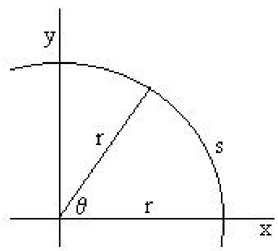
Section 1-8 - Rotational Motion -

Rotational Kinematics
Torque
Newton's Second Law for Rotation and the Moment of Rotational Inertia
Rotational Kinetic Energy
Angular Momentum
A Note on Rolling Objects
Mastery Question
Correlation to your Textbook

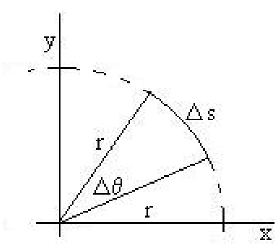
Rotational Kinematics

Let's consider a point mass free to move about a circle of radius r. First, we need to be able to specify the object's position. For this we will return to our convention of measuring angles CCW from the x-axis, however, we will for now on think of such angles in *radians*, not degrees.

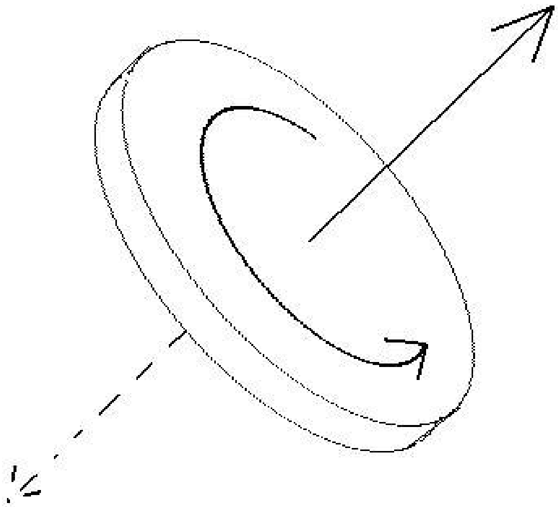


A radian is the angle such that the *arclength* s subtended is equal to the radius r, or about 57.3^o. Clearly, if we halve the angle, we also halve the distance along the arc, so that θ and s are proportional by the factor r: $s = \theta r$.
So, there are then 2π radians in a circle, since the circumference is $2\pi r$.

We should next find a way of describing changes in the position, or the *angular displacement*, $\Delta\theta$,



so that $\Delta s = \Delta\theta\, r$.
Since linear displacement was a vector, we might want to assign a direction to the angular displacement as well. First, we must define the plane in which the rotation occurs, which we can do almost unambiguously by picking a vector perpendicular to that plane. However, there are two such vectors:



Again, by convention, we'll pick one vector to represent rotation in one direction, and the other to represent the reverse rotation. Use a variation of the *right hand rule* (RHR): curl your fingers in the direction of rotation, and your thumb will point in the direction of the vector $\Delta\boldsymbol{\theta}$.

Then we continue with the *angular velocity*, the angular displacement *per* unit time:
$\omega_{ave} = \Delta\theta/\Delta t$; $\omega_{instantaneous} = \lim_{\Delta t \to 0} \Delta\theta/\Delta t$; direction of $\boldsymbol{\omega}$ also given by RHR.

Now, the speed around the circle, the *tangential velocity*, will be $v_T = \Delta s/\Delta t$. We can see that if $\Delta s = \Delta\theta\, r$, then $v_T = \omega r$.

Likewise, we can define the angular acceleration as the time rate of change of the angular velocity:
$\alpha_{ave} = \Delta\omega/\Delta t$; $\alpha_{instantaneous} = \lim_{\Delta t \to 0} \Delta\omega/\Delta t$; direction of $\boldsymbol{\alpha}$ also given by RHR.

Once again, we see a relationship between the angular quantity and the tangential quantity:
$a_T = \Delta v_T/\Delta t = (\Delta\omega\, r)/\Delta t = (\Delta\omega/\Delta t)\, r = \alpha\, r$.

