Section 5 – Waves and Particles Behaving Badly I

"Not only is the Universe stranger than we think, it is stranger than we can think."

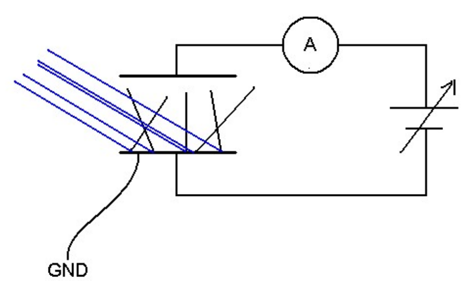
Werner Heisenberg, Schritte über Grenzen

In this section, we'll examine some experiments that put us on the road to modern quantum mechanics, including the photo-electric effect and Compton's X-ray scattering experiments.

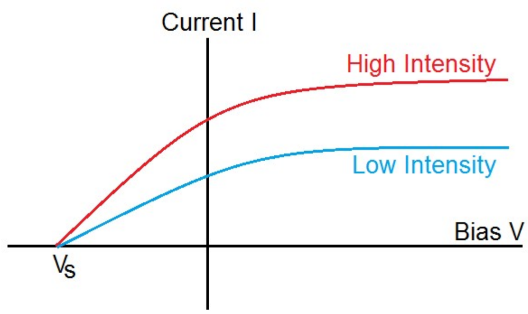
The Photo-electric Effect

The competition of the wave and particle models for light has been discussed already in class. We left the topic with Hertz's experiments and were quite confident that light is indeed a wave. We mentioned that Hertz's experiments in creating electromagnetic waves showed an anomaly when the room lights were on *vs* off. It was inferred that small particles were liberated for the electrodes as a result of exposure to light. In fairly quick order, these particles were characterized and eventually named *electrons*. In 1902, Lenard investigated this process, and discovered the following:

1) Metals emit electrons when exposed to light.
2) The maximum kinetic energy of the electrons is independent of the light intensity.
3) The maximum kinetic energy is related to the frequency of the light.

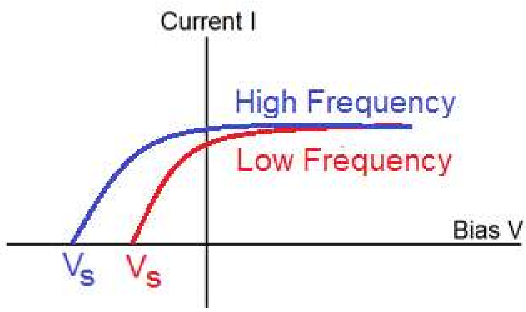


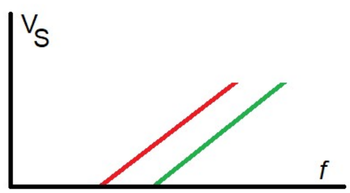
First of all, how can we measure such things? A schematic of the apparatus is shown in the figure. Light is incident on a metal *cathode* and electrons are released. For now, let's just assume that some fixed number of electrons are ejected from the metal. Some of the electrons will happen to arrive at the upper plate, and flow around the circuit, though the ammeter where they are counted, and back to the cathode. Consider several situations. Let's apply a bias voltage to the plates such that the upper plate is positive; more electrons will arrive there because they are being attracted and the current will increase. Eventually, the bias will be large enough to attract all of the emitted electrons, and the current will become constant. On the other hand, if we apply a reverse bias, *i.e.* make the upper plate negative, electrons will be repelled by the upper plate. Some electrons will make it and be measured, but only if their initial kinetic energies are larger than the potential energy difference between the plates, *i.e.*, if $K_i > qV$. As the reverse bias is increased, fewer and fewer electrons will make the trip across until even the very fastest fail and the current drops to zero. This value of the



reverse bias is called the *stopping potential*. Although I'll show you some real data, here for now is a schematic of the results. Here, we see curves generated by the same frequency of light, but with different intensities. Brighter light results in more electrons, but not faster electrons (the stopping potential remains the same).

