Section 3 - The Structure of the Atom

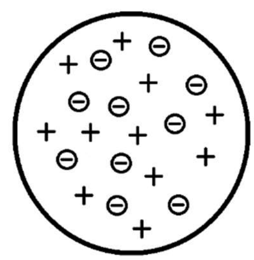
"Fruits, each in its season, are the cheapest, most elegant and wholesome dessert you can offer your family or friends, at luncheon or tea. Pastry and plum-pudding should be prohibited by law, from the beginning of June until the end of September."

Marion Harland, Breakfast, Luncheon, and Tea

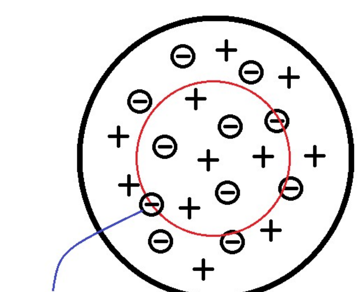
We've taken a shot at determining how big atoms are (2-5 Å) using a number of different methods. Now, let's try to determine the structure of the atom. By this point, we can be fairly sure that atoms include electrons and some type of positive charge in their makeup, but the distribution of these parts is unclear. Let's consider two models (among the many presented at the time), the predictions they make, and how these compare with reality.

The Thomson Model

The Thomson model originated with William Thomson (Lord Kelvin) and was developed by J.J. Thomson. Often known as the *plum pudding model*, it postulates that atoms include electrons and positive charges (not yet identified as protons!) but are over-all electrically neutral. Like raisins in a plum pudding, the electrons are spread throughout a positively charged goopy sphere with a diameter of several Ångstroms. Unlike a raisin, an electron can move through the positively charged atom, subject to the electrostatic force it experiences. Let's take a look at that motion in a bit more detail.



Consider a sphere of radius R and charge +Q. Now, let's add in enough electrons to make the atom have a net charge of +1e, that is, we're short one electron from having the atom be electrically neutral. Furthermore, let's assume that these electrons move around inside the atom and are roughly evenly spread throughout the sphere so that the overall positive charge possesses a uniform density. Now, we'll add in the last electron, but keep track of it separately from the others. The electron is located a distance r from the center of the sphere; as such, it experiences an electric force towards the center of the sphere. According to Gauss's Law, not all of the positive charge is considered to act on the electron; rather, only the charge contained within a gaussian sphere of radius r will contribute to the force.



$$F(r) = \frac{k_e Q_{Enclosed} q_{Electron}}{r^2} \text{ inward.}$$

That is, this is a *restoring force*, always trying to bring the electron back to its equilibrium point at the center of the atom. If the charges (other than our electron) are distributed uniformly, we can write a proportion:

$$\frac{Q_{Enclosed}}{1e} = \frac{\frac{4\pi}{3}r^3}{\frac{4\pi}{3}R^3} \quad \rightarrow \quad Q_{Enclosed} = +e\,\frac{r^3}{R^3} \quad \rightarrow \quad F(r) = \frac{k_e e^2}{R^3}\,r\,.$$

The force is proportional to the electron's distance from the equilibrium point at the center. We've seen similar restoring forces, namely the spring force, F = (-) kx. Any classical object moving under such a force will exhibit simple harmonic motion with a well-defined frequency,

$$\omega_o = \sqrt{\frac{k}{m}}\,.$$

Applying this concept to the electron, it should also oscillate within the atom with a frequency

$$\omega_o = \sqrt{\frac{\frac{k_e e^2}{R^3}}{m_{electron}}} \approx 3 \times 10^{15}\,\frac{rad}{sec}\,.$$

This value is independent of the amplitude of oscillation within the atom (as with all classical oscillators), and fairly independent of the size of the atom ($\sim R^{-3/2}$). Classically, oscillating charges emit light with the same frequency as that of the oscillation, which means that all atoms of an element should emit one frequency, and the frequencies emitted by all types of atoms[1] should be of about the same value ($\lambda = 2\pi c/\omega_o \approx 600$ nm). Of course, this contradicts the spectroscopic observations made of the elements, each of which has very distinct sets of multiple emission lines, as for example, you saw in Semester Two with mercury. So, one strike against the Thomson model.

HOMEWORK 3-1

> In the last Section, we estimated the diameter of a helium atom to be 2.4 Å and that of a xenon atom to be 5.2 Å. Calculate the single wavelength expected to be emitted from each, using the model described above.

The Rutherford Model

This model may seem more familiar to you; it's taught to most elementary school children. In this picture, the positive charge is concentrated in a small region at the center of the atom (the *nucleus*), while the negative electrons form a 'cloud' surrounding the nucleus. Since electrons have a mass thousands of times smaller than the atoms themselves, the nucleus must be, relatively speaking, the bulk of the atom. In such a structure, the atom would not at first glance be expected to emit any particular frequency of light. Strike One here too.