If we assume that there are situations where the angular acceleration is constant, we can derive some kinematic relationships. Since there is an analogy between the the definitions of θ and s, ω and v_T, and α and a_T, we need not actually perform these derivations, but simply replace each linear quantity with the analogous rotational quantity:

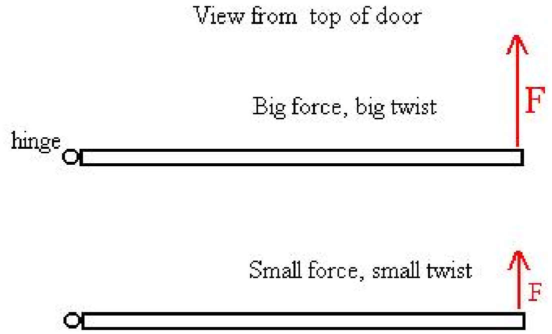
$v_T = v_{To} + a_T t$	$\omega = \omega_o + \alpha\, t$
$v_{T\,ave} = [v_{To} + v_T]/2$	$\omega_{ave} = [\omega_o + \omega]/2$
$s = s_o + v_{To}t + {}^1/_2 a_T t^2$	$\theta = \theta_o + \omega_o t + {}^1/_2 \alpha\, t^2$
$v_T^2 = v_{To}^2 + 2a_T(s - s_o)$	$\omega^2 = \omega_o^2 + 2\alpha\,(\theta - \theta_o)$

Torque

Back in Section Four, we saw that Newton's second law says that a net force is necessary in order for an object to have an acceleration. We might expect a similar necessary condition in order for an object to have an angular acceleration. Instead of a 'push' or 'pull,' it requires a 'twist.' The technical term for a twist is *torque*. So, we might guess that, in analogy with NII,
$\Sigma_i\, \tau_i = I\, \alpha$,

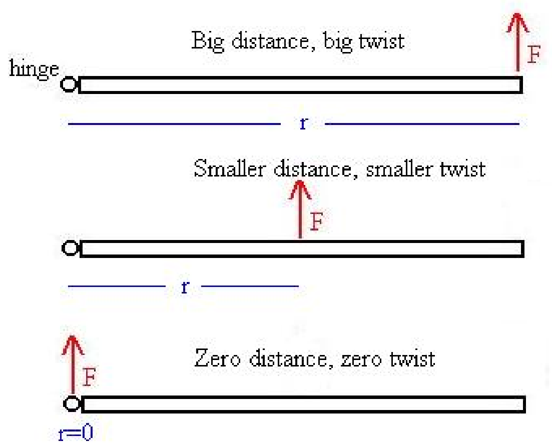
where I is some measure of the object's rotational inertia (how hard it is to accelerate rotationally) in the same way that the mass m is a measure of its translational inertia (how hard it is to accelerate linearly).

Before we try to justify this relationship, let's see if we can work out exactly what the torque means. Consider an object free to rotate around a particular axis, such as a door about its hinges. To get the door to begin to accelerate rotationally, it seems clear that a force must be applied.



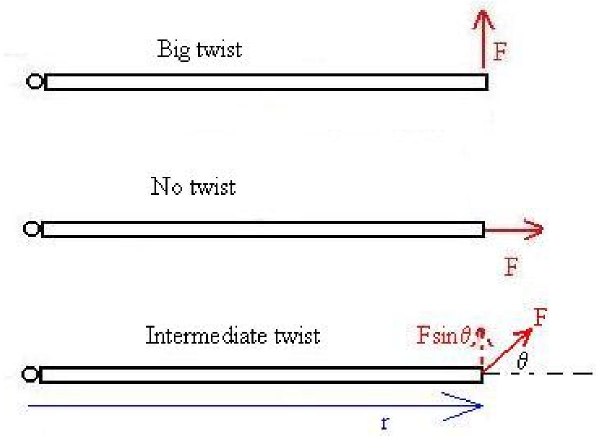
The larger the force, the bigger the twist applied. So we might guess that
$\tau \sim F$.

Where the force is applied also seems to matter. Try pushing the door near the end, then with the same force near the centre. See how the former results in more twist than the latter. Pushing near the hinge (axis) results in no twist at all.



So, now we might think that
$\tau \sim Fr$,
where r represents the distance from the axis of rotation to the point of application of the force.

