Section 1 - A Peek Outside of the Newtonian Box

"Philosophers play with the word, like a child with a doll. It does not mean that everything in life is relative."

— Albert Einstein[1]

Introduction

When we started this discussion in Semester One, a point was made that physics is always an approximation to reality, and we specifically restricted ourselves to the Newtonian regime of objects not too large or small and moving fairly slowly with respect to the speed of light. In this last section, we will take a peek outside of the 'Newtonian box' and look at some of the implications of *special relativity*. Historically, special relativity was developed to answer some nagging questions regarding electro-magnetism, which you may discuss in a more advanced E&M course. Special relativity concerns 'ordinary sized objects' moving at high speeds.[2] The relationships we are about to derive match experimental observations much better than our Newtonian approximations do, but remember that these relativistic relationships should always agree with Newtonian physics when velocities tend toward zero. This last notion is called the *correspondence principle*, and we will require it to hold for quantum mechanics, as well.

Now, much of the subsequent discussion requires a passing acquaintance with the properties of *light*. For the purposes of this discussion, light can be thought of as an *electro-magnetic wave*, in some ways similar to sound. The speed of light in vacuum was measured quite accurately by the 1840s and found to be approximately 3×10^8 m/s, a velocity now referred to with the symbol, c. In the 1860s, a synthesis of the laws of electro-magnetism resulted in the prediction of the existence of *electro-magnetic waves* with a speed in vacuum matching that of the known speed of light, and the 1880s experiments with artificially generated EM waves showed that they have the same properties of light. Eventually, we jump to the conclusion that light is an electro-magnetic wave. As we know, mechanical waves must travel through a medium, but light can travel through vacuum (as from the distant stars to the earth, or through an evacuated bell jar), so clearly no material medium is <u>necessary</u> for the propagation of light waves. Apparently, these notions are difficult to dispel, so an *æther* was proposed as the medium in which light waves propagate. We won't discuss the experiment that indicated the non-existence of the æther; we'll simply assume it doesn't exist, since no such super-natural material is required by the equations that predict EM waves.

We will make two important assumptions, though, often called the *postulates of special relativity*: the speed of light is the same for all *inertial frames of reference* (one in which the 'observer' is not accelerating), and the laws of physics are the same for all such frames. The second of these

---

[1] Like all quotes on the internet, subject to verification.

[2] We consider objects roughly between the size of the atom to that of a star. The boundaries are a bit fuzzy.

postulates we have been assuming to be implicitly true throughout these notes. However, note that the second postulate says the rules are the same, but not that the results of measurements need be the same for different observers.

Relative motion

Let's start by considering two 'observers,' A and B, each traveling along the x-axis with constant velocity. Since neither experiences any acceleration, each is in an inertial frame and thinks that he is at rest, while the other is moving with speed v. Making use of the notation of Section 3 from semester one, we can write that

$$\vec{v}_{B,A} = -\vec{v}_{A,B} \quad .$$

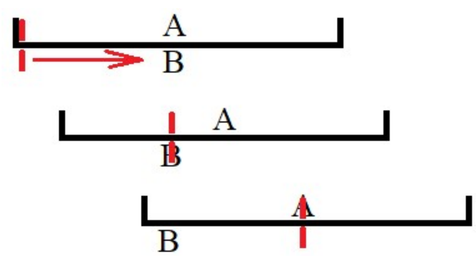
Simultaneity

Before we discuss what does happen to moving objects, let's discuss a few things that don't happen. Let's suppose we have two events that occur at the same time at the same place. An example might be two lamps placed next to one another that flash briefly. Since the light pulse from each lamp travels right next to the pulse from the other lamp, the two pulses will be seen at the same time by any observer regardless of his motion. Other observers will agree that the two flashes happened at the same instant, although they may not all agree on when that instant was.

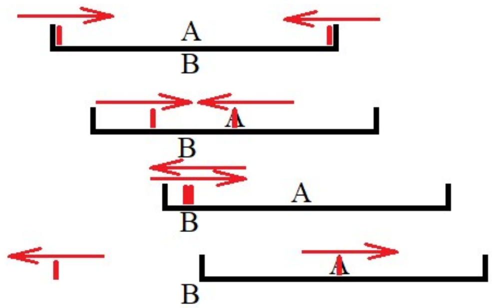
Now, consider what happens if the two flashes did not occur at the same place at the same time.

EXAMPLE 1-1

A standard example is to consider two observers, A and B. Let's say A is standing at the center of a moving flatcar of length L which is moving to the right at some speed, v, while B is standing track-side. Suppose that two lightning bolts hit the left end of the car at the same time. The diagram indicates that, as the light moves to the right from the location of the strikes, first B, then A, will see both flashes together. They will agree that the two strikes occurred simultaneously, although they may not agree on when.

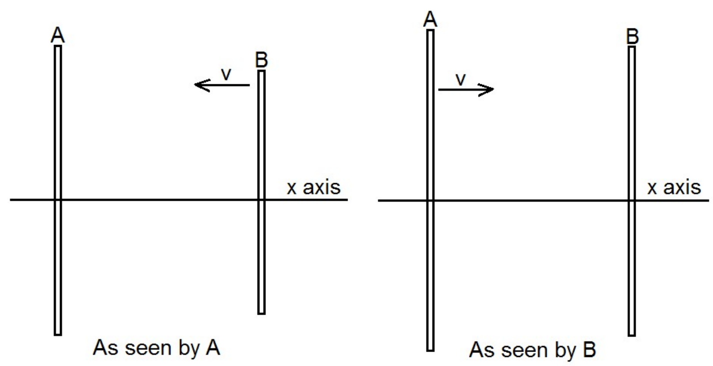


On the other hand, suppose that there is a lightning strike at each end of the car, at what we will for the moment call the same moment. A and B will disagree on which strike occurred first. B will of course say that the strikes occurred at the same time, because light from each must travel the same distance (in B's case, L/2). However, A is moving so as to meet the light from the right end strike and running away from the light from the left end strike; A will say that the right hand strike occurred first.



Length Contraction I

It is traditional to make use of a meter stick as a convenient object to be observed. Let's suppose that A considers himself to be stationary, and he holds his meter stick perpendicular to the constant motion of B. B considers himself to be stationary and holds his meter stick perpendicular to the constant motion he sees A performing.



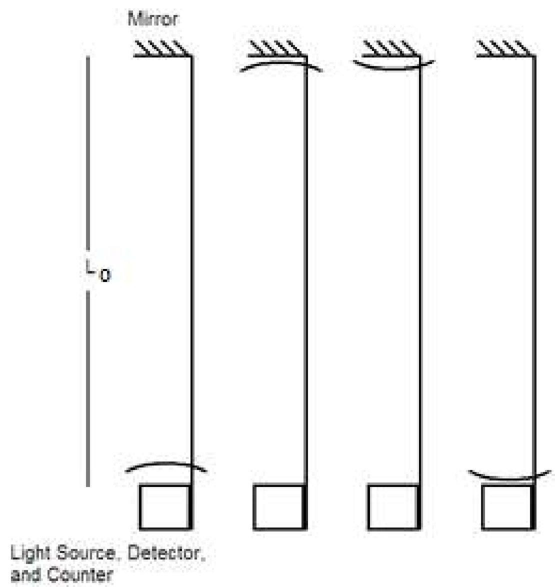
It's indisputably true that one of three things will happen as the sticks pass each other, A will measure the length of B's stick to be either shorter than a meter, equal to a meter, or longer than a meter. The same will be true of B measuring A's stick. Now, let's make use of a concrete example.[3] Suppose that, as the sticks pass one another, A sees the 100 cm end of B's stick to be located at his 95 cm mark, and the zero end of B's stick to be located at his 5 cm mark. That is, A thinks that B's stick has shrunk in length. But, B must agree with A that the 100 cm mark of B's stick passed the 95 cm mark of A's stick, since the two marks were in the same location at the same time. The same is true of course for the other end; B must agree that his 0 cm end passed A's 5 cm mark. However, since B correctly maintains that his stick is 100 cm long and at rest, he would conclude that A's stick lengthened. This violates the second postulate, in that A says such a moving stick shortens and B says such a stick lengthens; the laws of physics would need to be different for A and for B. We conclude then that sticks in this situation remain one meter long for all observers.[4]

Time Dilation

[3] Mermin, N. David, Space and Time in Special Relativity, McGraw-Hill, New York (1968) pp27-32.
[4] Of course, this assumes that our postulates are valid. As we make more predictions based on the postulates that turn out to be correct, we gain confidence that they are indeed valid.

DERIVATION 1-1

Consider the classic *light-clock* scenario. A rod of length L_o has a pulsed light source at one end and a mirror at the other. Periodically, a very short light pulse is emitted by the left end, reflected at the right end, and detected at the left end. Detection of the returned pulse trips a counter and the process is repeated.



