cycle is ½. This also is true for a quarter cycle. The area under such a curve is the average value times the interval, or ½ × $^{\pi}/_{2}$.

[3] Bayram, S. B., and M. V. Freamat, “Vibrational spectra of N_2: An advanced undergraduate laboratory in atomic and molecular spectroscopy,” *Am. J. Phys.* **80** 8 (2012) p664.

Transition	Emitted Wavelength (nm)	Photon energy = - ΔE (eV)	Difference between adjacent levels (eV)
$C^3\pi_u\ 0 \rightarrow B^3\pi_g\ 0$	337	3.6888	-
$C^3\pi_u\ 0 \rightarrow B^3\pi_g\ 1$	358	3.4724	0.2164
$C^3\pi_u\ 0 \rightarrow B^3\pi_g\ 2$	381	3.2628	0.2096
$C^3\pi_u\ 0 \rightarrow B^3\pi_g\ 3$	406	3.0619	0.2009
$C^3\pi_u\ 0 \rightarrow B^3\pi_g\ 4$	435	2.8578	0.2041

And so, we see that adjacent vibrational energy levels are indeed evenly spaced.

HOMEWORK 8-1

Consider a 'classical' mass on a spring system. If the mass is 2 kg and the spring constant is 60 N/m, what is the spacing between adjacent energy levels? If the amplitude of oscillation is 0.3m, what *per centage* higher is the next energy level?

While we're talking about diatomic molecules, it's also possible for such objects to rotate, in which case we would expect the angular momentum to follow the Wilson-Sommerfeld quantization rule. What quantity takes the place of linear momentum in rotation? What takes the place of position in rotation?

EXAMPLE 8-4

$$\oint L\, d\theta = nh\,.$$

In the absence of external torques, L will be constant, and

$$L \oint d\theta = L\, 2\pi = nh \quad \rightarrow \quad L = \frac{nh}{2\pi}\,.$$

Be aware, we're talking about rotation about <u>one</u> axis. The angular momentum about an additional, perpendicular axis would also be quantized, in a more complicated way. The energy of such an object is then

$$E_n = \frac{1}{2} I\, \omega^2 = \frac{L^2}{2I} = n^2 \frac{h^2}{8\pi^2 I}\,.$$

In a later Section, we will return to some of these examples and treat them more carefully.

HOMEWORK 8-2

Consider the moon orbiting the earth. Let's assume that the earth exerts no torque on the moon (not quite true!). Find the value for n for the moon's orbit.

With the results of this problem, we can see that the reason we don't observe any quantum effects in everyday life is that the gaps between permitted energy levels are so small, any transition appears to be continuous, or classical in nature. For example, a quick calculation shows that n for a turning LP with an energy of 0.02 joules is about 5×10^{31}. The next higher allowed energy level would then be about 4×10^{-32} Joules more.

Summary

We've used the 'old' quantum mechanics, based on the Wilson-Sommerfeld quantization rule for cyclic processes, to take a first look at several important system. In a later section, we'll use the Schrödinger picture to examine several of these systems more carefully.