Lastly, we se that there is a dependence on the orientation of the force with respect to the door. Namely, if we pull or push along the door, there is no twist, and we obtain the maximum twist when the force is at right angles to the door.



At intermediate angles, it seems clear that we need to take the component of the force which is perpendicular to the r-vector, namely Fsinθ, where θ is the angle as shown between the force vector and the r vector. So, we might now guess that

$\tau \sim F\, r \sin\theta$.

We also need to define a direction for the torque (after all, we can twist a bottle cap on or off). Assuming that the door starts from rest in the example above, then starts to turn CCW as a result of the applied force shown. Then, $\Delta\theta$ is out of the page, ω_{ave} is out of the page, and α is out of the page. Since for translational motion, the net force and the acceleration point in the same direction, we will require the net torque and the angular acceleration to do so as well. We see that we can get this result by defining the torque as the *cross-product* of **r** and **F**:

$\tau = r \times F$

or in our notation:

$|\tau| = r\ F \sin\theta_{r,F}$ (RHR).

The magnitude of the torque is found from the formula, and the direction through use of the right hand rule (RHR). In this case, the result of the cross product is out of the page, just as we wanted.

Now, this result is still tentative, since although we know on what factors the torque depends, we don't know the exact dependence. Only testing of the usefulness of this definition will vindicate our work here.

There is no special unit for torque; it's clear then that we can write it in terms of newton-metres.

Work and energy are also written in newton-metres, or joules.

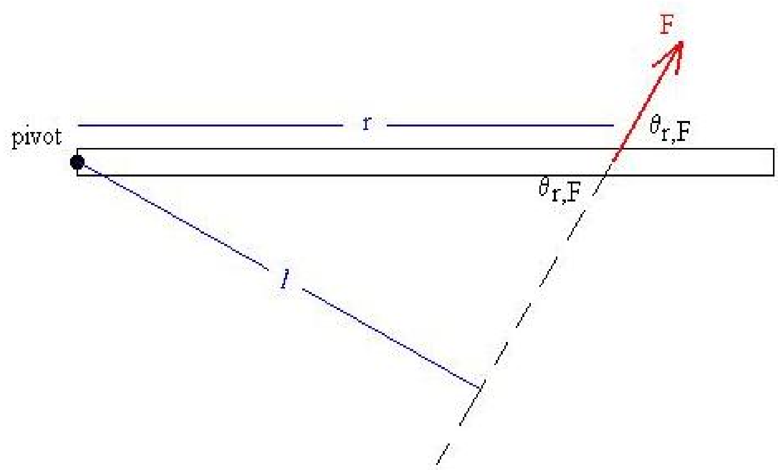
No. Even though the dimensions of energy and torque are the same, they are NOT the same type of quantity.

In the English system, these quantities are distinguished by putting energy in foot-pounds and torque in pound-feet.

Occasionally, the torque will be expressed as the product of a force and its *lever arm*, *l*.

$\tau = Fl$ (RHR).

The lever arm is found by extending the line of action of the force and finding the perpendicular distance (the lever arm) from the pivot to this line.



Clearly, $l = r \sin\theta$, so this definition is equivalent to, and sometimes more useful than, the one given above.

Center of Mass

Until the discussion immediately above, we have been treating objects as if they were *point masses*. For example, the location of a car could be given by $x_{CAR} = 34.7$m, but what does that actually mean? We could mean that the nose of the car is there, or tip of the shift lever; for displacements, there really isn't any import to any difference. However, now we will start to think of masses as *extended objects*, and on occasion we will want to describe the average position of the object. Now, by average position, we don't mean the average over time, but rather the average over the object at any given instant in time. For example, the average position of a meter stick can be said to be at the 50cm mark, halfway into the thickness of the wood and halfway down the side of the stick. This position is known as the *center of mass*.

The average of the grades on an exam are calculated by multiplying each grade G_i by the number of student who earned that grade N_i, then dividing that sum by the total number of students:

$$G_{AVE} = [\Sigma_i N_i G_i]/[\Sigma_i N_i]$$

The average position is found the same way. If the constituent mases are themselves point masses, then:

$$x_{CM} = [\Sigma_i m_i x_i]/[\Sigma_i m_i]$$

If the object is a continuous distribution of mass, then a similar calculation must be done using calculus.

