Section 4 - h

"*Die Wahrheit triumphiert nie, ihre Gegner sterben nur aus*."

- Attributed to Max Planck

The purpose of this section of notes is to introduce you to a new fundamental value on the scale of importance of G and c, *Planck's constant* (symbol, h). Historically, the need for Planck's constant stems from the theory of objects called *black bodies*. A black body by definition absorbs 100% of the radiation incident upon it. Once the object achieves thermal equilibrium, it must also emit radiation, and does so with a characteristic spectrum. Most objects behave approximately as black bodies, although for everyday objects, the vast majority of the radiation is released in the far infrared part of the spectrum. Two examples you may be familiar with are the photosphere of the sun, with its peak in the yellow, and perhaps the embers of a wood fire, with its output in the near IR. We're going to examine two failures in explaining black body radiation, and one exciting success, and learn some additional thermo-dynamics along the way.

Model of a Black Body Radiator

Consider as an example a hollow metal cube with interior dimensions L×L×L. The cube has a small hole connecting its interior to the outside universe. If light (or other EM radiation) were to enter the hole, it would presumably bounce around inside with very little chance of exiting, *i.e.*, the interior would absorb pretty much 100% of the energy that enters. This is our model for a black body.

Let's consider the waves bouncing around inside. In PHYS I, we wrote the equation for a one dimensional mechanical wave as

$$Y(x,t) = A\cos(kx - 2\pi f t + \varphi).$$

where $k = 2\pi/\lambda$. Let's turn this into a three dimensional wave by making k a vector, such that:

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z,$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} \quad \text{and} \quad r = \sqrt{x^2 + y^2 + z^2}.$$

Switching this over to an EM wave, we obtain

$$\vec{E}(\vec{r},t) = \vec{E}_o\cos\left(k_x x + k_y y + k_z z - 2\pi f t + \varphi\right) = \vec{E}_o\cos\left(\vec{k}\cdot\vec{r} - 2\pi f t + \varphi\right).$$

Now, since the inside surface of the cube is conducting, we know that the transverse electric fields of the EM wave there must be zero. That is, a node must exist at each surface. This is the same situation as when we talked about standing waves on a string, fixed at both ends. In that situation, we found that $L = n\lambda/2$ or $k = n\pi/L$, with n a positive integer. In three dimensions, this requirement should be met in each of the directions, although we're really talking about the projections of the wavelength in each direction:

$$k_x = \frac{n_x \pi}{L} \;;\; k_y = \frac{n_y \pi}{L} \;;\; k_z = \frac{n_z \pi}{L}$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{2\pi}{\lambda} = 2\pi\frac{f}{c} \;.$$

So, the frequencies f allowed in the cavity as standing waves are given by

$$f = \frac{c}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2} \;.$$

Now, let's let

$$n = \sqrt{n_x^2 + n_y^2 + n_z^2} \,.$$

$$f_{n} = \frac{nc}{2L} \,.$$

Well, this should be no surprise; it looks just like the frequencies allowed as standing waves on our string. The difference is that there are different restrictions on the values n can take.[1] These frequencies again are the ones allowed to exist in the interior of the cube. Now, since we have a small hole in the cube, we would expect these frequencies to leak out of the opening and become our blackbody radiation.

Now comes the tough part. We want to count how many standing wave modes have a frequency within some narrow range from f to $f + df$. We'll use a trick often used in Physics. Consider a three dimensional axis system, but instead of x, y, and z, use n_x, n_y, and n_z. Let's put a point at every possible combination of positive integer values of n_x, n_y, and n_z; each such point represents <u>two</u> distinct modes of standing wave (there are two possible independent polarizations for each set of n values) and each corresponds to a specific 1×1×1 cube. Now, n has some actual meaning; it's the 'distance' from the origin to some spot (n_x, n_y, n_z) in our 'n-space,' much like r is the distance to (x, y, z) in real space. Consider some volume in this n-space; if the volume is much, much larger than the volume of our unit cube, then the number of points enclosed should be equal to the volume, and the number of standing wave modes should be double the volume.