For now, let's let the clock be stationary with respect to Observer A. What is the time interval Δt_A between the ticks of the clock? The pulse is emitted and travels distance L_o at speed c to the mirror, is reflected, and travels distance L_o again to reach the detector. So,

$$\Delta t_A = \frac{L_o}{c} + \frac{L_o}{c} = \frac{2L_o}{c}.$$

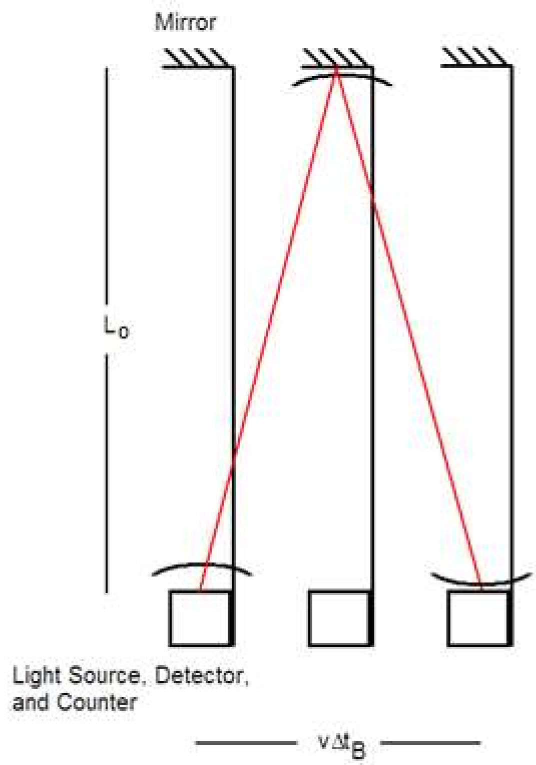
Now, according to Observer B, who thinks himself to be stationary, the clock is moving with speed v (specifically for this derivation, the length is perpendicular to the direction of motion). The pulse is emitted, reflected, detected, and counted, as above. However, the distance traveled by the pulse is no longer $2L_o$. If Δt_B is the time interval for the clock to tick once as seen by B, then the clock has moved a distance v Δt_B in that interval. Assuming *per* Postulate One that the speed of light is the same for both A and B, the time to make the trip shown in the figure will be

$$\Delta t_B = 2\frac{\sqrt{L_o^2 + \left(v\frac{\Delta t_B}{2}\right)^2}}{c}.$$

Solving this for the time interval results in

$$\Delta t_B = \frac{2L_o}{c}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{\Delta t_A}{\sqrt{1-\left(\frac{v}{c}\right)^2}} > \Delta t_A$$

Note that this interval is longer than the interval as measured by Observer A! Observer B would say that the clock is running slowly when compared with his own, identical clock.



So, if an object undergoes some process whose duration as measured by an observer at rest relative to the object is Δt_o (often referred to as the *proper time*[5]), the duration Δt as measured by another observer who sees the object moving at velocity v relative to himself will be[6]

$$\Delta t = \gamma \, \Delta t_o .$$

The term $^v/_c$ is often replaced with the symbol *beta* (β) and $(1 - (^v/_c)^2)^{-1/2}$ is often replaced with the symbol *gamma* (γ).

Since this argument will work for either observer being stationary while the other is moving, we obtain the mind-boggling result that <u>each</u> thinks his own clock keeps good time and that the other's clock runs slowly.

Now, this may convince you that these special light clocks run slowly, but what about wrist watches or chemical reactions or biological processes. Well, since we can correlate the 'ticking' of the light clock to the 'ticking' of any other type of clock, these other processes will run slowly as well. As B whizzes past a 'stationary' A, A will observe B to age more slowly, his chemical reactions to take longer, and so on.

EXERCISE 1-1

Do the algebra to solve for Δt_B above.

HOMEWORK 1-1

The HMCSS Clark passes by Planet X-372 at 0.99c. As they pass, Buzz Cutter, marooned on the planet, sets off a rescue beacon that lasts for 0.1 seconds. How long does the crew of the Clark measure the beacon pulse to be?

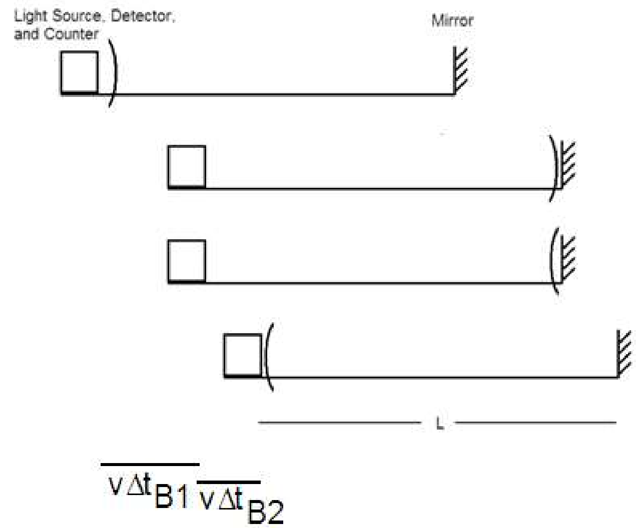
Length Contraction II

[5] 'Proper' in the sense of the French '*ses propres mains'* as opposed to *'ses mains propres.'*

[6] Be careful. Other texts may invert the sense of the time intervals described here, resulting in a reversed formula.

Well, we've asserted that the speed of light for each of our observers is the same. It stands to reason that if time intervals can be measured to be different, then distances may also be different. Again, we'll make use of the *light clock*, this time oriented parallel to the velocity of Observer B.

DERIVATION 1-2



Nothing has changed for Observer A; he is still at rest with respect to his clock, and once again we know that

$$\Delta t_A = \frac{L_o}{c} + \frac{L_o}{c} = \frac{2L_o}{c},$$

where L_o is the *proper length* of the rod, or the length when measured by an observer at rest relative to the clock. Now, let's at least consider the possibility that the length as seen by B (= L) is <u>not</u> the same as seen by A (= L_o). Let Δt_{B1} be the time necessary for the light to leave the source and strike the mirror; since the mirror moved distance v Δt_{B1} during this process, the total distance traveled is L + $v\Delta t_{B1}$ and the time required is

$$\Delta t_{B1} = \frac{L + v\Delta t_{B1}}{c} \quad \rightarrow \quad \Delta t_{B1} = \frac{L}{c-v}.$$

On the return trip, the distance is shorter since the detector moves to meet the returning light pulse:

$$\Delta t_{B2} = \frac{L - v\Delta t_{B2}}{c} \quad \rightarrow \quad \Delta t_{B2} = \frac{L}{c+v}.$$

The total time is then

$$\Delta t_B = \Delta t_{B1} + \Delta t_{B2} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2cL}{c^2 - v^2}.$$

We know from the time dilation discussion above that this time interval equals γ Δt_A, so

$$\Delta t_B = \frac{2cL}{c^2 - v^2} = \gamma \Delta t_A = \gamma \frac{2L_o}{c}$$

$$\frac{L}{1 - \left(\frac{v}{c}\right)^2} = \gamma L_o$$

$$\gamma^2 L = \gamma L_o$$

$$L = \gamma^{-1} L_o \quad .$$

So, Observer B sees A's rod shortened from its proper length L_o by factor γ. However, if we just exchange the frames of reference in the argument above, we see that A will see B's rod shortened by the same factor! Again, rather mind-boggling.

Before we look at an example, we should check that these results are consistent with what we have done in the previous sections of these notes. In both cases, if the velocities are very small compared to the speed of light, each of these results simplifies back to what we expected in the Newtonian world; the length of an object appears the same in any frame as do the time intervals.

EXAMPLE 1-2

Suppose B contends that he is holding a stick of length L_o at an angle of 45° from the x axis. As B moves past A along the x-axis at 0.3c, what will A say is the length of the stick and the angle from the x-axis?

From B's viewpoint, the projection of the stick along the y-axis will be $L_o/\sqrt{2} = 0.707L_o$, and the same for the x-axis. From the discussion above, we know that A will see the perpendicular projection of the stick to be the same as what B sees ($L_{yA} = 0.707L_o$), but the projection along the x-axis will be contracted by factor $\gamma^{-1} = 0.954$, or $L_{xA} = 0.954\, L_o/\sqrt{2} = 0.675L_o$. The length of the stick can be found with the Pythagorean theorem:

$$L = \sqrt{L_{yA}^2 + L_{xA}^2} = \sqrt{(0.707L_o)^2 + (0.675L_o)^2} = 0.977\ L_o.$$

The angle observed as the stick passes by can be determined with the arc tangent:

$$\theta = \arctan\left(\frac{L_{yA}}{L_{xA}}\right) = \arctan\left(\frac{0.707L_o}{0.675L_o}\right) = 46.3^{\text{o}}\ .$$

EXAMPLE 1-3

Here's an experiment that tests both time dilation and length contraction, depending on how it's solved.[7] *Muons* are sub-atomic particles somewhat similar to *electrons*; however, unlike electrons, they *decay* or break down into several other particles fairly soon after their creation. Sophisticated statistics allow us to define a *decay lifetime* of 2.2 microseconds when the particle is at rest in a laboratory. To simplify the question, we'll just assume that every muon lasts exactly 2.2 μsec, even though many last less time and many last longer. Muons are created high in the earth's atmosphere (~ 10 km above the earth's surface) by cosmic rays and subsequently travel at a speed close to c (we'll assume 0.999c).