The Geiger-Marsden Experiment

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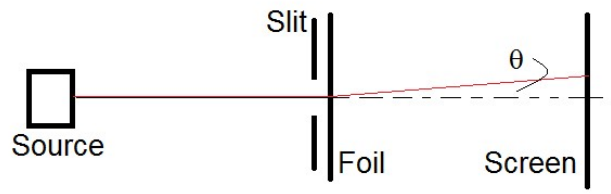
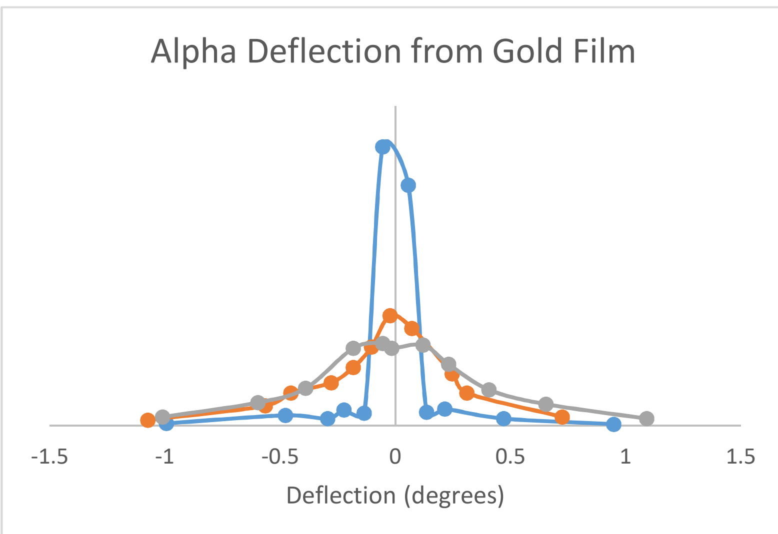
[1] Strictly speaking, this works for atoms with larger amounts of charge, large enough for the screening argument to work.

Secondly, let's consider the Geiger-Marsden experiment, which was designed to validate the Thomson model. A stream of alpha particles (charge = +2e) from radio-active decay[2] is aimed at a thin gold foil.[3] Gold was used for two reasons: it is *malleable* and so can be made into a very thin sheet, and gold atoms are fairly massive compared to alpha particles. Being charged, alpha particles could be deflected by the electric interaction with the charges in the atom, although most alphas would probably not be deflected.

The experimental setup is shown (very simplified) in the figure. Radium is placed at one end of an evacuated tube. Emitted alpha particles traverse the length of the tube, pass through a slit, through the foil, and ultimately strike a ZnS screen. The energy of each alpha causes the screen to fluoresce, and the strikes are observed with a microscope through a mica endcap. If the Thomson model were correct, there should be some deflection of particles when the foil is present and none when the foil is absent. Here are some experimental data.[4] As expected, there is no scattering without a foil (blue curve), to the resolution of the device, but scattering occurs for one thin foil (orange), and somewhat more scattering for two foils (grey). This is understood to be the result of multiple scatterings from many atoms in the foil before the alpha exits. There is a range of scattering angles because some small number of particles are always deflected to, say, the left, while others experience perhaps equal numbers of left and right deflections and emerge at 0°. Most particles end up somewhere in between.





Cooper[5] presents a quick estimate of the <u>maximum conceivable deflection</u> of an alpha particle from a Thomson gold atom. First, we'll assume that the electrons, being very light, play no part, and the full charge of +79e acts on the alpha particle. From Gauss's law, we know that the

[2] Alpha particles from radio-active decay generally have specific kinetic energies that are characteristic of the decaying element. These values range from roughly 2.5 to 9 MeV.

[3] At this point, it was not yet known that the charge of such an atom is +79e. However, we know that now and will use the fact to analyze these early data.

[4] Geiger, H., "On the Scattering of the α-Particles by Matter," *Proceedings of the Royal Society of London A.* ***81*** *(546)* (1908) pp 174–177.

[5] Add Reference

maximum force on the alpha from the gold atom will occur when the alpha is at the surface of the atom, or about 2Å from the center:

$$F_{max} = \frac{k_e Qq}{R^2} .$$

We'll also let the repulsive force act at right angles to the initial direction of motion of the alpha particle to maximize the momentum change perpendicular to the original direction of motion. The time interval over which this force is applied is the transit time of the alpha across the diameter of the atom, $\Delta t = 2R/v$; we'll ignore interactions beyond this interval because the alpha will then be under the influence of some other gold atom. So, making use of the impulse relationship, the change in momentum will be

$$\Delta p = F\,\Delta t = \left(\frac{k_e Qq}{R^2}\right)\left(\frac{2R}{v}\right) .$$

The resulting deflection angle will be

$$\theta = \arctan\left(\frac{\Delta p}{p}\right) = \arctan\left(\frac{k_e Qq}{R^2}\frac{2R}{v}\frac{1}{mv}\right) = \arctan\left(\frac{k_e Qq}{RK}\right).$$

Here, K is the alpha particle's kinetic energy. We'll even let the alpha particle have a low kinetic energy for this estimate:

$$\theta = \arctan\left(\frac{(9\times 10^{9})\times(79\times 1.6\times 10^{-19})\times(2\times 1.6\times 10^{-19})}{(1.5\times 10^{-10})\times(8\times 10^{-14})}\right) = 0.02^{o} .$$