The center of mass has one especially remarkable property. Consider a system of masses m_i with total mass $M = \Sigma_i m_i$:

$x_{CM} = [\Sigma_i m_i x_i]/[\Sigma_i m_i]$

$[\Sigma_i m_i] x_{CM} = \Sigma_i m_i x_i$

$M x_{CM} = \Sigma_i m_i x_i$

Now, that the time rate of change of each side, twice:

$M v_{CM} = \Sigma_i m_i v_i$

$M a_{CM} = \Sigma_i m_i a_i$

Now, let's consider all of the forces acting on any of the masses. For each one, we have

$F_{TOTi} = m_i a_i$

And if we add up all of those equations, we get

$F_{TOT} = \Sigma_i m_i a_i$

which is of course to say that

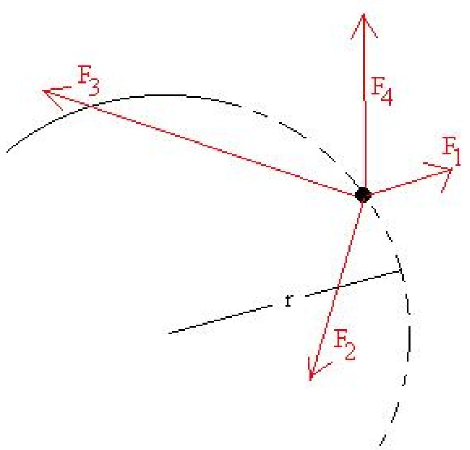
$F_{TOT} = M a_{CM}$.

In other words, all of the forces acting on any of the parts of the collection of masses will accelerate the secter of mass as if it were a sungle particle of mass M.

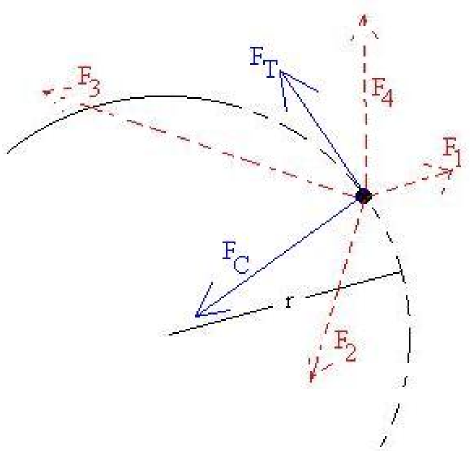
Newton's Second Law for Rotation and the Moment of Rotational Inertia

Following up on the notion of the existence of an analogy between linear and rotational motion, we might suspect that that there is a relationship similar to Newton's second law,
$\Sigma_i \mathbf{F}_i = m\mathbf{a}$,
perhaps of the form
$\Sigma_i \tau_i = I\,\alpha$,
where I is a constant whose meaning we still need to divine, but which we suspect might be a measure of how difficult it is to accelerate rotationally some object, in the same way that an interpretation of the mass is as a measure of the difficulty of altering an object's linear velocity.

Consider an object (point mass) constrained (for now) to move along a circular path, to which forces are applied:



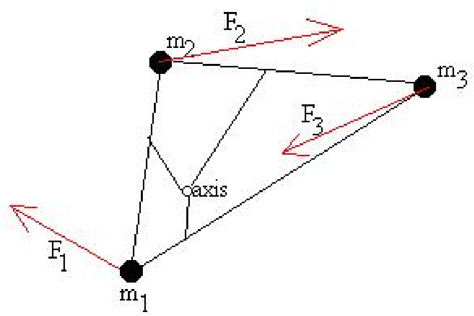
However many forces are applied, they can be added and resolved into components which are either centripetal or tangential, resulting in net force components like this:



The centripetal component is what keeps the object moving in a circle, and is of no particular interest to us just now. The tangential component, however, will accelerate the object <u>along</u> the circle, that is, tangentially:
$\Sigma_i F_{Ti} = ma_T$.