How many modes have the same n value? We remember that the volume of one-eighth of a thin spherical shell of radius r and thickness dr centered on the origin in real space is

$$dV = \frac{1}{8} \times 4\pi r^2 dr \,.$$

We count one-eighth because we only want to include the positive values of the ns. Analogously, then, the number of modes with a frequency between f and $f + df$ will be

[1] For example, in one dimension, n = 1, 2, 3, 4, … . In this case, n = 1, 1.41, 1.73, 2, 2.23, 2.45, 2.83, 3, 3.16, … .

$$dN = 2 \times \frac{1}{8} \times 4\pi\, n^2 dn = \pi \left(\frac{2Lf}{c}\right)^2 \left(\frac{2L}{c}\, df\right) = \frac{8\pi L^3}{c^3}\, f^2\, df\ ,$$

since

$$\mathrm{n} = \frac{2\mathrm{L}f_n}{\mathrm{c}} \text{ and } \mathrm{n} = \frac{2\mathrm{L}}{\mathrm{c}}\ df_{\mathrm{n}}\,.$$

The extra factor of two is because each mode has two polarizations. We'll make use of this result in each of the three discussions to come.

HOMEWORK 4-1

Calculate the number of possible standing wave frequencies between $4.3\times10^{+14}$ Hz and $7.5\times10^{+14}$ Hz (the visible light region for humans) in a 1m×1m×1m cavity.

Rayleigh-Jeans Law

Once again, let us review a bit of thermo-dynamics. We examined the behavior of a gas in a closed container and concluded that the average kinetic energy of translation of the particles in the gas was $^3/_2\ k_BT$, where T is the absolute temperature and k_B is Boltzmann's constant. We then made use of the *equipartition of energy theorem* to assert that energy is also distributed, on average, evenly among all modes of motion of the particle. For example, if a diatomic molecule can translate, rotate, and vibrate, then

$^3/_2\ k_BT$	K translation in 3 dimensions
$^2/_2\ k_BT$	K rotation around two short axes
$^1/_2\ k_BT$	K vibration
$^1/_2\ k_BT$	U vibration
$^7/_2\ k_BT$	Total

The Rayleigh-Jeans approach is to assume that the EM waves in the cube act like oscillators, if not the waves themselves, then the electrons at the inside surface of our enclosure. Then, according to the equipartition of energy theorem, each mode possesses an average energy of k_BT (half potential and half kinetic).

Then, the density of the energy $\rho(f, T)$ (in Joules *per* m^3 *per* Hz) in the interior of the cube due to modes with frequencies between f and df will be

$$\rho(f,\mathrm{T}) = \frac{(\text{number of modes per frequency interval})(\text{average energy of those modes})}{\text{volume}}$$

$$= \frac{\frac{\mathrm{dN}}{\mathrm{df}}\ \mathrm{E_{AVE}}}{\mathrm{L}^3} = \frac{\left(\frac{8\pi\mathrm{L}^3}{\mathrm{c}^3}\, f^2\right)(\mathrm{k_BT})}{\mathrm{L}^3} = \frac{8\pi\mathrm{k_BT}}{\mathrm{c}^3}\, f^2\,.$$

Hmm. That looks dangerous. The higher the frequency, the more energy at that frequency, so lots of X-rays and gamma-rays. Let's compare this result to reality; we're not being cooked by high energy EM waves from our surroundings, so maybe not such a good model.

The Boltzmann Distribution

Let's review a bit of the material from the end of our discussion on thermodynamics. Historically, or if you prefer, macroscopically, entropy S was defined by this relationship:

$$\Delta S = \int \frac{\delta Q}{T} \quad \text{or better,} \quad \delta Q = T\, dS,$$

where T is the absolute temperature at which thermal energy transfer Q occurs. We know now that S is related to the number of microstates g corresponding to a given macrostate: $S = k_B \ln(g)$. As such, it is a measure of how likely a given macrostate is: $g = \exp(S/k_B)$.

For example, we might have a box containing four indistinguishable particles. There is only one way (microstate) in which all four particles can be in the right side of the box (macrostate), but four ways (microstates) in which one could be in the left half with three in the right half (macrostate). Continuing:

N_{Left}	N_{Right}	$g = (N_{Total})!/(N_{Left}!\ N_{Right})!$	$S = k_B \ln g$
0	4	1	0
1	3	4	1.914×10^{-23} J/K
2	2	6	2.47×10^{-23} J/K
3	1	4	1.914×10^{-23} J/K
4	0	1	0

Notice that the highest entropy corresponds to the situation which we might intuitively believe would be the most likely case: two balls in each side. It's sometimes said that systems evolve to the state in which entropy is maximized, but that's really just saying that systems are most likely to be found in the states that are most probable.