---

[7] *Time Dilation: an Experiment with mu Mesons*. Cambridge: Educational Services, Inc., 1962. Currently available at https://www.youtube.com/watch?v=3CeQXsIiGp8.

a) How far could a muon travel from its creation point towards the surface of the earth before decaying?

If the muon traveled at 0.999c, it could cover $d = vt_o = 0.999\times(3\times10^8)(2.2\times10^{-6}) = 659$ m.

Well, 660 m is a lot less than 10,000 m. We would expect the number of muons arriving at the earth's surface to be zero. In reality, quite a few of the muons survive the trip to the surface. However, remember that the internal clock that determines when a muon decays is running slowly as seen from an observer on earth.

b) What is the decay time as seen by such an observer?

$t = t_o[1-v^2/c^2]^{-1/2} = 2.2[1-(0.999c)^2/c^2]^{-1/2} = 49.2\ \mu s$ in the frame of an observer on earth.

c) How far would these muons travel at 0.999c in 49.2 μsec?

$d' = vt = 0.999\times(3\times10^8)(49.2\times10^{-6}) = 14{,}700$ m

This is certainly greater than 10,000 m, so an observer on earth would expect to see a fair number of muons arrive at the surface. The problem now is that the earth-bound observer and one riding along with the muons must agree on how many muons survive the trip (all of them *vs* none of then, in our simplification). Luckily, length contraction removes the paradox. From the muon's point of view, it's stationary with a lifetime of 2.2 μsec, and the earth's surface is rushing up towards it at 0.999c.

d) How much time does it take for the earth's surface to arrive at the creation point of the muons??

$L = L_o[1 - v^2/c^2]^{1/2} = 10^4[1 - 0.9992]^{1/2} = 447$ m
$t = L/v = 447/(0.999\times3\times10^8) = 1.49\times10^{-6}$ sec = 1.49 μsec.

This is considerably less than 2.2 μsec, so quite a few muons will still exist when the surface arrives.

HOMEWORK 1-2

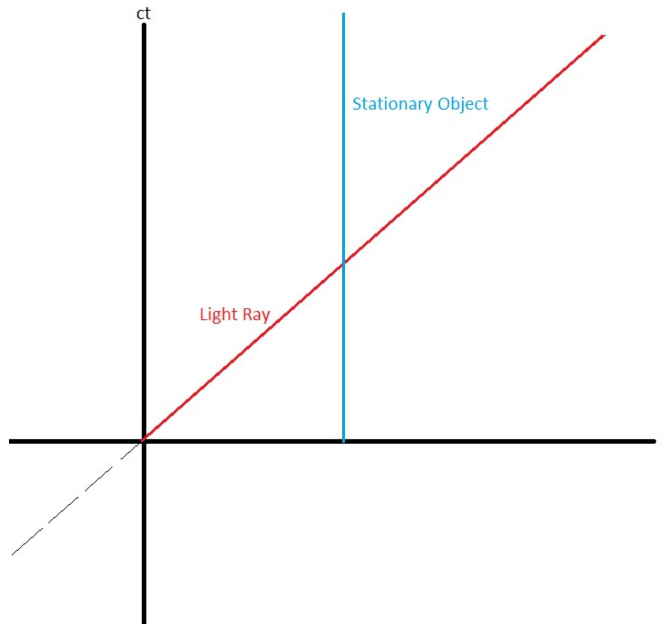
The proper half-life of the π+ particle is 2×10^{-8} seconds. How quickly would such a particle need to move to be able to traverse a distance of 25 meters in a laboratory?

HOMEWORK 1-3

Astrid is a colonist on an interstellar spaceship. If she left earth on her twentieth birthday and traveled the 12 light years to Tau Ceti at 0.99c, what age will she be when she arrives? Assume that her ship accelerates and decelerates quickly enough to ignore that effect.

Non-Simultaneity and the Order of Events

If both time intervals and distances depend on the motion of the observed and the observer, we might imagine that two observers may well disagree on when and where an event takes place. We can make use of a *Minkowsky diagram* to keep track of the times and places events occur. These can get very sophisticated, so we will use them in a few examples to determine only the order in which events occur.



Typically, for a one dimensional problem, the location x of an object is plotted along the abscissa, and the time (actually ct, so both axes are in units of distance) is plotted along the ordinate. Consequently, the graph of any light ray will be a line at 45° to the axes, since $x = x_o \pm ct$. The graph above depicts the curves for a light ray and a stationary object. Note that events that occur simultaneously in this frame of reference will lie along a horizontal line parallel to the x axis. Events that occur in the same position will lie along a line parallel to the ct axis.

If we want to see this from a different inertial frame moving at speed v with respect to ourselves, we tilt the axes by an angle theta equal to the arctangent of (v/c). Why? Consider an observer B that passes our origin at speed v when our time = 0. In our frame, his position would be

$x = x_i + vt = 0 + (v/c)\,ct,$

$ct = (c/v)\,x + 0.$

So, the slope of his path as seen by us on this graph would be $c/v = \tan\varphi$, and since θ and φ are complementary, $\tan\theta = v/c$. This red line is of course the x' = 0 line, or more importantly, the ct' axis, for the observer B in the moving frame, since he thinks himself to be stationary. Then, since the line representing the motion of light must bisect the two axes for B as it does for us, the x' axis will be tilted by angle theta as well.

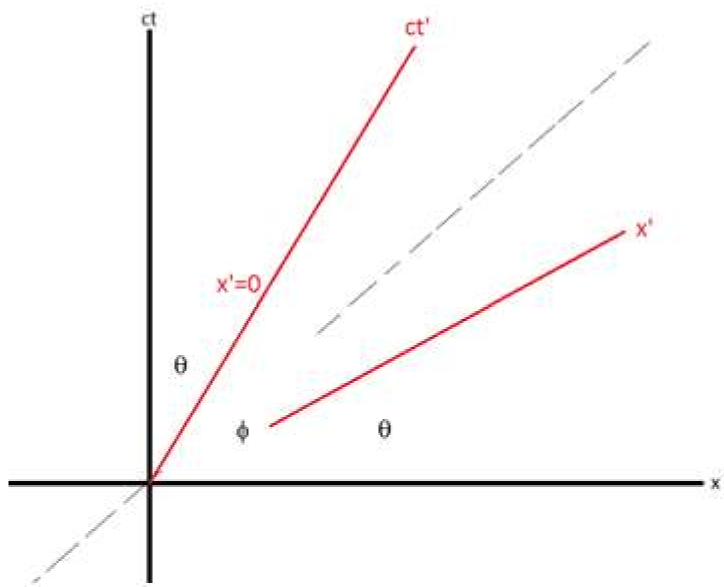


Figure V-1-6

To sum up, if A is the observer in the unprimed frame and B in the primed frame:

1) Any object stationary for A will trace out a line parallel to the ct axis.
2) Any object stationary for B will trace out a line parallel to the ct' axis.
3) Any events that occur simultaneously for A will lie along a line parallel to the x axis.
4) Any events that occur simultaneously for B will lie along a line parallel to the x' axis.

EXAMPLE 1-4

Let's make use of a Minkowsky diagram to help explain length contraction. How is it possible that each observer will see the other's meter stick as shorter than his own? Surely, at some point, the sticks lay next to each other and can be compared? The key is to realize that events happen in different orders for each of the observers. Let's contend that A is at rest and holds his meter stick parallel to the velocity of B, who is moving at 0.44c as seen by A and who holds his own stick parallel to his motion. B, of course, believes himself to be stationary. The length contraction relationship developed above says that A will measure B's stick to be 0.9 meters long, and that B will measure A's stick to be 0.9 m long.

The figure on the next page is drawn to scale. The heavy black lines are A's x and ct axes and the light, dotted, black line is the *light line*. The vertical red lines are the two ends of A's meter stick, each at a constant value of position x. Events that occur simultaneously for A will lie on a line parallel to the x-axis.

The blue lines represent the two ends of B's meter stick as seen by A. Notice that the length of B's stick has been adjusted to 0.9 times the length of A's stick. As time progresses, B's stick moves to the right as seen by A.