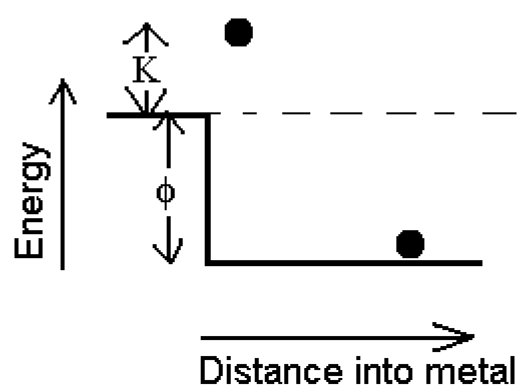
In the second figure, we have curves for light of different frequencies. Note that the stopping potential is different for different frequencies; higher frequency light imparts more kinetic energy to the electrons and so requires a larger potential difference to stop them.





In addition, there is a critical frequency of light below which no electrons are emitted (third graph). This *threshold frequency* depends on the type of metal used as the cathode, but the slope of the curves are the same for all metals.

Lastly, it is known that there is an energy barrier at the surface of metals; conduction electrons are fairly free to move around inside of the metal, but they generally don't simply seep out. The height of the barrier is called the work function of the metal, φ, and is generally of a few electron volts. When energy is transferred to an electron, there must be enough to overcome the work function; any extra may appear as kinetic energy as the electron moves away from the metal. We know this because, for example, electrons can be liberated from the metal surface by heating.



O.K., let's see what each of our models would predict as the behavior for the photo-electric effect.

The wave model allows light to carry energy, as described by the *Poynting vector*:

$$\vec{J} = \frac{(\vec{E} \times \vec{B})}{\mu_o} \quad .$$

Note that this quantity does not depend on the frequency of the light, only the strengths of the fields. The energy transferred to the electrons in the metal will depend on J, the area A and the length of time during which the light is incident:

$$\text{Energy transferred} = J_{AVE} \times A \times t \, .$$

So, we can make a number of predictions:

1) For very low light intensity levels, we might expect a delay between the onset of illumination and the electron's ejection from the metal as the electron's energy is slowly increased. Brighter light would imply shorter delays, since the rate of energy transfer would be greater. Let's estimate the delay:

EXAMPLE 5-1

How long should it take for an electron to absorb several electron-volts (~ 4×10^{-19} J) of energy from a light wave? We'll assume that the electron is within an atom, *i.e.*, a region approximately 3Å in diameter with cross-sectional area ~ 7×10^{-20} m^2. We'll assume that light hitting anywhere on the atom goes into the electron (best case scenario). Let's assume that the electron is exposed to light from a flashlight, such that J is approximately 10 W/m^2. Then,

$$t = \frac{E}{J\,A} = \frac{4 \times 10^{-19}}{10 \times (7 \times 10^{-20})} = \text{ 0.5 seconds.}$$

Here, we've again given our particle the benefit of many breaks, but this delay would certainly be noticeable. No such delay is actually observed.

2) Brighter light would be expected to increase the number of electrons ejected. This is seen.
3) There should be no frequency dependence. This is clearly not true.

In Einstein's model of light, energy is carried by particles called *photons*, each with an energy dependent on the frequency of the light:[1]

$$E_{Photon} = hf = \frac{hc}{\lambda} \, .$$

[1] We'll tie this into the Planck distribution discussion shortly.

When one of these particles is incident on a metal surface, it is absorbed by an electron in the metal, which acquires that photon's energy. If that energy is enough, the electron will surmount the barrier and leave the metal with kinetic energy, K:

$$E_{Photon} = K + \phi \ .$$

Since there may be losses of energy during the process, we usually write that

$$K_{Max} = E_{Max} - \phi_{Min} \ .$$

For those electrons that just barely traverse the gap between electrodes, arriving with no left-over kinetic energy, $K_{MAX} = qV_S$:

$$qV_S = hf - \phi_{Min} \ .$$

So, what are our predictions?