Now this is the <u>absolutely maximum deflection</u> from one Thomson atom, after we've given the process every possible break. Most likely, the actual deflection would be much less. If the alpha particle were to travel through a gold foil with a thickness of 200 atoms (860Å thick with a 4.1Å lattice constant), and <u>all 200 deflections</u> were in the same direction, the absolute maximum total deflection would be approximately

$$\theta_{max} = 200 \times 0.02^{0} = 4^{o} .$$

Let's take a closer look at the graph above. The resolution isn't very good, but it seems as if there are two maximums in the two-foil curve, one on each side of zero. This is actually to be expected. We presume that the alpha particles go through multiple scatterings on their way through the foil. Deflections to the right or left should be equally probable, so if we have a large number of particles, the <u>average</u> deflection should be zero. However, the <u>most probable</u> deflection will NOT be zero. Let's see why.

The Random Walk

Suppose a drunk starts at the origin. He takes steps of length L along the x-axis, but each step could just as easily be to the right as to the left. Let's represent his location after N steps as x(N). Over time, his average location should be of course at the origin ($x_{AVE} = 0$). What we want is his root-mean-square (r.m.s.) position,[6] which we would expect <u>not</u> to be zero, since it involves summing a collection of non-negative values. Now,

$$x(N+1) = x(N) \pm L \ ,$$

since the next step could be in either direction. Then,

$$(x(N+1))^2 = \ (x(N) \pm L)^2 = \left(x(N)\right)^2 \ \pm 2\, x(N)\, L + L^2 \ ,$$

$$((x(N+1))^2)_{AVE} = \left(\left(x(N)\right)^2\right)_{AVE} \pm 2L\left(x(N)\right)_{AVE} + L^2 = \left(\left(x(N)\right)^2\right)_{AVE} + L^2 \, ,$$

since $x(N)_{AVE} = 0$. Then,

$$\left(x(N+1)\right)_{rms}^2 = \left(x(N)\right)_{rms}^2 + L^2 \ .$$

Let's start at the origin and calculate:

$$\left(x(1)\right)_{rms}^2 = \left(x(0)\right)_{rms}^2 + L^2 = 0 + L^2 = L^2$$

$$\left(x(2)\right)_{rms}^2 = \left(x(1)\right)_{rms}^2 + L^2 = L^2 + L^2 = 2L^2$$

$$\left(x(3)\right)_{rms}^2 = \left(x(2)\right)_{rms}^2 + L^2 = 2L^2 + L^2 = 3L^2$$

or,

$$\left(x(N)\right)_{rms}^2 = NL^2 \quad \rightarrow \quad \left(x(N)\right)_{rms} = \pm\sqrt{N}L \ .$$

So, let's say we have a large number of drunks, all starting at the origin. After a fairly large number of steps, N, the farthest any of them could be from the origin would be NL, their average position would be zero, but the we would expect the most likely positions to be a distance $N^{1/2}L$ on each side of the origin.