Let's multiply both sides of the relationhip by the radius, r:
$[\Sigma_i F_{Ti}] r = ma_T r$.
Distribute the r:
$\Sigma_i [r F_{Ti}] = m a_T r$.
Since every tangential force component is (by definition) perpenducular to the radius r, we recognize the terms in the sum to be the torques exerted by each of the forces, and we remember that
$a_T = \alpha r$,
so that
$\Sigma_i \tau_i = m (\alpha r) r = [mr^2] \alpha$.
So, in this very special case, we see that a rotational form of Newton's second law holds true, if the proportionality constant is
$I_{point\ mass} = mr^2$.
Note that this quantity depends not only on the mass, but on the distribution of the mass. This last comment should become clearer after the next discussion.

Suppose we have an object that comprises several point masses which are somehow connected, perhaps with light rigid rods:



Without bothering to calculate each torque explicitly, we can safely assume that there will be some torques applied to each object, including external torques due to the forces shown (make each the net force, if more than one force is desired), and also due to internal torques from the other objects, mediated through the rods. For each mass, we can write that
$\tau_{1\ ext} + \tau_{1\ int} = I_1 \alpha_1$
$\tau_{2\ ext} + \tau_{2\ int} = I_2 \alpha_2$
$\tau_{3\ ext} + \tau_{3\ int} = I_3 \alpha_3$
and so on.
If the objects rotate as a single object about a common axis, the all the α 's are the same. Let's add the equations (three here, but there could be as many as we like...):
$\tau_{1\ ext} + \tau_{1\ int} + \tau_{2\ ext} + \tau_{2\ int} + \tau_{3\ ext} + \tau_{3\ int\ t} = I_1 \alpha_1 + I_2 \alpha_2 + I_3\alpha_3 = [I_1 + I_2 + I_3]\alpha$
The sum of all the internal torques should be zero, since the third law says that each force that one mass exerts on another will match up with a force that the other exerts on the one, which is equal in magnitude and opposite in direction, and because the lever arms of these torques will be equal as well; this means that the torques are also equal in magnitude but opposite in direction. The sum of the external torques is just the sum of the torques exerted on the masses as a unit, so we now have that
$\Sigma_i \tau_i = (\Sigma_i I_i) \alpha$,
from which we see that the moment of inertial of an extended, rigid object is the sum of the moments of its constituent parts:
$I_{TOT} = \Sigma_i I_i$.
More generally, we would break an extended object (or a group of objects connected rigidly) into a very large number of tiny point masses, m_i, for which we already know that the individual moments of inertia are

$m_i r_i^2$, so that
$I_{TOT} = \Sigma_i \; m_i r_i^2$.

NOTE: Because the value for the moment of inertia depends not only on the mass, but also on the distribution of the mass in an object, the value for I for a given object may well (and probably will) be different for different axes of rotation.

Example:
Find the moment of inertia of a hoop of radius R and mass M about an axis passing through the centre of the hoop, perpendicular to its plane.
$I_{HOOP} = \Sigma_i I_i = \Sigma_i m_i r_i^2$
Now in this case, each piece of the mass is the same distance R away from the axis, so
$I_{HOOP} = \Sigma_i m_i R^2 = [\Sigma_i m_i] R^2 = MR^2$.
Will the moment of this same hoop be greater, the same, or smaller if it were rotated about one of its diameters?

Smaller (in fact, half as big, see below) since now much of the mass is less than distance R from the axis.

Try this:
Pick up a metre stick at the centre and try to twist it back and forth. Now try the same thing, but while holding the stick near the end. Which was harder to do?

You should have found that it was harder to do in the latter case.

The mass of the stick didn't change when you changed your grip, but more of it was farther from the axis of rotation in the second case.

Compare the moments of the hoop, a disc, and a sphere, all of mass M and radius R about an axis passing through the centre and perpendicular to the plane of the object (that means nothing for the sphere, of course). Try to place them in order of decreasing moment of inertia:

Hoop, disc, sphere; each has more of the mass concentrated towards the centre than the previous one.