Macrostates might also be defined by how energy is distributed in the system. We previously did an example of two objects placed side by side, one with ten 'flippers' and the other with twenty. We placed nine units of energy in the smaller block (each flipper can have one unit of energy, or none). We can define the temperature of each object as the average energy *per* flipper. Let's allow the objects to share energy, and look at the microstates associated with each macrostate.

Macrostate	Number of microstates (g)	S
$T_1 = 0.9$; $T_2 = 0$	10	3.18×10^{-23} J/K
$T_1 = 0.8$; $T_2 = 0.05$	900	9.39×10^{-23} J/K
$T_1 = 0.7$; $T_2 = 0.1$	22,800	13.86×10^{-23} J/K
$T_1 = 0.6$; $T_2 = 0.15$	239,400	17.10×10^{-23} J/K
$T_1 = 0.5$; $T_2 = 0.2$	1,220,940	19.35×10^{-23} J/K
$T_1 = 0.4$; $T_2 = 0.25$	3,255,840	20.71×10^{-23} J/K
$T_1 = 0.3$; $T_2 = 0.3$	**4,651,200**	**21.20×10^{-23} J/K**
$T_1 = 0.2$; $T_2 = 0.35$	3,488,400	20.80×10^{-23} J/K
$T_1 = 0.1$; $T_2 = 0.4$	1,259,700	19.40×10^{-23} J/K
$T_1 = 0.0$; $T_2 = 0.45$	167,960	16.62×10^{-23} J/K

As we can see, the most likely state, and therefore the one with the highest entropy, is the one in which the two objects have reached thermal equilibrium, as expected. The more states, the narrower the range of likely states.

Let's see if we can twist this around to be useful in another context.[2] Consider a small object in thermal equilibrium with a reservoir, both at temperature, T. The number of microstates (g) associated with a particular macrostate of the reservoir is proportional to the probability *P* of that macrostate. Solving the microscopic definition of S for g, we obtain

$$P = C\, g = C\, e^{S/k_B}$$

with C being some unknown constant. Next, let's look at the First Law of Thermodynamics:[3]

$$\delta Q = \mathrm{P}\, dV + dU \,.$$

Substituting the macroscopic definition of entropy and assuming that *d*V is zero (as for solids, gases in a rigid container, *et c.*) gives us

$$T\, dS = dU \quad dS = \frac{1}{T} dU \,.$$

Now, when some energy is transferred from the reservoir to our object, we write that *d*U = - *d*E, where E is the energy of the object:

$$dS = -\frac{1}{T} dE \,.$$

and integration gives us

$$S - S_o = -\frac{1}{T}(E - E_o) \quad \rightarrow \quad S = B_o - \frac{1}{T} E \,,$$

with B_o some constant that depends on the initial state. If we always compare to this state, we'll see that our dependence on it eventually drops out.

Substituting back into the probability expression,

$$P = C\, e^{S/k_B} = C\, e^{(B_o - E/T)/k_B} = \left(C\, e^{B_o/k_B}\right) e^{-E/k_B T} = D\, e^{-E/k_b T} \,.$$

The probability of being in <u>any</u> state m is the sum of the probabilities of being in each state, and should equal 1.

$$1 = \sum_m P_m = \sum_m D\, e^{-E_m/k_B T} = D \sum_m e^{-E_m/k_B T} \,.$$

[2] This section on the Boltzmann distribution is based closely on a discussion in Schroeder, Daniel V., "An Introduction to Thermal Physics," Addison-Wesley, San Francisco (2000) pp222 – 225. This in turn is similar to a discussion published by Planck.

[3] Don't confuse P the pressure and *P* the probability.

Then, doubling back, the probability of a particular state j is given by

$$P(E_j) = \frac{D\ e^{-E_j/k_BT}}{1} = \frac{D\ e^{-E_j/k_BT}}{D\sum_m\ e^{-E_m/k_BT}} = \frac{e^{-E_j/k_BT}}{\sum_m\ e^{-E_m/k_BT}}\ .$$

This relationship is known as the *Boltzmann distribution*. Hence, the probability of being in a state with a particular energy E_j decreases as the value of E_j increases.

EXAMPLE 4-1

Suppose we have a system with three energy levels: 1 eV, 2 eV, and 3 eV. What is the probability that the system is each of the states when the temperature is 10^5 K? You may find this alternate value for the Boltzmann constant useful: $k_B = 8.62\text{x}10^{-5}$ eV/K.