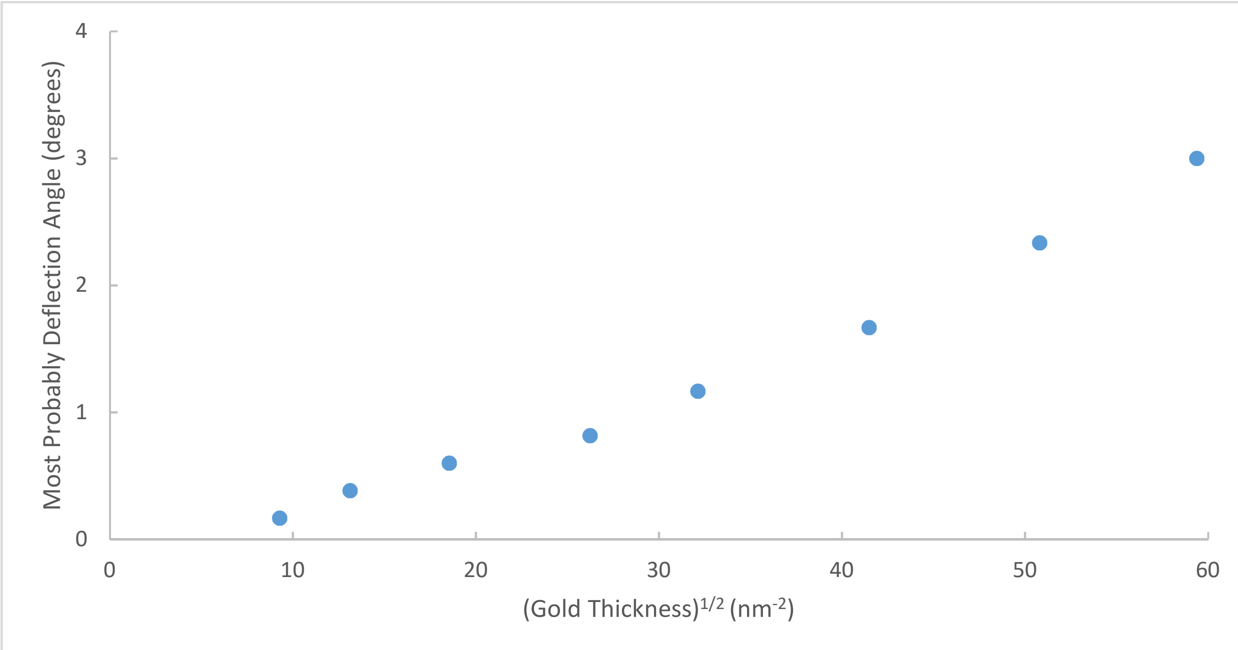
HOMEWORK 3-2

> Suppose I release an infinite number of bunnies from the origin at time t = 0. Bunnies hop about a half meter once every second. <u>Sketch</u> the distributions of bunnies along the x-axis for t = 0 seconds, t = 1 minute, and t = 10 minutes.

As an analogy, let's consider each subsequent deflection of an alpha particle as it passes through a gold foil to be similar to the steps of the random walk. As such, we might expect the most probable deflection to occur at an angle that is proportional to the square root of the number of atoms the alpha particle passes by, and therefor to of the thickness of the foil,. Here are some data

[6] As with rms voltages and currents, square the values, average them, then take the square root.

from Geiger.[7] We see a vaguely linear relationship between the deflection angle and the square root of the number of atoms encountered by the alphas.[8]



With these data, we should be able to estimate the actual deflection from a single atom. [9]

EXAMPLE 3-1

> A film of total thickness 8.6×10^{-8} m will be about 200 3Å atoms thick (lattice constant of gold is about 4.1Å). From the data, such a film causes a most probable 0.16° deflection, so that the deflection of one atom can be estimated to be
>
> $$\theta_{\mathrm{Film}} = 200^{0.73}\theta_{1\mathrm{atom}} \quad \rightarrow \quad \theta_{1\mathrm{atom}} = \frac{0.16^{\mathrm{o}}}{200^{0.73}} = 0.0033^{\mathrm{o}} \ll 0.02^{\mathrm{o}} \ .$$
>
> This value is much smaller than our outrageously generous estimate above (about 1/6th as much), and so may seem fairly reasonable.

Well, now things aren't looking too bad for the Thomson Model, but the interesting bits are always in the slight deviations from what we expect.

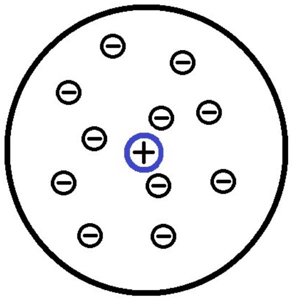
That Special 'WTF' Moment

Returning to the example above, we can also estimate the absolute maximum deflection from a thin gold sheet of thickness 8.6×10^{-8} m foil as[10]

[7] Geiger, Hans, "On the Scattering of the α-Particles by Matter." *Proceedings of the Royal Society of London A.* ***81*** *(546)* (1908) pp174–177.
[8] A better fit to these data is that the angle is proportional to the 0.73 power of the thickness. Geiger speculated that the slightly higher power dependence is due to the alphas slowing as they pass through the foil.
[9] Remember, in our calculation above, we made conditions for deflection as favorable as possible.
[10] Remember that this value is for one thin foil; the data in the graph are for many thin foils.

$$\theta_{max} = 200\theta_{1atom} = 200 \times 0.0033^0 = 0.7^o \ .$$

This is fairly consistent with the graph for one foil several pages back.



Statistically speaking, the probability of an alpha particle departing this film at an angle greater than even a few degrees is virtually zero. However, during the experiment, approximately 0.01% of the incident particles were deflected <u>backward</u>, that is, at angles larger than 90^o! Clearly, the repulsion experienced by the alpha particles is much larger than estimated above. We can accomplish this by making the positively charged part of the atoms much smaller, forming a *nucleus* within the atom, and thereby allowing some small number of alpha particles to pass much more closely to the nucleus while still feeling the full effect of the charge. How small?

EXAMPLE 3-2

Calculate the maximum radius of a gold atom (or at least the positive part of it) if an alpha particle of energy 5 MeV is stopped and returned along its original path, *i.e.*, the deflection angle is 180^o. Ignore the effects of the electrons.