1) In dim light, electrons will be ejected with the same kinetic energies as in bright light; there will just be more of them. This is consistent with experiment.
2) For sufficiently low frequencies, there will be no emission of electrons, regardless of how bright the light is. This is again consistent.

We've now arrived at the baffling conclusion that light is both definitively a wave and definitively a particle.

Plotting the stopping potential as a function of the incident light's frequency allows us to determine several values:

$$V_S = \frac{h}{q} f - \frac{\phi_{Min}}{q} \ .$$

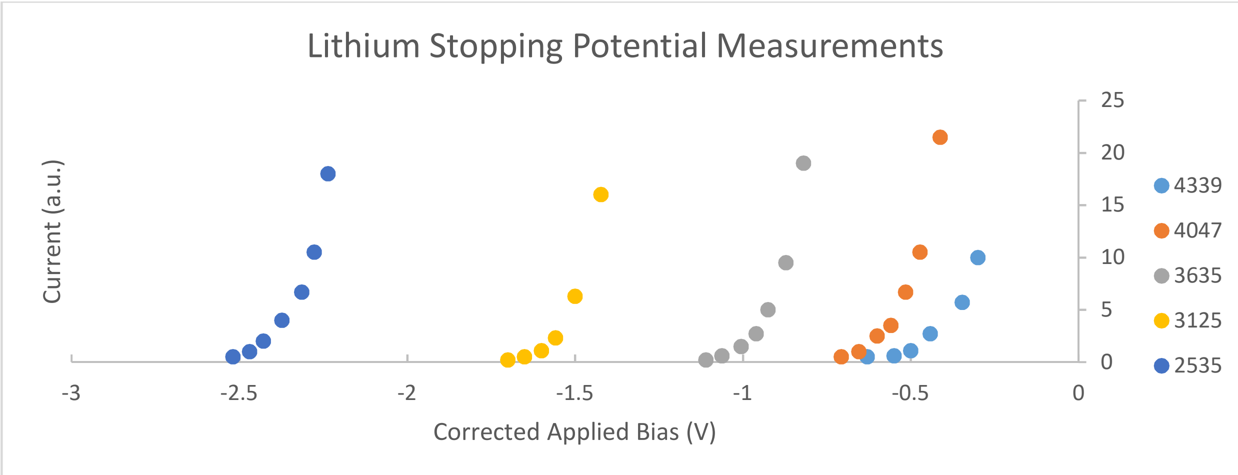
The slope of the line is Planck's constant divided by the electron charge, and the intercept gives the metal's work function.

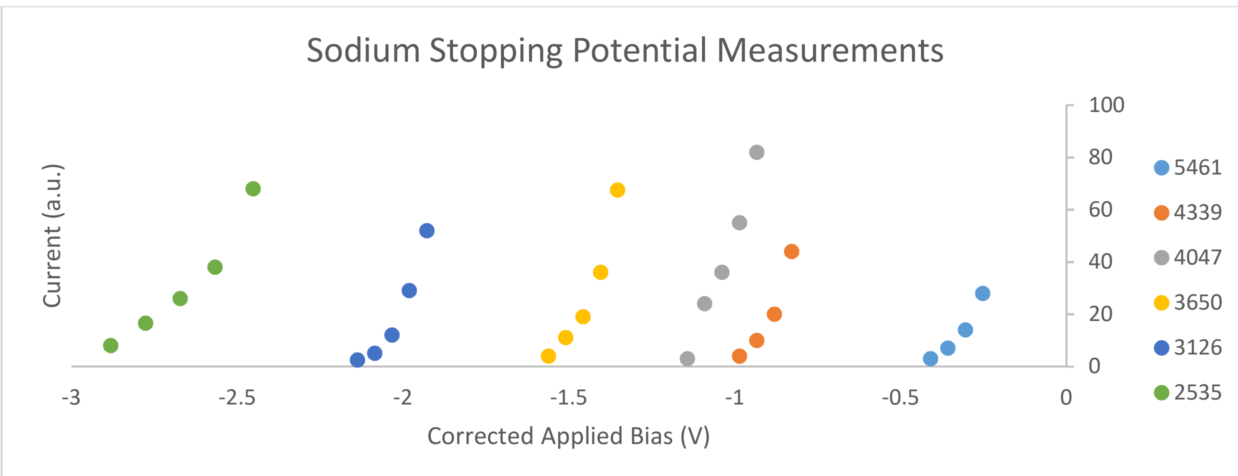
Let's look over some actual data measured by Millikan.[2] The first two graphs show the tail ends of the I-V curves for sodium and lithium when exposed to different wavelengths of light from a mercury lamp (wavelengths are in Å).