First, $k_BT = (8.62\text{x}10^{-5})(10^5) = 8.62$ eV. Then for each state, the Boltzmann factor is:

Level 1: $e^{-E_1/k_BT} = e^{-1/8.62} = 0.890$;

Level 2: $e^{-E_2/k_BT} = e^{-2/8.62} = 0.793$;

Level 3: $e^{-E_3/k_BT} = e^{-3/8.62} = 0.706$.

$$P(E_1) = \frac{e^{-E_1/k_BT}}{\sum_m\ e^{-E_m/k_BT}} = \frac{0.890}{0.890 + 0.793 + 0.706} = 0.373\ .$$

$$P(E_2) = \frac{e^{-E_2/k_BT}}{\sum_m\ e^{-E_m/k_BT}} = \frac{0.793}{0.890 + 0.793 + 0.706} = 0.332\ .$$

$$P(E_3) = \frac{e^{-E_3/k_BT}}{\sum_m\ e^{-E_m/k_BT}} = \frac{0.706}{0.890 + 0.793 + 0.706} = 0.295\ .$$

HOMEWORK 4-2

Repeat the calculation of Example 4-1 for a temperature of 100K.

DISCUSSION 4-X

What happens to the probabilities of each state when the temperature goes toward infinity? What about when the temperature heads toward zero?

HOMEWORK 4-3

Suppose we have a system with two energy levels. There are two states at the lower energy (0 eV), and eight at the higher (10.2 eV). Find the probability that the system is in one of the higher energy states if the temperature is 600K. Count each state as a separate term. Note: We're going to make use of this result later in the course.

Wien's Distribution

Whereas the Rayleigh-Jeans model assumes that all energy states are equally likely, the Wien model adds in the Boltzmann factor to account for the probabilities (the higher the energy of the state, the less likely that the state is 'occupied'). Now, what is the energy of a wave of frequency f? The Wien formula assumes that the energy carries by a wave is proportional to the frequency: $E = \eta f$. This is very non-obvious, but in the end, actually correct! The energy density expression then becomes

$$\rho(f,T) = \frac{\frac{dN}{df}\, E\, P(E)}{L^3} = D\,\frac{8\pi}{c^3}\, f^2(\eta f)\, e^{-\eta f/k_B T} = D\,\frac{8\pi\eta}{c^3}\, f^3\, e^{-\eta f/k_B T}\ .$$

The problem is, we don't have a value for eta, but in a while, we'll try to fit this function to some actual data. This does however fix the problem of being roasted; as the frequency increases, the probability of that frequency being emitted by the black body decreased rapidly due to the exponential term.

In contrast to the Rayleigh-Jeans relationship, which we'll see is a good fit at low frequencies but ridiculously incorrect for high frequencies, we will see that the Wien relationship is a good fit at high frequencies, but merely poor at low frequencies.

Planck's Distribution

Planck originally worked out an empirical relationship that fit the observed data points quite well, but spent ten years trying to justify the assumptions made. Eventually, he formulated what we now call *Planck's Postulate*. The oscillators in the cavity, whether we consider the EM waves themselves or the electrons on the inside edge of the cavity in the metal, have restrictions on the energy they may possess. Classically, the energy of an oscillator depends most directly on the amplitude of oscillation and the spring constant and can take on any value; the frequency of oscillation is not relevant. Planck asserted, somewhat similarly to Wien, that the oscillators are allowed energies only of this form:

$$E_m = jhf, \quad j = 0, 1, 2, 3, \ldots$$

with h a constant (similar to *eta* in the Wien approximation). Note the difference: Wien asserted the energy has one possible value ηf, while Planck allows integer multiple possible values, jhf. In Section 5, we'll explain how this is possible.

Remember the Boltzmann function. The probability of having a particular energy $E_j = jhf$ at a given frequency f will be

$$P(E_j) = \frac{e^{-E_j/k_B T}}{\sum_m e^{-E_m/k_B T}} = \frac{e^{-jhf/k_B T}}{\sum_m e^{-mhf/k_B T}}\,.$$

Let's find the average energy in our states for a given frequency.[4]

[4] This is just like finding an average score S on an exam. $S_{AVE} = (\Sigma\, S_i N_i)/N = \Sigma\, S_i (N_i/N) = \Sigma\, S_i P(N_i)$.

$$E_{AVE} = \sum_j E_j\, P(j) = \sum_j jhf\, P(j) = \sum_j jhf\, \frac{e^{-jhf/k_BT}}{\sum_m e^{-mhf/k_BT}} = \frac{\sum_j jhf\, e^{-jhf/k_BT}}{\sum_m e^{-mhf/k_BT}}$$
$$= \frac{\sum_j jhf\, e^{-jhf\beta}}{\sum_m e^{-mhf\beta}}.$$