The alpha starts a large distance from the atom with kinetic energy K_o and little potential energy. At its closest approach, it has no kinetic energy and some potential energy. Assuming no other effects, we have that

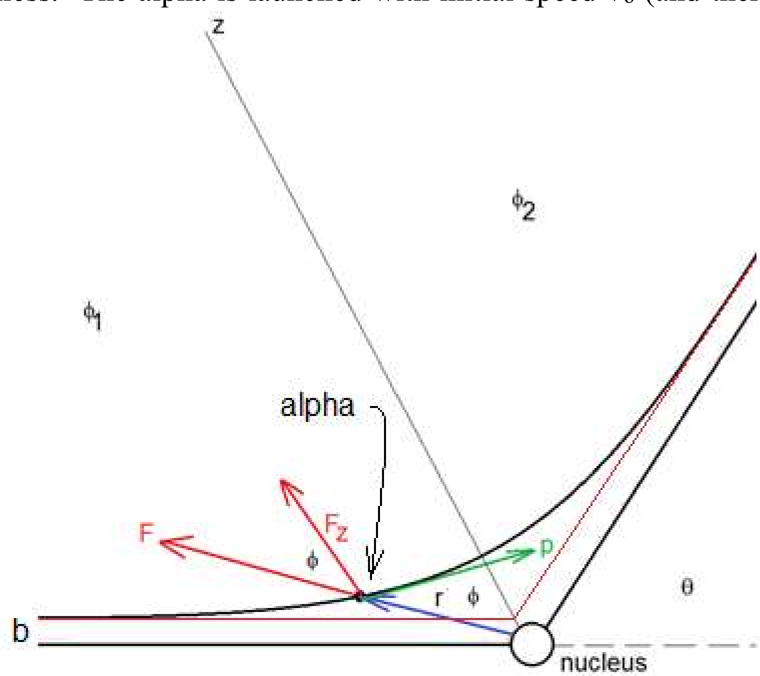
$$K_o + U_i = K_f + U_f \quad \rightarrow \quad K_o = \frac{k_e Q_{gold} q_{alpha}}{R_{MIN}}$$

$$R_{MIN} = \frac{k_e Q_{gold} q_{alpha}}{K_o} = \frac{9 \times 10^9 \times (79 \times 1.6 \times 10^{-1}\ ) \times (2 \times 1.6 \times 10^{-1}\ )}{5 \times 10^6 (1.6 \times 10^{-19})}$$
$$= 4.6 \times 10^{-14}\ \mathrm{m} \ .$$

This is approximately 1/10,000th the size of the atom itself. This gives an upper limit on the size of the gold nucleus, since we presume that such an alpha particle did not actually collide with the nucleus, however, it's not enough to be conclusive; first, this only gives an upper limit (the natural alpha particles have a limited range of energies), and second, there may be other forces at work within the atom, while we have assumed that only the coulomb force affects the motion of the alpha..

DERIVATION 3-1

Let's concentrate on these high-angle-scattered particles. Assume that the atom consists of a very small positively charged *nucleus* and electrons in a 'cloud' buzzing around the nucleus. The electrons are so light compared to the alpha particle (mass ratio of ~7500) that their interactions have virtually no effect on the motion of the alpha (think about a bus hitting a Superball; the bus is undeflected!). In addition, the gold nucleus is quite massive, so let's assume that it remains motionless. The alpha is launched with initial speed v_o (and therefor initial kinetic energy K_o), but not directly toward the nucleus. Its path is offset by a distance called the *impact parameter*, b. This is the distance the particle would miss the center of the nucleus by if its path were not deflected.



Let's keep track of the alpha's position with polar coordinates, r and φ, with the nucleus at the origin. The magnitude of the angular momentum of the alpha particle as seen from the nucleus is $L = |\mathbf{r}\times\mathbf{p}| = bmv_o$.[11] Since the Coulomb force is a central force, the torque on the alpha ($\mathbf{r}\times\mathbf{F}$) is zero and **L** is conserved.

We might imagine from Semester One that the shape of the path taken by the alpha will be one arm of a hyperbola; an object moving under the influence of a central, attractive $1/r^2$ force can trace out one of several curves: circle, ellipse, parabola, or one arm of a hyperbola. An object acted on by a central, repulsive $1/r^2$ force follows the other arm of the hyperbola. As a result, we expect there to be an axis of symmetry to the path; let's define that axis as the z-axis where $\varphi = 0$. Note the unusual orientation used in the figure. Then, the direction from which the alpha came is φ_1, the angle to which it heads is φ_2, and due to the symmetry of the situation, $\varphi_1 = -\varphi_2$. Lastly, the angle of deflection that is actually measured in the laboratory is $\theta = \pi - (\varphi_2 - \varphi_1) = \pi - 2\varphi_1$.

Use the impulse-momentum relationship:

$$\Delta\vec{p} = \int \vec{F}(\vec{r})\ dt \quad \text{with } F(r) = \frac{k_e Qq}{r^2},$$

[11] When the alpha particle is far for the nucleus, $L = |\mathbf{r}\times\mathbf{p}| = r(mv_o)\sin\varphi = (r\sin\varphi)\,mv_o = bmv_o$.

with Q = 79e and q = 2e. Now, any force component that is not parallel to the z-axis will average to zero because of the symmetry, so we only need to worry about the z-component, F(r) cosφ. We need one more piece before we can integrate. Consider again the angular momentum of a point mass about the nucleus,

$$L = I\omega = mr^2 \frac{d\phi}{dt}, \qquad \text{but } L = bmv_o.$$

So,

$$mr^2 \frac{d\phi}{dt} = mv_o b \quad \rightarrow \quad \frac{dt}{d\phi} = \frac{r^2}{bv_o}.$$