The events (black dots) labeled (1), (2), (3), and (4) are:

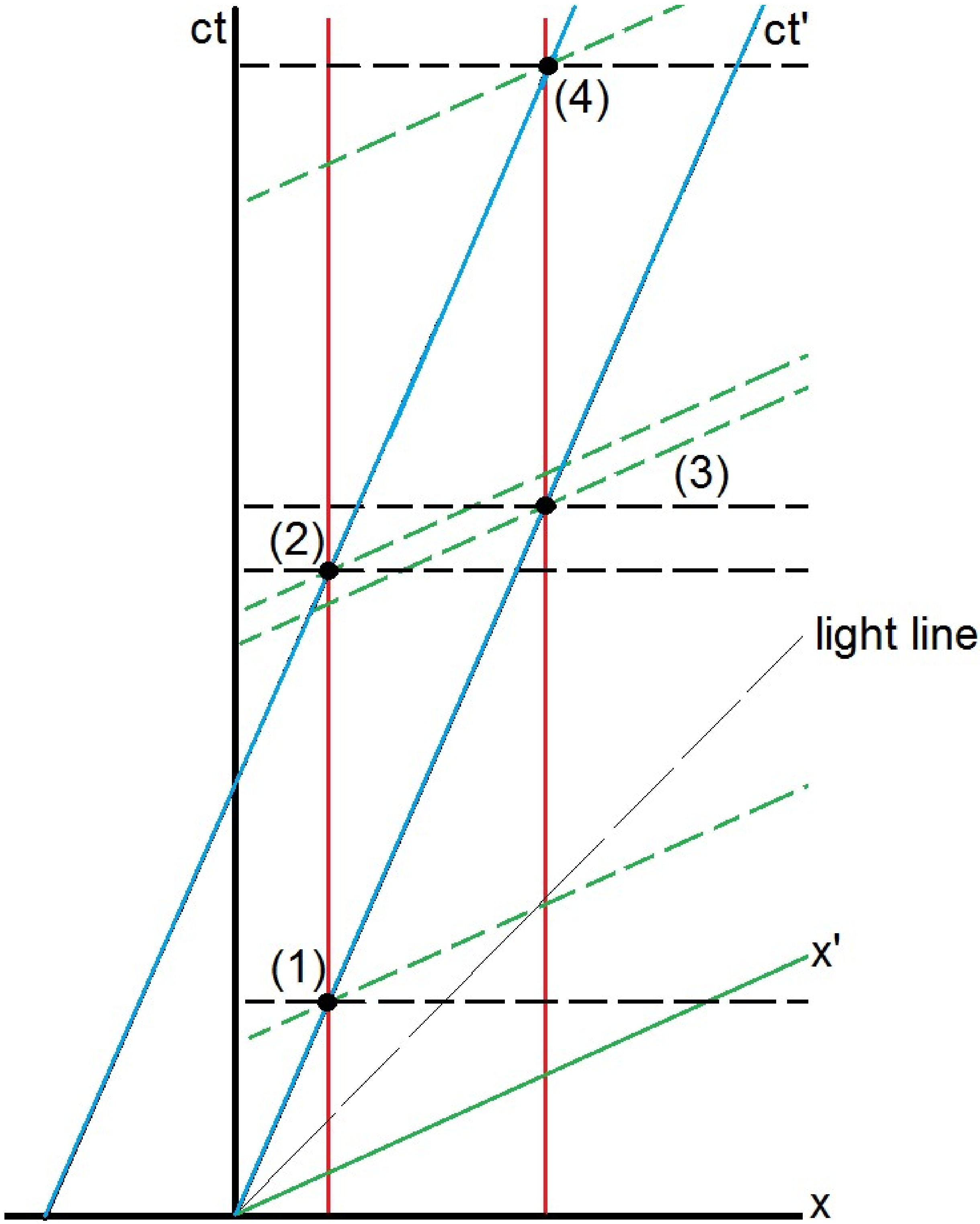
(1) The right end of B's stick passes the left end of A's stick.
(2) The left end of B's stick passes the left end of A's stick.
(3) The right end of B's stick passes the right end of A's stick.
(4) The left end of B's stick passes the right end of A's stick.

We know that this order of events is correct for A by looking at the dotted black lines from the bottom to the top (increasing time). The order of events (2) then (3) is consistent with B's stick being shorter than A's as seen by A.

However, if we examine the sequence of events as seen by B, the order is different. Remember that simultaneous events as seen by B will lie along a line parallel to the x' axis; these are the green dotted lines.

(1) The left end of A's stick passes the right end of B's stick.
(3) The right end of A's stick passes the right end of B's stick.
(2) The left end of A's stick passes the left end of B's stick.
(4) The right end of A's stick passes the left end of B's stick.

Here, the sequence of events (3) <u>then</u> (2) means that B will see A's stick as being shorter. Once the two right ends align, some additional time must pass before the left ends of the two sticks align, and in that time, A's stick's right end will have moved to the left.


ct
ct'
(4)
(3)
(2)
light line
(1)
x'
x

HOMEWORK 1-5

Here's a well-known problem. Suppose we have a shed that is 4 meters long with a door at each end. We'd like to put a 5 meter long ladder in the shed and have both doors shut. How? Well, if we were to have Eb run the ladder through the shed at a high velocity, length contraction would shorted the ladder enough, from our frame of reference, so that its length would be less than 4m. We see:

1) Door One opens and the ladder enters the shed.
2) Once the ladder is completely in the shed, Door One closes. Both doors are now shut.
3) Door Two opens and the ladder exits.
4) Once the ladder completely exits the shed, Door Two closes.

All well and good, except that, from the Eb's reference frame, the ladder is still 5m long and the shed's length has instead contracted by a factor of 0.6, and so appears to be 2.4 meters long! Explain qualitatively how this is actually consistent with what we saw.

Doppler Effect

DERIVATION 1-3

We derived the Doppler effect for sound in semester one:

$$f = \frac{\mathrm{v_{sound}} \pm \mathrm{v_{observer}}}{\mathrm{v_{sound}} \mp \mathrm{v_{source}}}\, f_o$$

where f_o is the frequency emitted by a source in its proper frame, f is the frequency heard by the 'observer,' the upper signs are used for 'approaching' objects and the lower for 'receding' ones.[8] You may remember an example in which we tried to use relative velocities to simplify the case of both a moving source and observer, but correspondingly had to change the speed of sound to include a 'wind' term. Since there is no analog to the air for light (no 'æther wind'), this correction is unnecessary and we can use the simplified form of assuming that the observer is at rest, the relative velocity v between the objects becomes $\mathrm{v_{source}}$, and the speed of the wave becomes c:

$$f = \frac{\mathrm{c}}{\mathrm{c} \mp \mathrm{v}_{source}}\, f_o.$$

However, remember that whatever clock that determines the frequency f_o emitted by the source runs slowly as seen by the observer due to the time dilation effect, so that the frequency is reduced by a factor $(1 - \mathrm{v}^2/\mathrm{c}^2)^{1/2}$, so that

[8] These terms were previously defined very specifically in Section 11.

$$f = \frac{c}{c \mp v}\left(\left(1 - \frac{v^2}{c^2}\right)^{1/2} f_o\right) = \frac{1}{1 \mp \frac{v}{c}}\left(\left(1 - \frac{v}{c}\right)^{1/2}\left(1 + \frac{v}{c}\right)^{1/2} f_o\right)$$

$$f = \frac{\left(1 \pm \frac{v}{c}\right)^{1/2}}{\left(1 \mp \frac{v}{c}\right)^{1/2}} f_o = \left(\frac{1 \pm \beta}{1 \mp \beta}\right)^{1/2} f_o,$$

where, once again, the upper signs are used for 'approaching' objects and the lower for 'receding' ones.

EXAMPLE 1-6

Distant galaxies often emit frequencies of light that are clearly identifiable as due to a specific element, such as for example calcium. In the laboratory, the wavelength of one such emission is 393 nanometers. However, in light from a distant galaxy, the observed wavelength is 572 nm.

What is the radial speed of this galaxy relative to the earth?

First, we assume that the light emitted by the galaxy has a wavelength λ_o of 393nm in its proper frame of reference.[9] Secondly, we might safely assume from the information given that the galaxy is receding from the earth. Recalling that $f\lambda = v = c$ for light waves, the relationship can be rewritten as

$$\frac{c}{\lambda} = \left(\frac{1 - \beta}{1 + \beta}\right)^{1/2} \frac{c}{\lambda_o}.$$

Squaring and cross multiplying results in

$$\lambda_o^2(1 + \beta) = \lambda^2(1 - \beta).$$

Re-arranging to solve for beta,

$$\beta = \frac{\lambda^2 - \lambda_o^2}{\lambda^2 + \lambda_o^2} = \frac{572^2 - 393^2}{572^2 + 393^2} = 0.36.$$

So, the galaxy is traveling at 36% of the speed of light, or 1.08×10^8 m/s, relative to the earth.

Relative Velocities

[9] We assume that each calcium atom follows the same laws of physics as does every other calcium atom in the universe (Postulate One) so that the emission wavelength is the same in each atom's proper frame of reference.

Let's consider three observers, A, B, and C, each moving along the x-axis.[10] From Semester One, we would possibly have written

$$\vec{v}_{C,A} = \vec{v}_{C,B} + \vec{v}_{B,A} \quad .$$

Suppose though that these velocities are quite high, *e.g.,* let A be a planet at rest (P), B a space ship heading away from the planet (S), and C be a missile (M) launched from the ship directly away from the planet, with $v_{S,P} = 0.6c$ and $v_{M,S} = 0.8c$. Would the velocity of the missile relative to the planet really then be 1.4c?