In the last step, I've changed $1/k_BT$ to β for convenience of calculation.[5] Let's deal with the denominator first. This lower sum can be written as[6]

$$\sum_m e^{-mhf\beta} = \sum_m \left(e^{-hf\beta}\right)^m = \frac{1}{1 - e^{-hf\beta}}\ .$$

The numerator is tougher.[7]

$$\sum_j jhf\, e^{-jhf\beta} = \sum_j -\frac{d}{d\beta} e^{-jhf\beta} = -\frac{d}{d\beta}\sum_j e^{-jhf\beta} = -\frac{d}{d\beta}\sum_j \left(e^{-hf\beta}\right)^j.$$

Next, we'll make use of our infinite series again, then take the derivative:

$$-\frac{d}{d\beta}\left(\sum_j \left(e^{-hf\beta}\right)^j\right) = -\frac{d}{d\beta}\left(\frac{1}{1 - e^{-hf\beta}}\right) = \frac{hf\, e^{-hf\beta}}{(1 - e^{-hf\beta})^2}$$

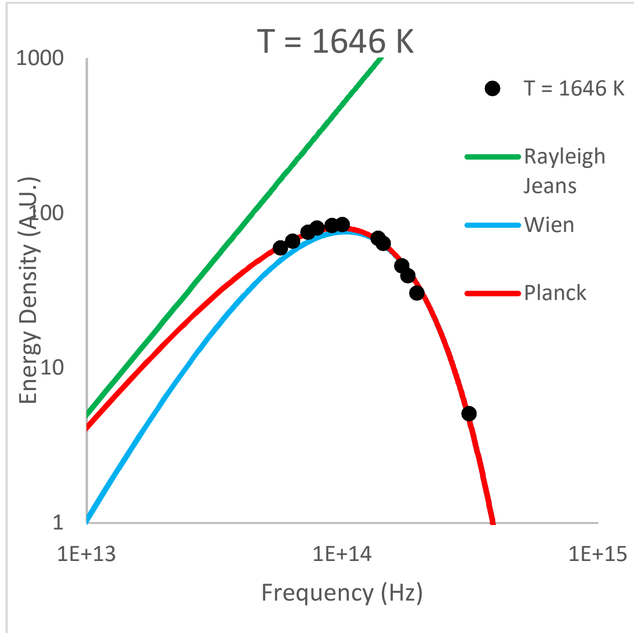
Let's put numerator and denominator back together:

$$E_{AVE} = \frac{\left(\frac{hf\, e^{-hf\beta}}{(1 - e^{-hf\beta})^2}\right)}{\left(\frac{1}{1 - e^{-hf\beta}}\right)} = \frac{hf\, e^{-hf\beta}}{(1 - e^{-hf\beta})} = \frac{hf}{e^{+hf\beta} - 1} = \frac{hf}{e^{hf/k_BT} - 1}.$$

[5] This is a common notation in thermodynamics. Don't confuse $\beta = 1/k_BT$ with $\beta = v/c$ from relativity.
[6] This infinite series, $1 + x + x^2 + x^3 + \ldots$ is known to equal $(1-x)^{-1}$ for $x<1$.
[7] The derivative has no physical meaning, it is a math trick. Eisberg, Robert and Robert Resnick, Quantum Physics pf Atoms, Molecules, Solids, Nuclei, and Particles, John Wiley & Sons, New York (1947) p20.

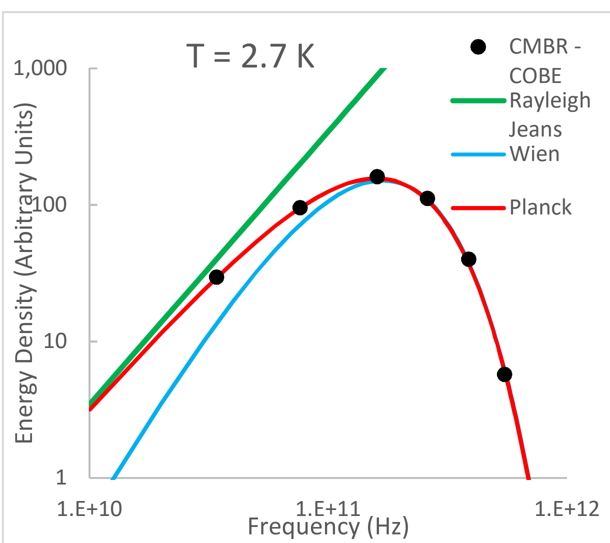
Then the energy density is given, as before, by