Put it all together:

$$\Delta p_z = \int F_z \, dt = \int F(r)\cos\phi \, dt = \int \frac{k_e Qq}{r^2}\cos\phi \frac{dt}{d\phi} d\phi = \int \frac{k_e Qq}{r^2}\cos\phi \frac{r^2}{v_o b} d\phi$$

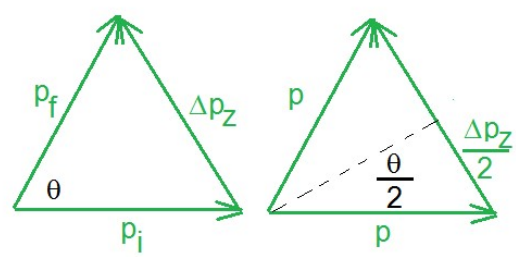
$$= \int_{\phi_1}^{\phi_2} \frac{k_e Qq}{v_o b}\cos\phi \, d\phi = \frac{k_e Qq}{v_o b} \sin\phi\Big|_{\phi_1}^{\phi_2} = \frac{k_e Qq}{v_o b}[\sin\phi_2 - \sin\phi_1]$$

$$= \frac{k_e Qq}{v_o b}[\sin\phi_2 - \sin(-\phi_2)] = \frac{k_e Qq}{v_o b}[2\sin\phi_2] = \frac{2k_e Qq}{v_o b}\sin\left(\frac{\pi - \theta}{2}\right)$$

$$= \frac{2k_e Qq}{v_o b}\cos\left(\frac{\theta}{2}\right).$$

Now, let's look at Δp_z from a different point of view. The Coulomb force is conservative, so the total energy of the alpha particle is conserved. The alpha starts out far from the nucleus with K_o and no potential energy ($U = k_e Qq/r$ and r is very large). It also ends far away with no potential energy, so $K_f = K_o$. That allows us to say that the magnitudes of the momentums at the start and finish are also equal, even if the directions are different; indeed, $\vec{p}_f$ has rotated from the direction of $\vec{p}_i$ by angle θ.

The change in p_z is represented by the 'base' of the isosceles triangle (in the figure, on the right). Splitting this into two right triangles allows us to write that



$$\Delta p_z = 2mv_o \sin\left(\frac{\theta}{2}\right).$$

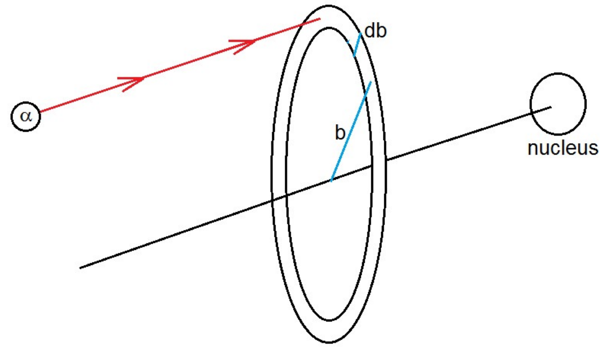
Equating these two relationships results in

$$b = \frac{k_e Qq}{mv_o^2}\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{k_e Qq}{2K_o}\cot\left(\frac{\theta}{2}\right).$$

So, given the scattering angle θ of a particular alpha particle, we can determine the impact parameter b that caused the scattering, or of course, we can predict the scattering angle for a given impact parameter.

O.K., deep breath.

We're now going to use this to determine the probability that some random alpha particle will be deflected at a particular angle, θ. Think of the nucleus as a bull's eye, surrounded by an annulus of radius b and width *d*b. We expect the probability of any given alpha to approach our nucleus with an impact parameter between b and b+*d*b, P(b)*d*b, to be proportional to the area of this ring.



$$\mathrm{P(b)}d\mathrm{b} = \mathrm{C}\ 2\pi\ \mathrm{b}\ d\mathrm{b} = \mathrm{C}\ 2\pi\ \mathrm{b}(\theta)\ \frac{d\mathrm{b}}{d\theta}\ d\theta\ .$$

Then, from above,

$$\frac{d\mathrm{b}}{d\theta} = \frac{\mathrm{k_e Qq}}{2\mathrm{K_o}} \frac{d\ \cot\left(\frac{\theta}{2}\right)}{d\theta} = \frac{\mathrm{k_e Qq}}{4\mathrm{K_o}} \sin^{-2}\left(\frac{\theta}{2}\right).$$

$$\mathrm{P}(\theta)\ d\theta = \mathrm{C}\ 2\pi \left(\frac{\mathrm{k_e Qq}}{2\mathrm{K_o}} \cot\left(\frac{\theta}{2}\right)\right) \left(\frac{\mathrm{k_e Qq}}{4\mathrm{K_o}} \sin^{-2}\left(\frac{\theta}{2}\right)\right) d\theta$$

$$= \mathrm{C}\ 2\pi \left(\frac{\mathrm{k_e Qq}}{2\mathrm{K_o}}\right)^2 \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \sin^{-2}\left(\frac{\theta}{2}\right)\ d\theta\ .$$