DERIVATION 1-4

Let's let the missile emit a series of radio pulses of frequency f_o in its proper frame. This radio wave will travel back toward the planet and be intercepted by the ship, who will measure a frequency given by the Doppler effect of

$$f_{M,S} = \left(\frac{1-\frac{v_{M,S}}{c}}{1+\frac{v_{M,S}}{c}}\right)^{1/2} f_o.$$

Now, let's let the ship emit its own radio pulse towards the planet every time it receives one from the missile, that is, at the frequency $f_{M,S}$ given above. These pulses will arrive at the planet with frequency

$$f_{S,P} = \left(\frac{1-\frac{v_{S,P}}{c}}{1+\frac{v_{S,P}}{c}}\right)^{1/2} f_{M,S} = \left(\frac{1-\frac{v_{S,P}}{c}}{1+\frac{v_{S,P}}{c}}\right)^{1/2} \left(\frac{1-\frac{v_{M,S}}{c}}{1+\frac{v_{M,S}}{c}}\right)^{1/2} f_o.$$

Of course, the planet will also receive pulses directly from the missile with a frequency given by

$$f_{M,P} = \left(\frac{1-\frac{v_{M,P}}{c}}{1+\frac{v_{M,P}}{c}}\right)^{1/2} f_o.$$

Since each pulse arriving at the planet directly from the missile corresponds exactly with a pulse emitted by the ship, it must be true that

$$f_{M,P} = f_{S,P}$$

and so,

[10] Mermin, N. David, <u>Space and Time in Special Relativity</u>, McGraw-Hill, New York (1968) pp27-32.

$$\left(\frac{1-\frac{v_{M,P}}{c}}{1+\frac{v_{M,P}}{c}}\right)^{1/2} = \left(\frac{1-\frac{v_{S,P}}{c}}{1+\frac{v_{S,P}}{c}}\right)^{1/2} \left(\frac{1-\frac{v_{M,S}}{c}}{1+\frac{v_{M,S}}{c}}\right)^{1/2},$$

or, after squaring and substituting in betas,

$$\left(\frac{1-\beta_{M,P}}{1+\beta_{M,P}}\right) = \left(\frac{1-\beta_{S,P}}{1+\beta_{S,P}}\right)\left(\frac{1-\beta_{M,S}}{1+\beta_{M,S}}\right).$$

Now, solve for $v_{M,P}$ by multiplying both sides by the left hand denominator, distributing, collecting the $\beta_{M,P}$ terms, and multiplying numerator and denominator by $(1 + \beta_{S,P})(1 + \beta_{M,S})$:

$$\beta_{M,P} = \frac{(1+\beta_{S,P})(1+\beta_{M,S}) - (1-\beta_{S,P})(1-\beta_{M,S})}{(1+\beta_{S,P})(1+\beta_{M,S}) + (1-\beta_{S,P})(1-\beta_{M,S})}$$

Multiply out the products in the numerator and denominator and simplify:

$$\beta_{M,P} = \frac{\beta_{M,S}+\beta_{S,P}}{1+\beta_{M,S}\,\beta_{S,P}}.$$

Then, convert back to velocities by multiplying each side by c:

$$v_{M,P} = \frac{v_{M,S}+v_{S,P}}{1+\frac{v_{M,S}\,v_{S,P}}{c^2}}$$

which is not quite what we might have expected from Section 3! In the example given above, the velocity of the missile as seen from the planet would be 0.95c, still less than the speed of light!

Let's look at a couple of extreme cases. If the velocities are very low, then the denominator of the equation above is about 1 and the result reduces to what we would expect from Newtonian mechanics. At the other extreme, let's suppose that the space ship is traveling at a speed so close to the speed of light as makes no numerical difference and launches the missile at a velocity relative to itself also close to the speed of light. Then, the speed of the missile relative to the Planet is <u>still</u> ever so slightly below c. This suggests that no material object can attain a velocity equal to that of light, regardless of the observer's reference frame. We'll discuss this later in more detail.

EXAMPLE 1-7

Suppose a spaceship (S) is moving at 0.999c toward Planet P. The ship fires a SuperGalacto-Corp. Mark V torpedo (T) at 0.998c relative to itself, directly at Planet P. With what speed will the torpedo approach Planet P?

We already have the relationship:

$$v_{T,P} = \frac{v_{T,S} + v_{S,P}}{1 + \frac{v_{T,S}\ v_{S,P}}{c^2}} = \frac{0.998c + 0.999c}{1 + \frac{0.998c \times 0.999c}{c^2}} = 0.999999c\,.$$

EXERCISE 1-2

Using the formula directly above, let $v_{T,S} = c - \varepsilon$ and $v_{S,P} = c - \iota$, where *epsilon* and *iota* are infinitesimally small numbers. Prove mathematically that, no matter how close to c these velocities get, $v_{T,P}$ will always be less than c.

HOMEWORK 1-4

Consider three spaceships. A is chasing B, which is chasing C. Each ship uses the Schockner drive that emits light of wavelength 200 nm. A sees B's emission at 180 nm, while B sees C's emission at 165 nm.

A) What is the velocity of B as measured by A?
B) What is the velocity of C as measured by B?
C) What is the velocity of C relative to A?
D) What wavelength will A see C's emission to be?

Momentum

In Semester One, we defined the momentum $\vec{p}$ of a particle as the product of the particle's mass and its velocity. We saw that if momentum is conserved in one frame of reference, then it is also conserved in other frames. Does that relationship still hold at high speeds?

Consider the very special case of a totally inelastic head-on collision of two particles (A and B) with equal proper masses m_o and equal speeds u (*i.e.*, we see them from the point of view of the center of mass, CM).[11] Here, u represents a specific, if unknown, speed.

$$\vec{v}_{A,CMi} = u; \ \ \vec{v}_{B,CMi} = -u; \ \ \vec{v}_{CM} = 0$$

$$(\vec{p}_{TOT})_i = m_o\vec{v}_{Ai} + m_o\vec{v}_{Bi} = m_o(u) + m_o(-u) = 0.$$

Then, by conservation of momentum,

$$(\vec{p}_{TOT})_f = m_o\vec{v}_{A,f} + m_o\vec{v}_{Bf} = 0,$$

and since, for an inelastic collision, $\vec{v}_{Af} = \vec{v}_{Bf}$, both final velocities are zero. That's actually pretty trivial; just about any reasonable vector function we could think up would exhibit conservation for this scenario.

[11] Adapted from an argument in Serway, R.A, C.J. Moses, and C.A. Moyer, Modern Physics 3rd Edition, Thomson, Belmont, (2005) pp41-43. There are several typographical errors in the solution to Example 2.6.

Now, let's switch to a frame of reference in which mass B is initially at rest. Then, using relativistic relative velocity relationships:

$$\vec{v}'_{Bi} = 0 \quad \vec{v}'_{CM} = -\vec{v}_{Bi} = u \quad \vec{v}'_{Ai} = \frac{v'_{CM}+v_{Ai}}{1+\frac{v'_{CM}\,v_{Ai}}{c^2}} = \frac{2u}{1+\frac{u^2}{c^2}}.$$

In this case, 'sticking together' means that each mass has a final velocity equal to that of the center of mass in the frame in which B was initially at rest:

$$\vec{v}'_{Af} = \vec{v}'_{Bf} = \vec{v}'_{CM} = u.$$

That is, we see the combined objects move away together at speed u. Now, let's tally up the momentum before and after to see if it is conserved:

$$(\vec{p}_{TOT})'_i = m_o\vec{v}'_{Ai} + m_o\vec{v}'_{Bi} = m_o\left(\frac{2u}{1+\frac{u^2}{c^2}}\right) + m_o(0) = \frac{2m_o u}{1+\frac{u^2}{c^2}}$$

$$(\vec{p}_{TOT})'_f = m_o\vec{v}'_{Af} + m_o\vec{v}'_{Bf} = 2m_o\vec{v}'_{CM} = 2m_o u.$$

So, the quantity $m_o\vec{v}$ is <u>not</u> conserved in the relativistic regime.

Well, momentum is a very useful concept, so perhaps we can do what we've done before and refine the definition of momentum so that we get conservation of momentum in the relativistic regime, but still get the results expected from Semester One for low speeds. Let's <u>guess</u> that, since lengths and time intervals are modified by the factor γ, the momentum should also be modified by the same factor. Let's try

$$\vec{p} = \gamma m_o\vec{v} = \frac{m_o\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

and see what happens. For small, v, we can expand the denominator (See Note One at the end of the Section.) to obtain