DISCUSSION 5-X

Note the absence of data at 5461Å for lithium. Why do you think that is?

[2] Millikan, R.A., 'A Direct Photoelectric Determination of Planck's h,' XXX VII 3 ().

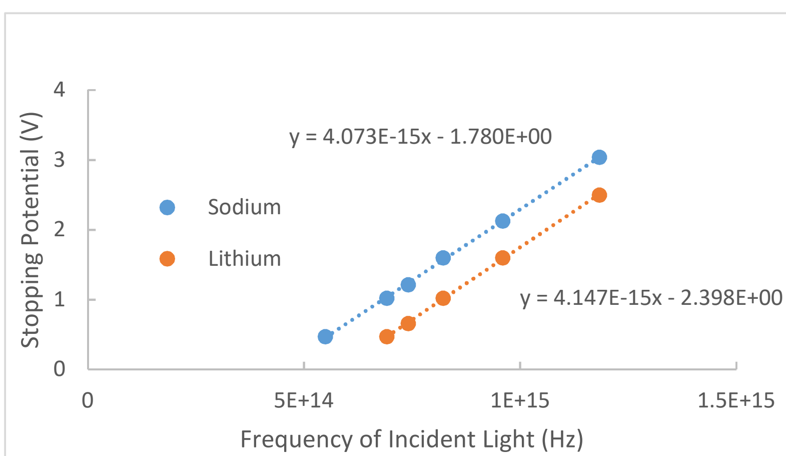




In each case, the stopping potential can be estimated. Plotting these data as

$$V_S = \frac{h}{q} f - \frac{\phi}{q},$$

we can find a value for Planck's constant and for the work functions of each metal. Using the currently accepted value for the electron charge, the sodium curve gives 6.53×10^{-34} J s and lithium 6.64×10^{-34} J s as values for h. The



currently accepted value is 6.63×10^{-34} J s.[3] These data, however, do not give quite accurate values for the work functions:

Element	Work function (from these data)	Work function (currently accepted value)
Lithium	2.40 eV	2.90 eV
Sodium	1.78 eV	2.28 eV

This may be explained, however, by the existence of an additional potential difference between the metal cathode and anode. Millikan made a strong effort to account for this effect, but found the correction factor to vary a bit. Luckily, this does not affect the values obtained for h.

Let's wrap this section up by revisiting the Planck distribution's postulate, that the energy of a standing wave of frequency f is jhf, with j some positive integer. If we mean literally a wave, there were only two polarizations that are possible. However, if we replace the wave with a set of j photons of frequency f, the energy becomes, as conjectured, jhf. Although we started our derivation for the black body radiator in terms of waves, it is necessary in the end to consider the radiation to be particles.

HOMEWORK 5-1

> The photo-electric effect is performed with aluminum as the cathode. The stopping potential is measured to be 2.14 volts when the incident light has wavelength 200 nm. What is the work function of aluminum and what is the cut-off wavelength (the wavelength above which no electrons are ejected)?