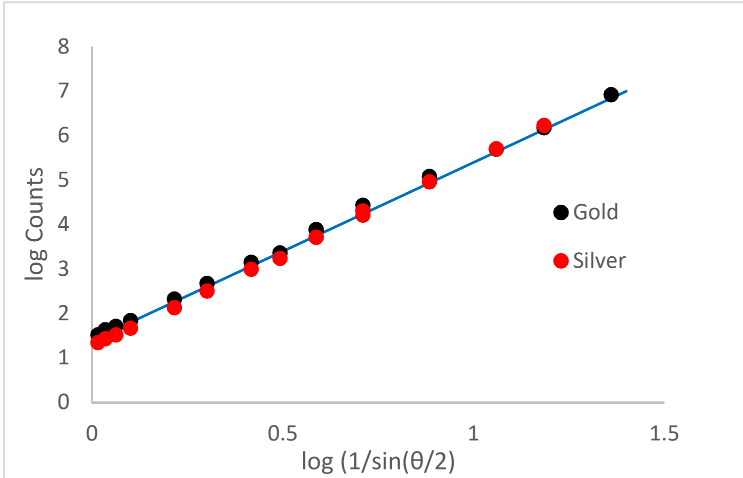
Use the trig identity sinθ = 2sin(θ/2)cos(θ/2).

$$\mathrm{P}(\theta)\ d\theta = \mathrm{C}\ 2\pi \left(\frac{\mathrm{k_e Qq}}{2\mathrm{K_o}}\right)^2 \frac{\left(\frac{\sin\theta}{2\sin\left(\frac{\theta}{2}\right)}\right)}{\sin\left(\frac{\theta}{2}\right)} \sin^{-2}\left(\frac{\theta}{2}\right)\ d\theta = \frac{\mathrm{C}}{8} \left(\frac{\mathrm{k_e Qq}}{2\mathrm{K_o}}\right)^2 \sin^{-4}\left(\frac{\theta}{2}\right)\ (2\pi \sin\theta\ d\theta)\ .$$

Since 2π sinθ *d*θ is the solid angle *d*Ω into which the alpha was scattered, we have that the probability *per* solid angle is proportional to

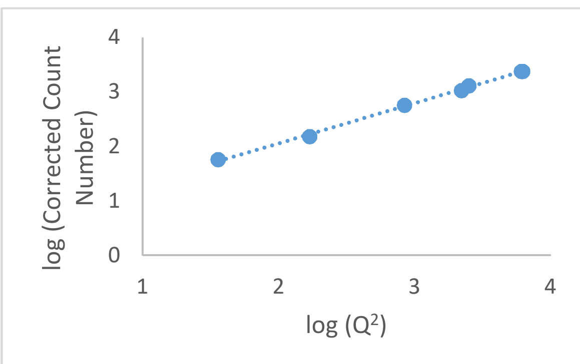
$$\mathrm{Q}^2\ \mathrm{K_o^{-2}} \sin^{-4}\left(\frac{\theta}{2}\right)\ .$$

Here, we have three testable results:



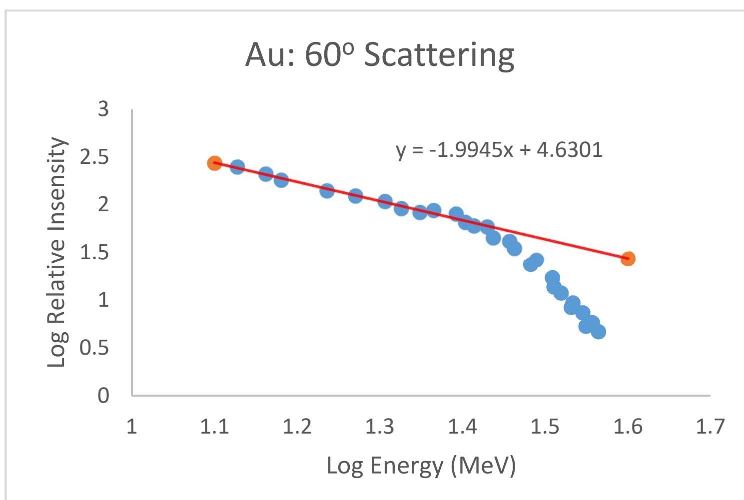
1. for a given initial energy K_o, the number of alphas observed at a given angle will vary as $\sin^{-4}(\theta/2)$.
2. if we perform this experiment on different types of nuclei, such that Q changes, we should see scattering proportional to Q^2.
3. for a given angle, the number should vary as K_o^{-2}.

Let's look at some data. In the first graph, we see the number of counts for each of gold and silver as a function of the scattering angle. The solid line has slope 4, showing that the scattering varies as $\sin^{-4}(\theta/2)$, as predicted.