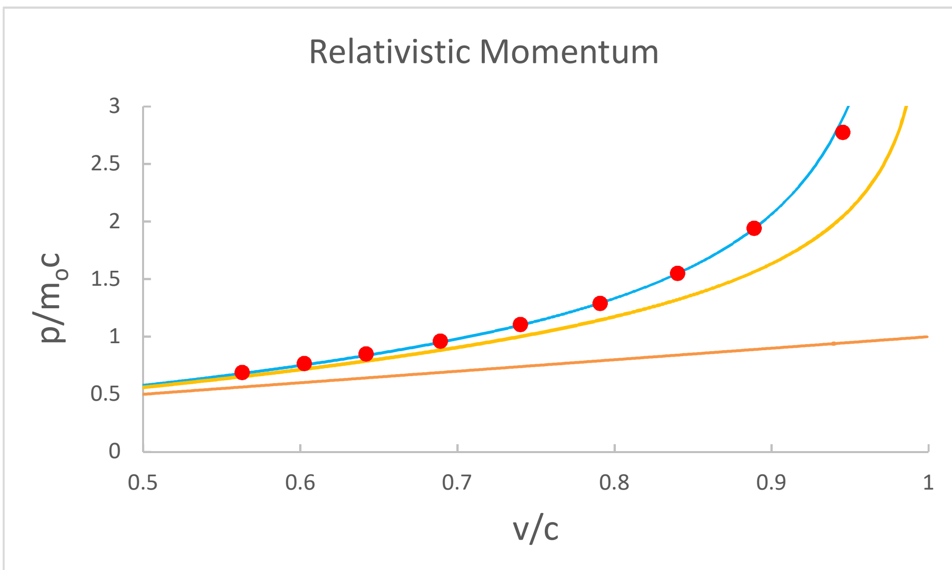
$$\vec{p} = \frac{m_o\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \approx m_o\vec{v}\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4+\frac{15}{48}\left(\frac{v}{c}\right)^6+\cdots\right).$$

Now, it's not enough to show that $\vec{p}\rightarrow 0$ as $\vec{v}\rightarrow 0$. We need to show that $\vec{p}\rightarrow m_o\vec{v}$ as $\vec{v}\rightarrow 0$. Indeed, since $(v/c)^{n+2} < (v/c)^n$, we can see that as v heads toward zero, all of the terms beyond the first become negligible compared to the first and we are left with the familiar Newtonian $\vec{p} = m_o\vec{v}$. Next, let's look at very high velocities. Let's re-arrange the equation above and solve for v:

$$v = \frac{pc}{\sqrt{m_o^2 c^2 + p^2}}.$$

This expression approaches c as the momentum increases towards infinity, consistent with our previous notion of a natural speed limit; you can give the object as much momentum as you like, but the speed will always be less than c. So, the correspondence principle is satisfied. Now, passing these two tests does not mean that our guess is correct, only that it is consistent with some of our expectations. Let's go with it for now.

Now, this definition for momentum is usually interpreted in one of two ways. In the first, the mass remains constant (m_o is called the *rest mass*, or in our terminology, *the proper mass*) and the <u>definition</u> of momentum changes from $\mathbf{p} = m_o\mathbf{v}$ to $\mathbf{p} = \gamma m_o \mathbf{v}$, as above. In the second, the formula remains $\mathbf{p} = m\mathbf{v}$, but the mass is redefined as $m = \gamma m_o$, that is, the *inertial mass* m is considered to increase as the object speeds up. Generally, we will stick with the former; however, the latter notion can be useful.



It would be nice if our guess about momentum were to be supported by experiment. The figure shows the *specific momentum* (p/m_oc) of high speed electrons as a function of beta. The blue curve is calculated using the function we guessed, the yellow line is a competing relationship from the early 20th century, [12] and the red line was calculated using Newtonian momentum. [13] We can see that our guessed function for momentum is almost certainty correct.

Let's return to the idea of conservation of momentum. Although it results in a mess, let's use our new notion of momentum in the example from above, keeping in mind that we must use the

[12] Abraham, M., 'Prinzipien der Dynamik des Elektrons,' *Annalen der Physik* **10**: 105-179 (1903).

[13] These experiments were conducted between 1901 and 1905 by Walter Kaufmann using an apparatus that was much more complicated than necessary. Kaufmann was actually trying to *dis*prove the relativistic behavior in favour of Abraham's relationship, but the data were inconclusive. In 1906, Planck gave a presentation re-examining some of Kaufmann's data and claimed that it was not possible to rule out either picture. That analysis made use of an incorrect value for the specific charge of the electron; I have corrected that error and the results are shown in the graph. Planck, M., 'Die Kaufmannschen Messungen der Ablenkbarkeit der β-Strahlen in ihrer Bedeutung für die Dynamik der Elektronen,' *Physikalische Zeitschrift,* **7**: 753–761 (1906).

relativistic relative velocity expressions. In the reference frame in which B is initially at rest, we have

$$(p_{TOT})'_i = \frac{m_o v'_{Ai}}{\sqrt{1 - \frac{{v'_{Ai}}^2}{c^2}}} + \frac{m_o v'_{Bi}}{\sqrt{1 - \frac{{v'_{Bi}}^2}{c^2}}} = \frac{m_o \frac{2u}{1 + \frac{u^2}{c^2}}}{\sqrt{1 - \frac{\left(\frac{2u}{1+\frac{u^2}{c^2}}\right)^2}{c^2}}} + m_o(0)$$

$$= \frac{2m_o u}{\left(1 + \frac{u^2}{c^2}\right)\sqrt{1 - \frac{4u^2}{c^2}\frac{1}{\left(1+\frac{u^2}{c^2}\right)^2}}} = \frac{2m_o u}{\sqrt{\left(1 + \frac{u^2}{c^2}\right)^2 - 4\frac{u^2}{c^2}}}$$

$$= \frac{2m_o u}{\sqrt{1 + 2\frac{u^2}{c^2} + \frac{u^4}{c^4} - 4\frac{u^2}{c^2}}} = \frac{2m_o u}{\sqrt{1 - 2\frac{u^2}{c^2} + \frac{u^4}{c^4}}} = \frac{2m_o u}{1 - \frac{u^2}{c^2}}.$$

The final total momentum is a bit easier:

$$(p_{TOT})'_f = \gamma_{Af} m_o v'_{Af} + \gamma_{Bf} m_o v'_{Bf} = 2\gamma_{CM} m_o v'_{CM} = \frac{2m_o u}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

Hmm. Not the same. Well, that's a big disappointment after so much calculation. Somehow, though, I think this idea may be salvageable. Let's put it aside for a while.

Energy

Next, let's consider the kinetic energy of an object moving at high velocity. Once again, let's make a guess as to the form of the kinetic energy in the relativistic regime, by returning to the notion that a moving object can be thought to have a larger mass than the same object at rest:

$$m = \gamma m_o = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

So, our guess might as well be that

$$K = \tfrac{1}{2}(\gamma m_o)v^2.$$

Let's check it out at low velocities with an expansion.

$$K = \frac{\frac{1}{2}m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} \approx \frac{1}{2}m_o v^2 \left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4+\frac{15}{48}\left(\frac{v}{c}\right)^6+\cdots\right).$$

So, as v approaches zero, K approaches $^1/_2 m_o v^2$. So far, so good. What happens at high velocities? Solve the proposed relationship above for v^2:

$$v^4 + \frac{4K^2}{m_o^2 c^2}v^2 - \frac{4K^2}{m_o^2} = 0.$$

This quadratic equation has one physical solution (*i.e.*, $v^2 \geq 0$) :

$$v^2 = \frac{-2K^2}{m_o^2 c^2} + \sqrt{\frac{4K^4}{m_o^4 c^4} + \frac{4K^2}{m_o^2}}$$

for which, as expected, v goes to c as the kinetic energy increases to infinity (See NOTE 2 at the end of the Section.). So, this is a possibility. However, if we make actual measurements of K *vs* v, we find that there is poor agreement (see the figure).

So, we need to look at things from scratch and use the Work-Energy theorem. Let's restrict ourselves to a situation where the force F is always in the direction of motion of the object (*i.e.*, a one dimensional problem), and let's require the object to start from rest (why not?). Also, let's assume that our guess of the function for momentum is correct; I think there's enough experimental evidence for that, even if we still have a few problems. Remembering the impulse-momentum relationship:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m_o \mathbf{v}) = \frac{d}{dt}\left(m_o \mathbf{v}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right)$$

$$\mathbf{F} = m_o \frac{d\mathbf{v}}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} + m_o \mathbf{v}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-1}{2}\right)\left(\frac{-2v}{c^2}\right)\frac{d\mathbf{v}}{dt}.$$

Cleaning up the mess, and realizing that **v** and d**v**/dt are in the same direction in our example, we obtain

$$\mathbf{F} = \left(1-\frac{v^2}{c^2}\right)^{-3/2} m_o \frac{d\mathbf{v}}{dt} = \left(1-\frac{v^2}{c^2}\right)^{-3/2} m_o \mathbf{a} = \gamma^3 m_o \mathbf{a}\,. \text{ (Eq. V} - 1 - 6)$$

So, Newton's Second Law is no longer **F** = m**a**! More on this later.