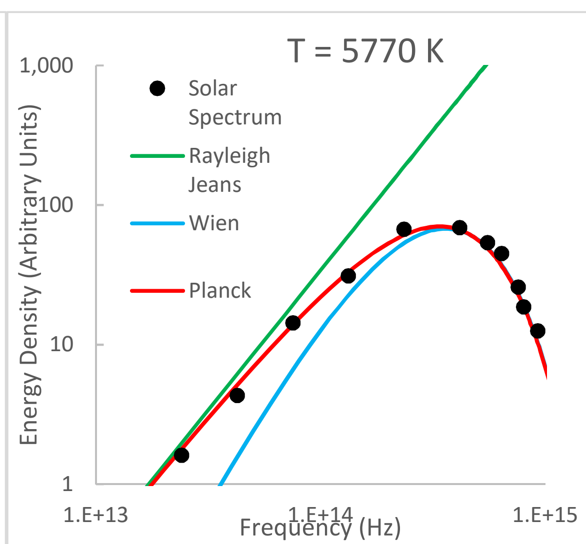
$$\rho(f,T) = \frac{\frac{dN}{df} E_{AVE}}{L^3} = \frac{8\pi L^3}{L^3 c^3} f^2 \frac{hf}{e^{hf/k_BT} - 1}$$
$$= \frac{8\pi h}{c^3} \frac{f^3}{e^{hf/k_BT} - 1} \ . \quad (*)$$



Let's compare this with some data. In the first graph, we have data taken by Lummer and Pringsheim[8] of a black body at 1646K. The colored lines represent our three proposed energy distributions. Of course, we see that the Rayleigh-Jeans curve is nowhere near correct. One <u>might</u> assert that the data lie more on the Planck curve than on the Wien curve, but the fit isn't that good, and data from Lummer's other temperatures are not as clear as even this. So, perhaps a toss-up.

The decisive experiments were performed by, among others, Rubens and Kurlbaum,[9] who measured well past 6μm out to 9, 24, 32, and 51 μm. Their experimental results are impossible to add to the figure, in that Lummer set a temperature and scanned the wavelength, but Rubens set a wavelength and scanned the temperature.[10] These experiments were, nonetheless, able to distinguish clearly the black body behavior from the Wein's prediction, and verify the validity of Planck's.





[8] O/ Lummer and E. Pringsheim, Verhandlung der Deutschm Phys. Ges.,I. Jahrg S.23 and 215 (1889).
[9] Rubens and Kurlbaum, 'Anwendung der Methode der Reststrahlen zur Prüfung des Strahlungsgesetzes,' Annalen der Physik 4:4 (1901) p 649.
[10] This is due to the incredibly difficult nature of measuring wavelengths in that range.

So, let's pull up some more recent data to illustrate the correctness of the Planck formula. First, here are data showing the cosmic microwave background radiation at about 2.7 K. In this figure, it is clear that the Planck distribution fits the data exceptionally well. Second, the output of the photosphere of our sun, with a temperature of about 5770 K. In these cases, it is clear that the Planck distributon fits the data better than either of the other models.

Let's continue to test the Planck formula. One characteristic of black bodies that was determined experimentally by Josef Stefan is that the total power output[11] is proportional to the fourth power of the temperature. Let's test this for the Planck distribution. We won't worry about any constants, we just want the temperature dependence. We'll integrate the energy density *per* Hz over the frequency range from zero to infinity:

$$P = C_1 \int_0^{\infty} \rho(f, T)\, df = C_2 \int_0^{\infty} \frac{f^3}{e^{hf/k_B T} - 1} df\,.$$