The Compton Effect

The Compton experiment, like the photoelectric effect, also indicates that light acts as if a particle. Consider an X-ray incident on a small particle, such as an electron. According to J.J. Thomson, these waves should be scattered in a classical manner by causing the electron to oscillate at the frequency of the light, thus re-radiating the light at the same frequency as that at which it was absorbed. Compton, on the other hand, assumed that the X-Rays behave as particles (photons) that collide elastically with the electrons, much as two pool balls might collide. The scattered photon transfers energy to the electron, and is thereby shifted to a longer wavelength by an amount that depends on the final direction of the X-Ray. A moderately long derivation (done below) results in this relationship:

$$\lambda_{\mathrm{Scattered}} = \lambda_{\mathrm{Incident}} + \frac{h}{m_e c}(1-\cos\phi)\,,$$

where m_e is the rest mass of the electron and ϕ is the scattering angle, the angle between the initial and final paths of the X-Ray.

[3] A useful alternative value is $h = 4.136\times10^{-15}$ eV s.

The first thing we need to do is develop an expression for the momentum of a photon. We saw in the last section that the energy is given by $E = hf = hc/\lambda$. Let's make use of some results from relativity. For particles, we saw that

$$E^2 = p^2c^2 + m_o^2c^4 \ .$$

Since photons travel at the speed of light, they must not have any mass, so

$$E = pc \quad \rightarrow \quad hf = \frac{hc}{\lambda} = pc \quad \rightarrow \quad p = \frac{h}{\lambda} = \frac{hf}{c} \ .$$

Before we start, we should show that the X-ray photon is indeed scattered from the electron and not absorbed, as happened in the photo-electric effect. We can treat this as a one dimensional problem with the photon moving along the x-axis toward a stationary, but free to move, particle. If all the energy of the photon were to go into the particle, then

$$E_{\text{Photon}} + E_{\text{Rest Particle}} = E_{\text{Final Particle}}$$

$$hf + m_oc^2 = \gamma_f m_o c^2$$

The term on the right of course included rest mass energy and kinetic energy.

$$hf = (\gamma_f - 1)\, m_oc^2$$

and of course, conservation of momentum requires that

$$p_{\text{Photon}} + p_{\text{Initial Particle}} = p_{\text{Final Particle}}$$

$$\frac{hf}{c} + 0 = \gamma_f m_o v_f \, .$$

$$hf = \gamma_f m_o v_f c \, .$$

Combining the two equations results in

$$\gamma_f v_f = (\gamma_f - 1)c \ .$$

$$\frac{v_f}{c} = \frac{(\gamma_f - 1)}{\gamma_f} = 1 - \frac{1}{\gamma_f} = 1 - \sqrt{1 - \frac{{v_f}^2}{c^2}}$$

$$1 - \frac{v_f}{c} = \sqrt{1 - \frac{{v_f}^2}{c^2}}$$

$$1 - \frac{v_f}{c} = \sqrt{1 - \frac{v_f}{c}}\sqrt{1 + \frac{v_f}{c}}$$

$$\sqrt{1 - \frac{v_f}{c}} = \sqrt{1 + \frac{v_f}{c}}$$

By inspection, we see that the solution is $v_f = 0$. In order for v_f to be zero, and to have both conservation of mass/energy and conservation of momentum, the energy of the photon must be zero, that is, there was no photon collision. So, some scattering must take place.

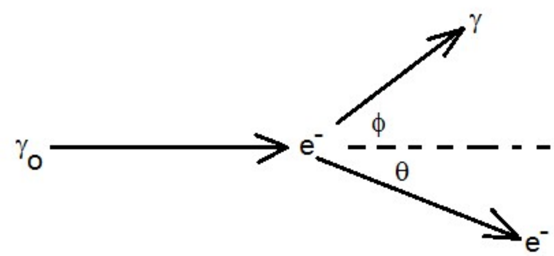
DERIVATION 5-1

We'll assume that the electron is initially at rest. The two conditions for an elastic collision are conservation of energy and conservation of momentum:

$$E_{\gamma i} + E_{ei} = E_{\gamma f} + E_{ef}$$

$$p_{\gamma i} = p_{\gamma f}\cos\phi + p_{ef}\cos\theta$$

$$0 = p_{\gamma f}\sin\phi - p_{ef}\sin\theta \quad .$$



Let's re-arrange and square the two momentum equations, then add them:

$$\left(p_{\gamma i} - p_{\gamma f}\cos\phi\right)^2 = {p_{ef}}^2\cos^2\theta$$

$$\frac{{p_{\gamma f}}^2\sin^2\phi = {p_{ef}}^2\sin^2\theta}{}$$

$$p_{\gamma i}^2 - 2p_{\gamma i}p_{\gamma f}\cos\phi + p_{\gamma f}^2 = p_{ef}^2$$

Next, we'll solve the energy equation for E_{ef} and square:

$$\begin{aligned} E_{ef}^2 = {} & \left(E_{\gamma i} + E_{ei} - E_{\gamma f}\right)^2 \\ & = p_{\gamma i}^2c^2 + m_O^2c^4 + p_{\gamma f}^2c^2 + 2p_{\gamma i}m_Oc^3 - 2p_{\gamma f}m_Oc^3 - 2p_{\gamma i}p_{\gamma f}c^2 \, . \end{aligned}$$

Next, we'll place each of these relationships into the relativistic equation for the electron:

$$E_{ef}^2 = p_{ef}^2c^2 + m_O^2c^4 \ ,$$

$$p_{\gamma i}^2c^2 + m_O^2c^4 + p_{\gamma f}^2c^2 + 2p_{\gamma i}m_Oc^3 - 2p_{\gamma f}m_Oc^3 - 2p_{\gamma i}p_{\gamma f}c^2 = \left(p_{\gamma i}^2 - 2p_{\gamma i}p_{\gamma f}\cos\phi + p_{\gamma f}^2\right)c^2 + m_O^2c^4 \ .$$

Cancelling like terms on each side reduces this to

$$+p_{\gamma i}m_Oc^3 - p_{\gamma f}m_Oc^3 - p_{\gamma i}p_{\gamma f}c^2 = \left(-p_{\gamma i}p_{\gamma f}\cos\phi\right)c^2 \ .$$

Dividing through,

$$\frac{1}{p_{\gamma f}} - \frac{1}{p_{\gamma i}} = \frac{(1-\cos\phi)}{m_Oc} ,$$

$$\frac{+\lambda_{\gamma f}}{h} - \frac{\lambda_{\gamma i}}{h} = \frac{(1-\cos\phi)}{m_Oc} ,$$