Finding the moment of inertia for objects about different axes usually requires calculus (there is a table of common moments in your textbook), but there are some special cases (such as the hoop) and some useful techniques.

The *parallel axis theorem* states that, if one knows the moment of inertia about an axis passing through the *centre of mass* of an object (I_{CM}), then the moment about any other axis parallel to that one is given by
$I = I_{CM} + Mh^2$,
where h is the distance the second axis is displaced from the first.
For simplicity of calculation, align the x axis along the direction of the displacement of the axis of rotation and place the origin at the centre of mass.
FIGURE
The moment about the centre of mass is
$I_{CM} = \Sigma_i m_i r_i^2 = \Sigma_i m_i [x_i^2 + y_i^2]$
The moment about the new axis is
$I = \Sigma_i m_i (r_i')^2 = \Sigma_i m_i [(x_i - h)^2 + y_i^2] = \Sigma_i m_i [x_i^2 - 2hx_i + h^2 + y_i^2]$
$= \Sigma_i m_i [x_i^2 + y_i^2] + \Sigma_i m_i h^2 - \Sigma_i m_i 2hx_i =$
$= \Sigma_i m_i [x_i^2 + y_i^2] + [\Sigma_i m_i] h^2 - 2h \Sigma_i m_i x_i$
The first term we recognize as I_{CM}, the second is Mh^2, and the third is 2h/M times the x co-ordinate of the centre of mass, which we specified was at the origin, so that term is zero.

So,
$I = I_{CM} + Mh^2$.

Here is an example:
The moment of inertia of a thin rod of length L and mass M, about an axis through its centre perpendicular to its length, is found (using calculus) to be
$I_{CM} = {}^1/_{12}\, ML^2$.
What is the moment about an axis passing through the end of the rod, perpendicular to the length of the rod? This fulfills the necessary condition for the theorm, so we can write that
$I_{END} = I_{CM} + Mh^2 = {}^1/_{12}\, ML^2 + M\,(L/2)^2 = {}^1/_{12}\, ML^2 + {}^1/_4\, ML^2 = {}^1/_{12}\, ML^2 + {}^3/_{12}\, ML^2 = {}^4/_{12}\, ML^2 = {}^1/_3\, ML^2$.

The *perpendicular axis theorem* is valid for objects which are two dimensional, that is, flat and with zero thickness. This theorem states that, by knowing the moment about each of two axes in the plane of the object that are perpendicular to each other, we can find easily the moment about a third axis, perpendicular to the other two that passes through the intersection point of those first two.
$I_Z = I_X + I_Y$.
Let's prove it. For simplicity of calculation, let the two axes in the plane of the object be the x and y axes; they do <u>not</u> need to intersect at the centre of mass. The z axis comes out perpendicular to the plane of the object.
FIGURE
$I_X = \Sigma_i\, m_i\, y_i^2$
$I_Y = \Sigma_i\, m_i\, x_i^2$
$I_Z = \Sigma_i\, m_i\, r_i^2 = \Sigma_i\, m_i\, [x_i^2 + y_i^2] = \Sigma_i\, m_i\, x_i^2 + \Sigma_i\, m_i\, y_i^2 = I_X + I_Y$

Example:
Suppose that we want to know the moment about the diameter of a hoop. We already know the moment about an axis through the centre, perpendicular to the hoop:
$I_Z = MR^2 = I_X + I_Y$.
By symmetry, it's clear that I_X and I_Y are equal, so
$I_Z = MR^2 = 2I_X$
$I_X = {}^1/_2\, MR^2$.