In the second figure, we see scattering at a fixed angle and incident energy for different elements, *i.e.*, different nuclear charges. Our prediction was that the number of counts should be proportional to the square of the charge; this isn't quite true, although a linear relationship is arguable.[12]

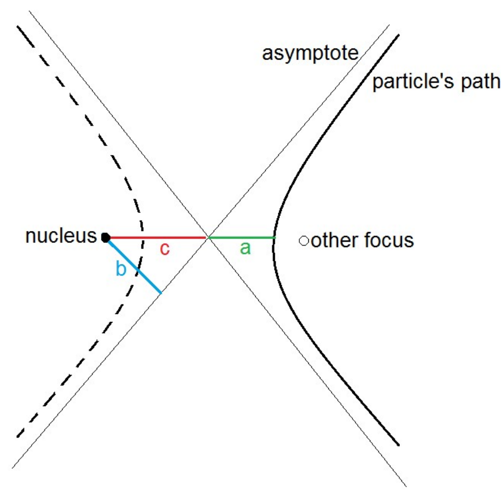
In the last figure,[13] the alpha particles were artificially accelerated to energies higher than naturally available (>9MeVs). At lower energies, the data follow the expected behaviour, as seen by the line of slope -2. That is, the amount of scattering is proportional to K_o^{-2}. However, note that there is a point around 26 MeV where the data leave that line; at that distance of closest approach, something other than the coulomb force is affecting the trajectories of the alpha particles. We can imagine that either the alpha collides with or passes into the nucleus, or perhaps some other force becomes important. From this, we should be able to find a new upper limit value for the radius of a gold atom.



[12] A better fit is actually around $Q^{3/2}$.
[13] Insert footnote.

First, we'll need to know the impact parameter b for 26 MeV at 60°:

$$
\begin{aligned}
\mathrm{b} &= \frac{\mathrm{k_e Q q}}{2\mathrm{K_o}}\cot\left(\frac{\theta}{2}\right) \\
&= \frac{9\times10^{9}\times(79\times1.6\times10^{-19})\times(2\times1.6\times10^{-19})}{2\times(26\times10^{6}\times1.6\times10^{-19})}\cot\left(\frac{60}{2}\right) \\
&= 7.58\times10^{-15}\ \mathrm{m}\,.
\end{aligned}
$$



Reviewing the properties of hyperbolas (see Semester One), the eccentricity of a Keplerian orbit is given by

$$
\mathrm{e} = \sqrt{1+\frac{2\mathrm{EL}^2}{\mathrm{G}^2\mathrm{M}^2\mathrm{m}^3}} \;\rightarrow\; \mathrm{e} = \sqrt{1+\frac{2\mathrm{K_o L}^2}{\mathrm{m}_\alpha(\mathrm{k_e Q q})^2}} = 2\,,
$$

but it's also given (and more easily!) for hyperbolas by

$$
\mathrm{e} = \frac{1}{\cos(\varphi_{\max})} = \frac{1}{\cos(60^{\mathrm{o}})} = 2.
$$

The eccentricity can also be expressed in terms of the *semi-major axis* a (half the distance between the vertices), the *semi-minor axis* b (the smallest distance between the asymptotes and either focus, also the impact parameter in our application), and the *linear eccentricity* c (half the distance between the focuses):

$$
\mathrm{e} = \sqrt{1+\frac{\mathrm{b}^2}{\mathrm{a}^2}} \;\rightarrow\; \mathrm{a} = \frac{\mathrm{b}}{\sqrt{\mathrm{e}^2-1}} \quad \text{and } \mathrm{e} = \frac{\mathrm{c}}{\mathrm{a}} \;\rightarrow\; \mathrm{c} = \mathrm{ea} = \frac{\mathrm{eb}}{\sqrt{\mathrm{e}^2-1}}\,.
$$

The distance of closest approach will be c + a:

$$
\mathrm{r_{min}} = \mathrm{c}+\mathrm{a} = \frac{\mathrm{eb}}{\sqrt{\mathrm{e}^2-1}} + \frac{\mathrm{b}}{\sqrt{\mathrm{e}^2-1}} = \sqrt{\frac{\mathrm{e}+1}{\mathrm{e}-1}}\,\mathrm{b} = \sqrt{\frac{2+1}{2-1}}\,\mathrm{b} = 1.73\mathrm{b} = 1.31\times10^{-14}\ \mathrm{m}.
$$

The currently accepted value for the radius of a gold nucleus is about half that, 7×10^{-15}m. We'll show how to get that accuracy later in the course.

Most of the alpha particles in this experiment do not come particularly 'close' to the nucleus. The ratio of the cross sections of the atom to the nucleus is approximately

$$
\frac{\pi \mathrm{r}^2_{\mathrm{Nucleus}}}{\pi \mathrm{r}^2_{\mathrm{Atom}}} = \left(\frac{10^{-1}}{10^{-1}}\right)^2 = 10^{-8}.
$$

So, the majority of such particles will be scattered in a manner similar to the Thompson model. However, the only mechanism that will scatter alpha particles at such large angles is to have a very small, compact nucleus of charge, as in the Rutherford model. We still have a lot of unanswered questions, though. For example, what is the nucleus made of, and why do different elements radiate at very different frequencies?