and finally,

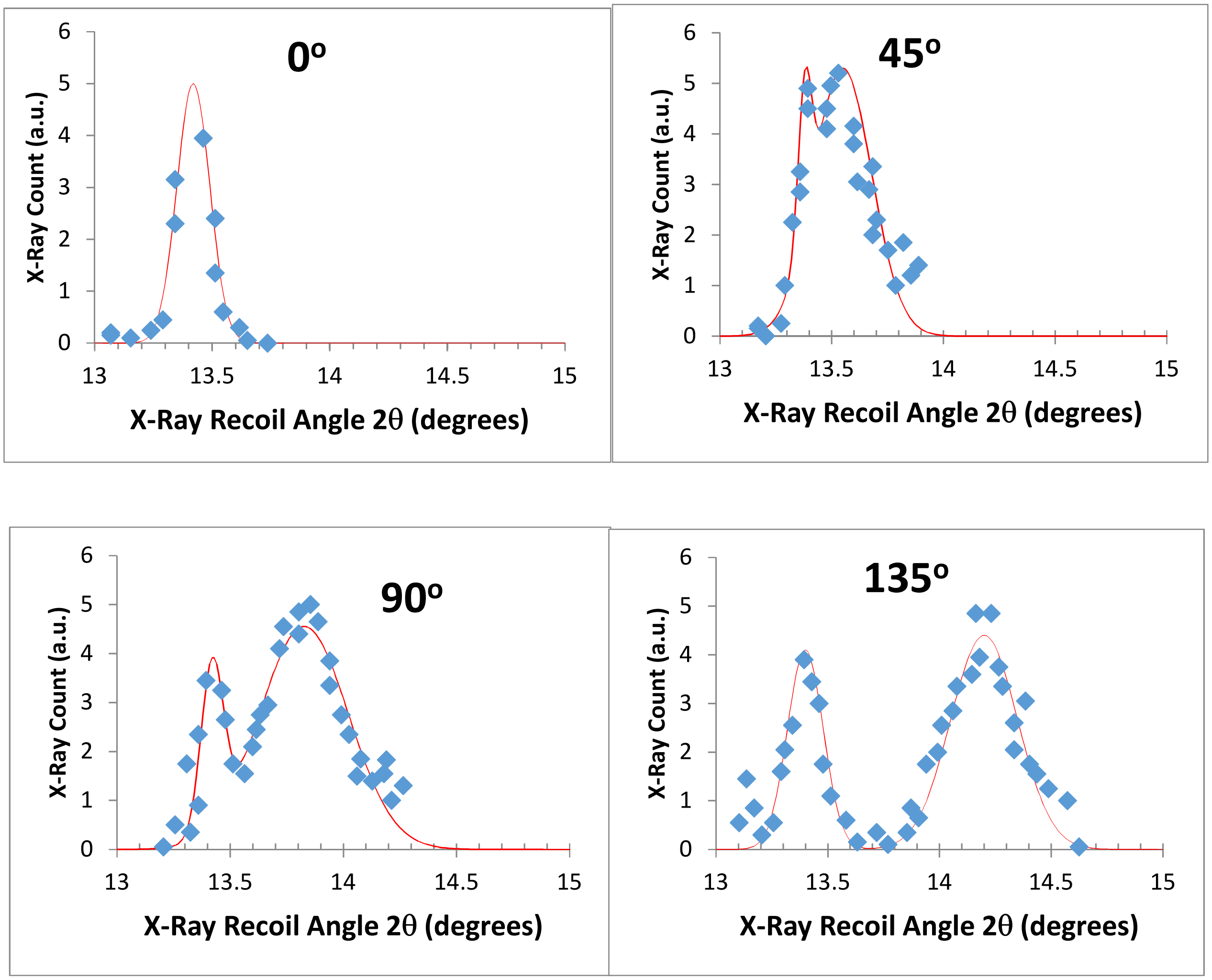
$$\lambda_{\gamma f} - \lambda_{\gamma i} = \frac{h}{m_Oc}(1-\cos\phi). \ ^4$$

For a separate paper, Compton[5] scattered X-rays from the Kα line of a molybdenum target (wavelength $\lambda_{\text{Incident}} = 0.711\text{Å}$) from a graphite (carbon) target.[6] Scattered X-Rays were measured at 45°, 90°, and 135° from the direction of the incident rays. The wavelengths of the scattered rays were measured by diffracting them from a calcite crystal (rhombohedral structure, distance between planes d = 3.036Å). Compton's data are presented in the figure. The shifting peak corresponds to Compton scattering (as if a particle) from the outer electrons of the carbon atoms, while the unshifted peak corresponds to 'classical' wave scattering (as if a wave) from the more tightly bound inner electrons of the atoms.

[4] An interesting factsicle is that the absolute shift at any given angle is the same for any energy of incoming X-ray.

[5] Compton, Arthur H., 'The Spectrum of Scattered X-Rays,' *Phys. Rev.* **22** 5 p409 (1923). These data were transcribed for these notes by Mr Russell Scott.

[6] The energy of the X-rays is much higher than the energy holding the electrons to the graphite, so we might presume that they are essentially 'free.'



HOMEWORK 5-2

What is the recoil wavelength of a molybdenum Kα X-ray incident on a muon if diffracted at 60º from its initial direction?

SUMMARY

We see that, occasionally, waves must be treated as particles in order to explain some peculiar effects. What about the other way 'round?