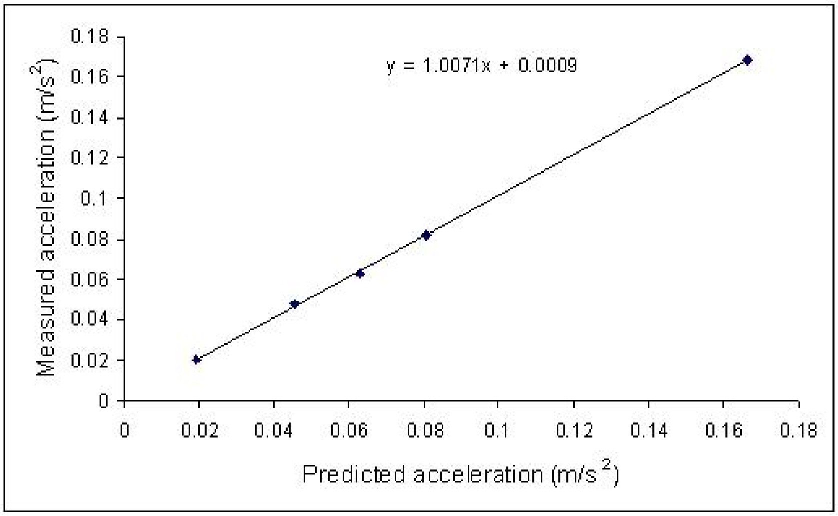
More examples:

Find the moment of inertia of a disc (mass M and radius R) about an axis in the plane of the disc, passing tangentially through the rim of the disc.
FIGURE

(5/4)MR^2

First, use the perpendicular axis theorem to find the moment of inertial about a diameter; the method is similar to that used in the preceding example. Then use the parallel axis theorem.

Here are the results of an experiment to verify NII for rotation. A hanging mass pulled a string wrapped around the hub (radius r) of a horizontal disc of moment I. The frictional torque was estimated by allowing the mass to drop at constant speed. You derived the expression:
$a_{CALC} = [m*a_g*r - \tau_F]/[I/r - mr]$



Mastery Question

This one will be a real test for you: Consider a thin *annulus* (or ring) of mass M, outer radius R_O, and inner radius R_I.
FIGURE
What is the moment of inertia about an axis in the plane of the annulus passing tangentially to the inner radius?
I'll post a solution to this someday.

Rotational Kinetic Energy

Continuing with the notion of there being quantities in rotational motion which are analogous to quantities in translational motion, we might expect that there is such a thing as *rotational kinetic energy*, which we might guess has the form
$KE_{ROT} = {}^1/_2\, I\omega^2$,
since ω is analogous to v and I is analogous to m.
Let's prove it. Consider a rigid object rotating about some axis. Each particle of the object, m_i, will have kinetic energy by virtue of its motion, and the total KE will be the sum of the individual KEs:
$KE = \Sigma_i {}^1/_2\, m_i\, v_i^2$
FIGURE

As seen from the axis of rotation, these $\mathbf{v}_i$s are tangential velocities, $\mathbf{v}_{Ti}$. We saw previously that there is a relationship between the angular velocity and the tangential velocity,
$v_{Ti} = \omega_i\, r_i$,
so we can substitute
$KE = \Sigma_i {}^1/_2\, m_i\, [\omega_i\, r_i]^2$
But all the ω's are the same, since it's a rigid body, so factor it (and the half) out of the sum:
$KE = {}^1/_2\, [\Sigma_i m_i\ r_i^2]\, \omega^2$
The quantity in parentheses we recognize as the moment of inertia for the object, and so
$KE_{ROT} = {}^1/_2\, I\, \omega^2$
as expected. The unit of rotational KE is still the Joule.

Note that we could have thought of the kinetic energy in the preceding example as either translational or as rotational. In a sense, it all depends on your point of view.