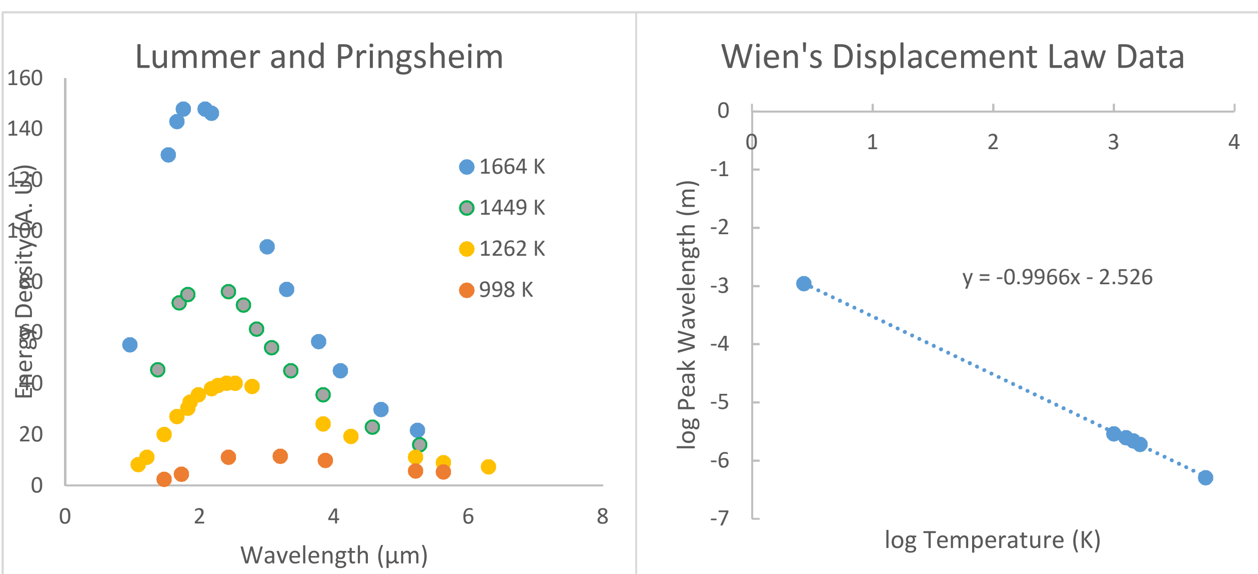
Let $u = hf/k_B T$, so $f = (k_B T/h)u$ and $df = (k_B T/h)\, du$

$$P = C_3 T^4 \int_0^{\infty} \frac{u^3}{e^u - 1} du\,.$$

Now, we don't even need to evaluate the integral; it will just be some particular number, which we'll slip into the constant:

$$P = CT^4$$

as required.



[11] Another P! Don't confuse them all.

The figure above left shows the output of a black body at four temperatures. As the temperature increases, the total energy output, as indicated by the area under the curve, increases.

The graph also demonstrates a second property of black bodies: the wavelength of the peak output varies inversely with the temperature:

$$\lambda_{MAX} = C\,T^{-1}.$$

This was derived theoretically by Wien using thermodynamics, and can be seen in the experimental data shown in the figure on the right. There, the Lummer data, the COBE data, and the solar spectrum peaks are plotted. The slope of -1 indicates that the peak wavelength and the temperature are indeed inversely proportional. We can even get an idea of the value of C.

$\log(\lambda_{\mathrm{MAX}}) = \log \mathrm{C} - \log \mathrm{T} \quad \rightarrow \quad \log \mathrm{C} = -2.526 \quad \rightarrow \quad \mathrm{C} = 10^{-2.526} = 0.00298\ \mathrm{mK}^{-1}$.

A more careful analysis reveals that C = 0.002898 mK^{-1}.

To test the Planck distribution, we should first convert ρ(*f*, T) to ρ(λ, T):

$$f = \frac{c}{\lambda}\ ;\ df\ =\ (-)\frac{c}{\lambda^2}\,d\lambda;$$

Requiring ρ(λ, T) *d*λ = ρ(*f*, T) *df* results in

$$\rho(\lambda, T) = \ C_1\lambda^{-5}\left(e^{hc/k_BT\lambda} - 1\right)^{-1}.$$

In general, ρ(λ, T) = ρ(*f*, T)/λ^2. Note that $f_{\mathrm{MAX}} \neq c/\lambda_{\mathrm{MAX}}$!