The Work-Energy Theorem (for one dimension) states that

$$W = \int_{x_i}^{x_f} F(x)dx = \Delta K\,.$$

Substituting the penultimate expression above for the force results in

$$W = \int_{x_i}^{x_f} \left(1 - \frac{v^2}{c^2}\right)^{-3/2} m_o \frac{dv}{dt}\, dx = \int_{x_i}^{x_f} \left(1 - \frac{v^2}{c^2}\right)^{-3/2} m_o \frac{dx}{dt}\, dv = \int_0^v \left(1 - \frac{v^2}{c^2}\right)^{-3/2} m_o v\, dv =$$
$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} m_o c^2 \,\Big|_0^v = \gamma(v)\; m_o c^2 - \gamma(0) m_o c^2 = (\gamma - 1) m_o c^2 = \Delta K.$$

Since we set $K_i = 0$, we obtain the general result that

$$K = (\gamma - 1) m_o c^2\,,$$

which is quite different than our guess. Let's test the extremes of this function to see if it jibes with our requirements. Let's expand gamma as we did before:

$$K \approx \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \frac{15}{48}\left(\frac{v}{c}\right)^6 + \cdots - 1\right) m_o c^2$$

$$K \approx \left(\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \frac{15}{48}\left(\frac{v}{c}\right)^6 + \cdots\right) m_o c^2$$

$$K \approx \frac{1}{2} m_o v^2 + \frac{3}{8} m_o c^2 \left(\frac{v^4}{c^4}\right) + \frac{15}{48} m_o c^2 \left(\frac{v^6}{c^6}\right) + \cdots$$

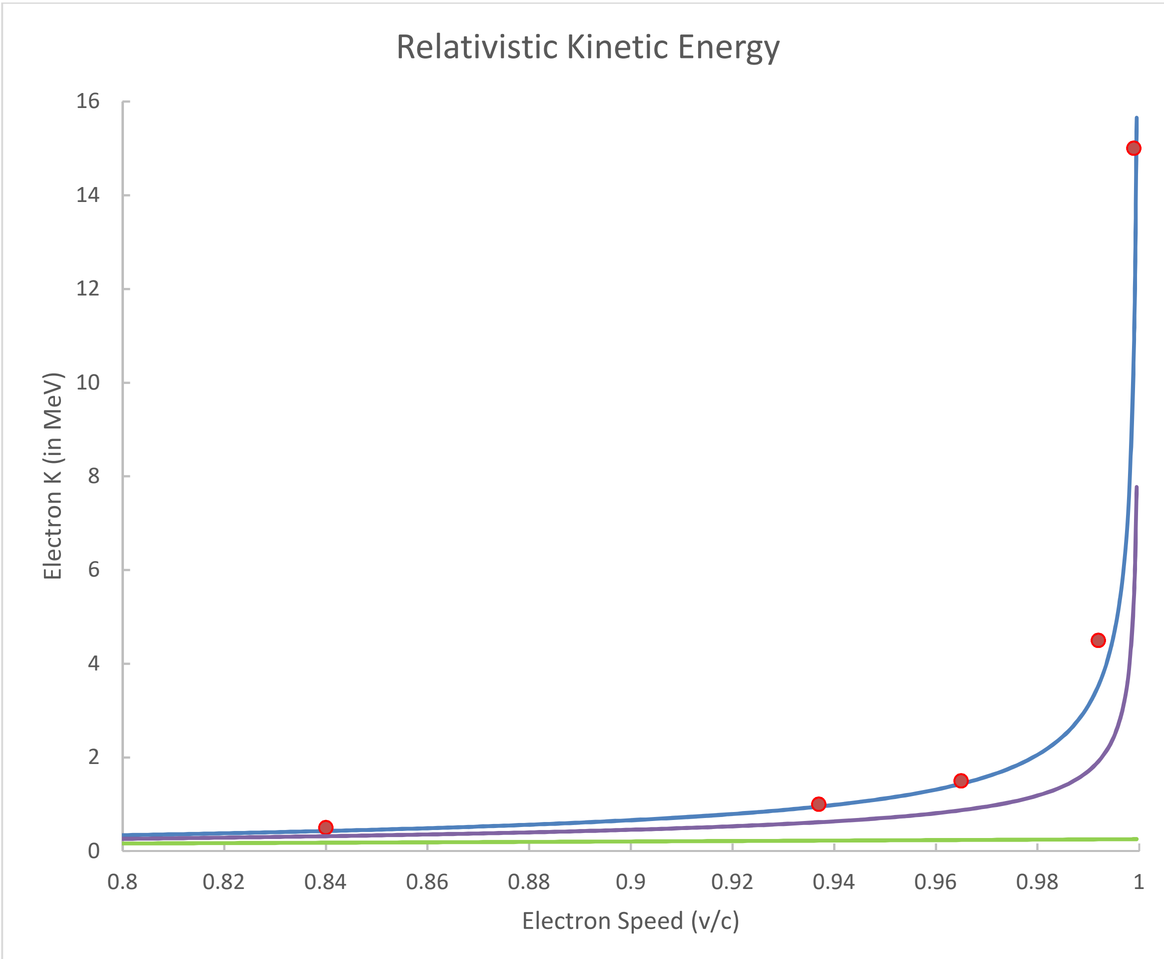
We can see that as v becomes much smaller than c, $K \rightarrow {}^1/_2 mv^2$, as expected.

Let's look at the other extreme. Solve for v as a function of K:

$$v = c\sqrt{1 - \frac{1}{\left(1 + \frac{K}{m_o c^2}\right)^2}}$$

And so we see that as $K \rightarrow \infty$, $v \rightarrow c$, as expected.

We would also like to see if experimental data support <u>this</u> relationship. See the figure.



Here, the blue curve is drawn using the function calculated above, the violet curve is our 'bad' guess $\frac{1}{2}\gamma\, m_o\, v^2$, and the green curve is the Newtonian $K = \frac{1}{2}\, m_o\, v^2$. These particular data points are interesting in that the experiment was filmed.[14] Recommended viewing.

Before we move on, let's look at another special case of Newton's second law, one where the force is perpendicular to the velocity. As discussed in Physics I, no work in done if the force and velocity are perpendicular. We would then expect the kinetic energy, and therefor both the speed and gamma, to remain constant. Repeating our calculation above, we obtain

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m_o \mathbf{v}) = \gamma m_o \frac{d\mathbf{v}}{dt} = \gamma m_o \mathbf{a} \quad .$$

Note that this is different than when the force and velocity are parallel. In fact, we can make another extraordinary statement. Unlike in Newtonian mechanics, the direction of a force on an object and the resulting acceleration <u>do not have to be in the same direction</u>.

[14] *The Ultimate Speed*. Cambridge: Educational Services, Inc., 1962. Currently available at: https://www.youtube.com/watch?v=B0BOpiMQXQA.

EXAMPLE 1-8

Consider an object of mass m_o moving in the +x direction at speed 0.5c. Let's apply a force F in the x-y plane at an angle $\theta_F = 30^o$ from the x-axis. What is the direction θ_a of the resulting acceleration?

$$\tan\theta_F = \frac{F_y}{F_x} = \frac{\gamma m_0 a_y}{\gamma^3 m_o a_x} = \gamma^{-2}\frac{a_y}{a_x} = (1-\beta^2)\tan\theta_a$$

We can see that for low speeds ($\beta \to 0$), the direction angles become the same, as we expect from Newtonian physics. However, to continue,

$$\tan\theta_a = \frac{1}{1-0.5^2}\tan(30^O) = 0.77 \qquad \theta_a = 37.6^O.$$

HOMEWORK 1-5

Consider a 1 kg mass. Calculate how much work must be done to accelerate the mass from

A) 0.1c to 0.12c using relativistic relationships.
B) 0.1c to 0.12c using newtonian relationships.
C) 0.95c to 0.97c using relativistic relationships.
D) 0.95c to 0.97c using newtonian relationships.

Return to Momentum

Now, let's take another crack at the problem we had earlier with the inelastic collision. We found that the total initial momentum was

$$\frac{2m_o u}{1-\frac{u^2}{c^2}},$$

while the final total was

$$\frac{2m_0 u}{\sqrt{1-\frac{u^2}{c^2}}}.$$

We can perhaps salvage our notion about momentum if we assume a really wild thing: suppose that <u>there is more mass in the final state than in the initial state</u>. Let M_o (> $2m_o$) be the mass of the combined objects after the collision. What would we need to make M_o be in order for momentum to be conserved? We would have to require that

$$M_o = \frac{2m_o}{\sqrt{1-\frac{u^2}{c^2}}}$$

and not $M_o = 2m_o$ in order to make our initial and final momentums equal. Then,

$$p_i = \frac{2m_o u}{1-\frac{u^2}{c^2}} \text{ and } p_f = \frac{M_o u}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{\left(\frac{2m_o}{\sqrt{1-\frac{u^2}{c^2}}}\right)u}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{2m_o u}{1-\frac{u^2}{c^2}}.$$

The amount of mass created as a result of the collision would then be

$$\Delta m = M_o - 2m_o = \frac{2m_o}{\sqrt{1-\frac{u^2}{c^2}}} - 2m_o = (\gamma(u)-1)2m_o.$$

This extra mass has to come from somewhere. What do we have less of after the collision than we had before the collision? Well, we know from Semester One that we have less kinetic energy after an inelastic collision. Let's find out how much is lost in our example (again, in the frame in which Object B is initially at rest and using relativistic relative velocities).

