Between the Lectures I

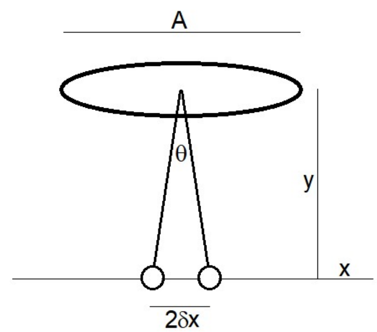
The Heisenberg Uncertainty Principle: Scattering of a Photon

The uncertainty principle states that there are fundamental physical restrictions on how accurately we can simultaneously know the position and momentum of an object, that is, the product of the uncertainties of these two quantities must be larger than a specific value:

$$\delta p_x \, \delta x \geq \frac{\hbar}{2} \, .$$

There are a number of ways in which the uncertainty relationship is derived, not all of which are entirely appropriate. However, we will examine this topic several times as applications of some of the concepts discussed.

Let's suppose that there is a particle at rest at the origin. If we want to 'see' this particle in order to make measurements, at least one photon must interact with the particle and be collected for analysis. In this example, we will use a microscope objective of aperture A to collect the photon.

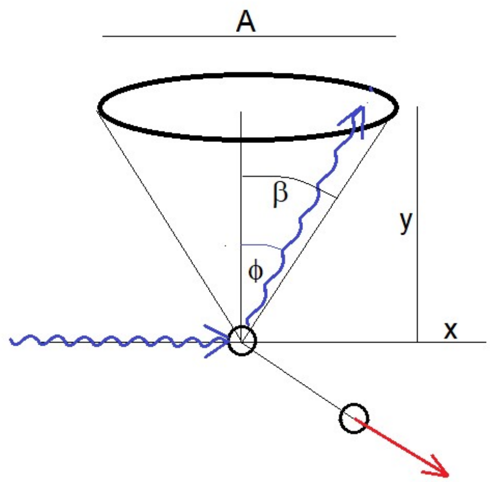


First, let's make use of Rayleigh's criterion to estimate how well we could possibly determine the particle's position through the microscope. We'll assume that the angle θ is small. If the particle nominally is at x=0, then the range of locations (shown by the circles) not resolvable by the lens is 0 ±δx, and so, referring back to the section on optics, we have that

$$\theta \approx \frac{2(\delta x_{\text{Particle}})}{y} \geq 1.22 \frac{\lambda_{\text{Photon}}}{A}$$

$$\delta x_{\text{Particle}} \geq 0.61 \frac{\lambda_{\text{Photon}} y}{A} \quad .$$

Next, let's look at the interaction of the photon and particle. Consider a photon incident on the particle from the left, along the negative x-axis, with momentum components $(p_{Photon})_{xi}$ and $(p_{Photon})_{yi}$, the latter of which is zero. Let's say that the scattered photon travels towards the lens in the direction indicated by the angle φ with momentum $(p_{Photon})_f$. The particle of course moves off in some other direction dictated by the conservation of momentum and of energy. Note that if the tangent of φ is larger than ½A/y, the photon will not be collected and observed. What's more, if tan φ is less than ½A/y, we can't tell where on the lens this photon is incident. This introduces an uncertainty in the x-momentum of any collected photon, $\delta(p_{Photon})_{xf}$, which can vary between $-(p_{Photon})_f \sin\beta$ and $+(p_{Photon})_f \sin\beta$:



$$\delta(p_{Photon})_{xf} = (p_{Photon})_f \sin\beta \approx (p_{Photon})_f \tan\beta \leq (p_{Photon})_f \frac{A/2}{y} = \frac{h}{\lambda_{Photon}} \frac{A}{2y} .$$

Then,

$$\delta(p_{Photon})_{xf}\, \delta x_{Particle} \leq \left(\frac{h}{\lambda_{Photon}} \frac{A}{2y}\right) \left(0.61 \frac{\lambda_{Photon} y}{A}\right) = 0.3\ h.$$

Now, we have to clean up the mess a bit. First, this condition is for not being able to locate the particle, so let's turn the inequality around for the condition to be able to locate it. Second, we've related the uncertainty of the particle's location to the uncertainty in the photon's momentum and not to that of the particle. Conservation of momentum requires that

$$(p_{Photon})_{xi} = (p_{Photon})_{xf} + (p_{Particle})_{xf} \ .$$

Since the left hand side of this equation is fixed by the initial conditions, we might naively write that

$$(p_{Photon})_{xi} = (p_{Photon})_{xf} \pm \delta(p_{Photon})_{xf} + (p_{Particle})_{xf} \mp \delta(p_{Particle})_{xf} \ ,$$

which then leads to

$$\delta(p_{Photon})_{xf} = \delta(p_{Particle})_{xf}$$

and finally,

$$\delta(p_{Particle})_{xf}\, \delta x_{Particle} \geq 0.3\ h.$$

This result is about twice the assertion at the beginning of this discussion.

This example is often used to explain, or perhaps illustrate, the uncertainty principle, but it is contingent on the particle colliding with a photon; our next example will also require a similar special case. What we will eventually find is that the uncertainty principle is <u>inherent</u> to the wave picture and would apply even to unmolested particles.