Let z = hc/k_BT for convenience. Then,

$$\rho(\lambda,\mathrm{T}) = \ \mathrm{C}_1\lambda^{-5}\left(\mathrm{e}^{\mathrm{z}/\lambda} - 1\right)^{-1}.$$

$$\text{Set } \frac{d\rho}{d\lambda} = \ -\mathrm{C}_1(-5)\,\lambda^{-6}\left(\mathrm{e}^{\mathrm{z}/\lambda} - 1\right)^{-1} + \ \mathrm{C}_1\lambda^{-5}\left(\mathrm{e}^{\mathrm{z}/\lambda} - 1\right)^{-2}(-1)\,\mathrm{e}^{\mathrm{z}/\lambda}\left(\frac{-\mathrm{z}}{\lambda^2}\right) = 0\,.$$

$$5\ =\ \left(\mathrm{e}^{\mathrm{z}/\lambda_{\mathrm{MAX}}} - 1\right)^{-1}\mathrm{e}^{\mathrm{Z}/\lambda_{\mathrm{MAX}}}\left(\frac{\mathrm{z}}{\lambda_{\mathrm{MAX}}}\right)$$

Let u = z/λ_{MAX} and this becomes

$$5\ =\ (e^u - 1)^{-1}\,e^u\,u\quad .\quad (**)$$

Now, once again, we do not need to solve this for u, we only need to convince ourselves that there is a particular solution value, u_o. Then,

$$u_o = \frac{z}{\lambda_{MAX}}\ =\ \frac{hc}{\lambda_{MAX}k_BT} \rightarrow\quad \lambda_{MAX} =\ C\,T^{-1}\,,\ \ (***)$$

as expected.

HOMEWORK 4-4

Solve for u_o (**) using a numerical method, for example, using Excel. Then, find the constant C in the equation above (***). Compare with the accepted value of 0.002898 mK^{-1}.

HOMEWORK 4-5

Consider two stars that are the same size, but A has a surface temperature of 3000 K and B has a surface temperature of 50,000 K. What is the ratio of the total power output of the stars?

HOMEWORK 4-6

What is the peak wavelength of each spectrum of the stars in the previous question?

Additional Notes

We have examined three functions proposed to explain the output spectrum of black body radiation. Please don't think that these three were the only such proposals. A review article by Day and Orstand[12] lists a number of such functions. To follow up, we'll take a quick look at these others as well.

Let's see if they pass the two tests. P_{TOTAL} for each model is found by setting up an integration similar to the one above. The displacement relationships are found by differentiating, as above.

MODEL	$\rho(f, T)$	P_{TOTAL}	Pass/Fail	λ_{MAX}	Pass/Fail
Planck	$C_1 f^3 (\exp(C_2 f/T) - 1)^{-1}$	$C\,T^4$	Pass	$\sim T^{-1}$	Pass
Wien	$C_1 f^3 \exp(-C_2 f/T)$	$C\,T^4$	Pass	$\sim T^{-1}$	Pass
Michelson	$C_1\ T^{3/2} f^4 \exp(-C_2 f^2/T)$	$C\,T^4$	Pass	$\sim T$	Fail
Thiesen	$C_1\ T^{1/2} f^{3.5} \exp(-C_2 f/T)$				
Rayleigh	$C_1\ T\ f^2 \exp(-C_2 f/T)$	$C\,T^4$	Pass	$\sim T^{-1}$	Pass
Rayleigh-Jeans	$C_1\ T f^2$	Infinite	Fail	Zero	Fail
Weber	$C_1 \exp(\alpha T) \exp(-C_2 f^2/T^2)$	$C\,e^{\alpha T}$	Fail	$\sim T^{-1}$	Pass
Jahnke	$C_1\ T f^2 \exp(-C_2 f^{1.25}/T^{1.25})$	$\sim C\,T^5$	Fail	$\sim T^{-1}$	Pass

So, if we had the inclination, it might be worthwhile to compare the Thiesen and Rayleigh distributions to observed data. However, Day and Van Orstand mention that these are generally just attempts to fit the data; the Jeans, Wien, and Planck functions do have some physics behind them.

HOMEWORK 4-7

Thiesen proposed a spectrum function $\rho(f, T) = C_1\ T^{1/2} f^{3.5} \exp(-C_2 f/T)$. Apply the two tests for P_{TOTAL} and λ_{MAX}. Does this function pass both tests?

Summary

[12] Day, Arthur L., and C. E. Van Orstand, 'The Black Body and the Measurement of Extreme Temperatures,' *The Astrophysical Journal* XIX:1 (1904) pp 1 – 40.

We have introduced the notion that some quantities are *quantized*, that is, they can take on only certain values, not any value as is usual in classical physics, and that this is a necessary condition to explain some phenomena. Secondly, we have introduced Planck's constant, sometimes referred to as *the quantum of action*; the accepted value today is 6.626×10^{-34} Joule seconds. In a later section, we'll discuss the justification of Planck's assertion regarding oscillators.