$$KE_i = \left(\frac{1}{\sqrt{1-\frac{\left(\frac{2u}{1+\frac{u^2}{c^2}}\right)^2}{c^2}}} - 1\right) m_o c^2 + 0 = \left(\frac{\left(\frac{u}{c}\right)^2}{1-\left(\frac{u}{c}\right)^2}\right) 2m_o c^2$$

$$KE_f = \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right) M_o c^2 = \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right)\left(\frac{2m_o}{\sqrt{1-\frac{u^2}{c^2}}}\right) c^2$$

$$= \left(\frac{1}{1-\left(\frac{u}{c}\right)^2} - \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right) 2m_o c^2$$

$$\Delta \mathrm{KE} = \mathrm{KE_f} - \mathrm{KE_i} = \left(\frac{1}{1-\left(\frac{u}{c}\right)^2} - \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right)2\mathrm{m_o}c^2 - \left(\frac{\left(\frac{u}{c}\right)^2}{1-\left(\frac{u}{c}\right)^2}\right)2\mathrm{m_o}c^2$$

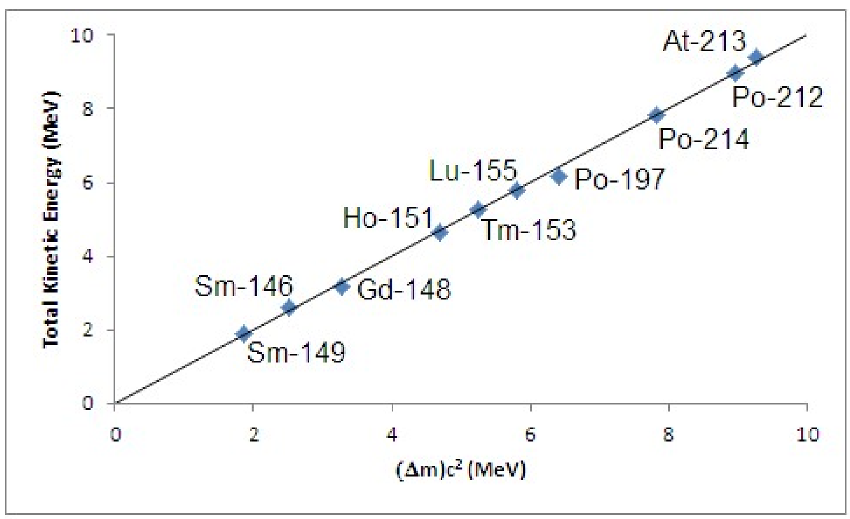
$$= \left(\frac{1-\left(\frac{u}{c}\right)^2}{1-\left(\frac{u}{c}\right)^2} - \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right)2\mathrm{m_o}c^2 = -(\gamma(u)-1)2\mathrm{m_o}c^2 = -(\Delta \mathrm{m_o})c^2.$$

So, the lost kinetic energy is equal to the extra mass after the collision, times c-squared. The kinetic energy was converted into the additional mass seen after the collision!

Let's go back to the original scenario, where we are at the masses' center of mass. In that case, all of the kinetic energy is lost, since the two masses come to rest, and presumably converted to the additional rest mass. This is seen easily to be true in our initial frame of reference as well:

$$\Delta K = K_f - K_i = 0 - 2(\gamma(u)-1)m_o c^2 = -(\Delta m_o)c^2,$$

same as for when we are in B's initial frame of reference.



One of the best examples of the conversion of energy to mass, or actually *vice versa*, is the *alpha-decay of radioactive nuclei*. The basic idea is that a large nucleus of atomic mass A is composed of Z protons and A-Z neutrons. Occasionally, an *alpha particle* (two protons and two neutrons) will break away from the nucleus at high speed. The masses of the remaining *daughter nucleus* and that of the alpha particle add up to less than that of the original nucleus. This missing mass Δm is converted into the kinetic energies of the alpha particle and the daughter nucleus. The figure[15] displays the missing mass times c^2 against the final kinetic energy for ten nuclei[16] (chosen

[15] Kinetic energies were calculated from information in Enge, H., Introduction to Nuclear Physics, Addison-Wesley Publishing Company, Reading (1966) pp528-568. Masses of nuclei were taken from Audia, G., A.H. Wapstrab, and C. Thibault, 'The AME2003 Atomic Mass Evaluation,' *Nuclear Physics A* **729** (2003) pp337–676.
[16] The unit of energy here is the mega-electron-volt (MeV), which you should have encountered in Semester Two. One MeV = 1.6×10^{-13} Joules.

semi-randomly to cover the range of all alpha decays). A line of slope 1 is included for comparison. Clearly, experiment confirms the conversion of mass into energy.

Now, this notion in turn suggests that we could not only convert part of the mass of an object into energy, but also in principle turn the object completely into energy, or conversely, that there is a *rest energy* associated with an object due entirely to the fact that it has mass:

$$\mathrm{E_o} = \mathrm{m_o c^2} \quad .$$

Now, our formula for kinetic energy makes more sense. The *total energy* of an object (excluding any potential energy) is the sum of its rest energy and its kinetic energy:

$$\mathrm{E} = \mathrm{E_o} + \mathrm{K} = \mathrm{m_o c^2} + (\gamma - 1)\mathrm{m_o c^2}\,,$$

$$\mathrm{E} = \gamma \mathrm{m_o c^2}\,.$$

This result also vindicates our guess regarding the functional form of the momentum; we have a nice, consistent set of definitions for relativistic momentum, energy, and mass, each of which agrees with its Newtonian approximation.

HOMEWORK 1-5

> When a particle of matter and a corresponding particle of *anti-matter* interact, they both disappear (annihilate) and produce high energy electro-magnetic waves. If a proton and an anti-proton slowly drift into each other, how much energy will be released? How many such collisions would be necessary to power a 100 watt bulb for an hour, assuming 100% conversion?

Lastly, we would like a relationship between the energy and the momentum, along the lines of the Newtonian relationship

$$\mathrm{KE_{Newt}} = \frac{\mathrm{p_{Newt}^2}}{2\mathrm{m_o}}\,.$$

Let's start with the relationships we know,

$$\mathrm{E} = \gamma \mathrm{m_o c^2} \quad \text{and} \quad \mathrm{p} = \gamma \mathrm{m_o v}.$$

Square both sides of the energy relationship. Multiply the momentum relationship by c, then square it.

$$E^2 = \gamma^2 m_o^2 c^4 \quad \text{and} \quad c^2 p^2 = \gamma^2 m_o^2 c^4 \frac{v^2}{c^2}.$$

Subtract the two equations to obtain

$$E^2 - c^2p^2 = \gamma^2 m_o^2 c^4 - \gamma^2 m_o^2 c^4 \frac{v^2}{c^2} = \gamma^2 \left(1 - \frac{v^2}{c^2}\right) m_o^2 c^4 = \gamma^2 \gamma^{-2} m_o^2 c^4 = m_o^2 c^4 = E_o^2$$

$$E^2 = E_o^2 + c^2p^2.$$

This is itself is an interesting result, in that the rest energy E_o of an object is of course independent of the frame of reference in which the object is observed, so that the quantity E^2- c^2p^2 is the same for an object in any inertial frame of reference:

$$E^2 - c^2p^2 = E'^2 - c^2p'^2 .$$

Conclusion

So, in this section, we have developed relationships for objects moving at high speed. It's true that several of the arguments were based on a special case, however, the results agreed with experimental results well enough that we may takes them as more generally correct.

NOTE 1 – We'll be making use of particular *Taylor expansion* quite a bit in this section:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{6}x^3 + \cdots .$$

In this case, we will expand gamma:

$$\gamma = \left(1+\left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \frac{15}{48}\left(\frac{v}{c}\right)^6 + \cdots$$

NOTE 2

$$v^2 = \frac{-2K^2}{m_o^2c^2} + \sqrt{\frac{4K^4}{m_o^4c^4} + \frac{4K^2}{m_o^2}}$$

Rewrite this using $\beta = v/c$ and $\varepsilon = K/m_oc^2$.

$$\beta^2 = -2\varepsilon^2 + \sqrt{4\varepsilon^4 + 4\varepsilon^2}$$

$$\beta^2 = -2\varepsilon^2 + 2\varepsilon^2\sqrt{1+\frac{1}{\varepsilon^2}}$$

We can expand the root for large epsilon to get

$$\beta^2 \approx -2\varepsilon^2 + 2\varepsilon^2\left(1+\frac{1}{2\varepsilon^2}\right) = 1\,.$$

Now, as $\varepsilon \to \infty$ infinity, we can see that $\beta \to 1$, as expected.

EXERCISE 1-1 Solution

Following the example and the hint, let $v_{S,P}$ and $v_{T,S}$ be slightly below the speed of light by different amounts, *e.g.,* $v_{T,S} = c - \varepsilon$ and $v_{S,P} = c - \acute{\iota}$. Then,

$$v_{T,P} = \frac{v_{T,S} + v_{S,P}}{1+\frac{v_{T,S}\ v_{S,P}}{c^2}} = \frac{(c-\varepsilon)+(c-\acute{\iota})}{1+\frac{(c-\varepsilon)\,(c-\acute{\iota})}{c^2}} = c^2\frac{2c-\varepsilon-\acute{\iota}}{c^2+(c-\varepsilon)\,(c-\acute{\iota})} = c\frac{2c^2-c\varepsilon-c\acute{\iota}}{2c^2-c\varepsilon-c\acute{\iota}+\varepsilon\acute{\iota}}$$
$$< c\ .$$