What happens when an object (mass M) is rotating in addition to an overall translational motion?
Each particle of mass m_i will have a velocity vector $\mathbf{v}_i$, as seen by some outside observer, so that
$KE_{TOT} = \Sigma_i {}^1/_2\, m_i\ v_i^2 = \Sigma_i {}^1/_2\, m_i\ (\mathbf{v}_i \cdot \mathbf{v}_i)$
Now, we can use the concept of relative velocities to write $\mathbf{v}_i = \mathbf{v}_{RA} + \mathbf{v}_{Ti}$, where $\mathbf{v}_{RA}$ is the velocity of the rotational axis and $\mathbf{v}_{Ti}$ is the tangential velocity of m_i relative to an observer riding along with the rotational axis.
$KE_{TOT} = \Sigma_i {}^1/_2\, m_i\ (\mathbf{v}_{RA} + \mathbf{v}_{Ti}) \cdot (\mathbf{v}_{RA} + \mathbf{v}_{Ti}) = \Sigma_i {}^1/_2\, m_i\ (\mathbf{v}_{RA} \cdot \mathbf{v}_{RA}) + \Sigma_i {}^1/_2\, m_i\ (\mathbf{v}_{Ti} \cdot \mathbf{v}_{Ti}) + \Sigma_i {}^1/_2\, m_i\ (\mathbf{v}_{RA} \cdot \mathbf{v}_{Ti})$
$KE_{TOT} = \Sigma_i {}^1/_2\, m_i\, \mathbf{v}_{RA}^2 + \Sigma_i {}^1/_2\, m_i\, \mathbf{v}_{Ti}^2 + \Sigma_i {}^1/_2\, m_i\ (\mathbf{v}_{RA} \cdot \mathbf{v}_{Ti})$
$KE_{TOT} = {}^1/_2[\Sigma_i m_i]\, \mathbf{v}_{RA}^2 + {}^1/_2[\Sigma_i m_i\ r_i^2]\omega^2 + {}^1/_2\, \mathbf{v}_{RA} \cdot [\Sigma_i m_i\, \mathbf{v}_{Ti}]$
$KE_{TOT} = {}^1/_2 M\, \mathbf{v}_{RA}^2 + {}^1/_2 I\omega^2 + {}^1/_2\, \mathbf{v}_{RA} \cdot [\Sigma_i m_i\, \mathbf{v}_{Ti}]$
The third term is a bit tricky to deal with. However, we remember that $\mathbf{v}_{Ti}$ represents the velocity vector of the i^{th} mass relative to the axis of rotation, and so we can write the term in the brackets as
$[\Sigma_i m_i\, \mathbf{v}_{Ti}] = \Delta[\Sigma_i m_i\, \mathbf{r}_i]/\Delta t = M\, \Delta\mathbf{r}_{CM}/\Delta t = M\mathbf{v}_{CM,\,RA}$,
that is, the total mass of the object times the velocity of the centre of mass <u>relative to the rotational axis</u>.
So, we now have that
$KE_{TOT} = {}^1/_2 M\mathbf{v}_{RA}^2 + {}^1/_2 I\omega^2 + {}^1/_2\, M\mathbf{v}_{RA} \cdot \mathbf{v}_{CM,RA}$

Now, let's consider a very common special case, that of an object which is translating while at the same time rotating about an axis passing through the centre of mass. In that case, $\mathbf{v}_{CM,RA} = 0$ and $\mathbf{v}_{RA} = \mathbf{v}_{CM}$, so that this reduces to:
$KE_{TOT} = {}^1/_2 M\ v_{CM}^2 + {}^1/_2 I\omega^2$,
that is, the total kinetic energy is the sum of the translational kinetic energy (as if the object were not rotating) and the rotational kinetic energy (as if the object were not translating). We see also that the distinction between translational KE and rotational KE is really arbitrary, a mere bookkeeping device for our convenience; all KE is fundamentally translational in nature.

How do we transfer energy into or out of a rotating (or rotatable) object? We need to apply a net torque and do work. Let's derive an expression for the work done by a torque. Let's restrict ourselves to a two dimensional case:
FIGURE
The net force is applied a distance r from the pivot point, and acts through a distance s, as shown.
Remembering several of the relationships we've derived so far, the work done becomes
$W = F_{net}\ s\ \cos\theta_{F,s} = F_{net}\ [r\ \Delta\theta]\ \sin\theta_{r,F} = \tau\ \Delta\theta$.
Since the torque τ and the angular displacement $\Delta\theta$ are both vectors, we still need to introduce a dependence on their relative orientations so that we can determine the sign of the work done. Taking a hint from the work defined for linear motion, we'll assert that
$W = \tau\ \Delta\theta\ \cos\theta_{\tau,\Delta\theta}$.
Don't confuse the two thetas in this relationship, they represent different angles!

The power delivered is then
$P = \tau\ \omega\ \cos\ \theta_{\tau,\omega}$.

We might also be able to define a potential energy associated with rotation. An example is that of a *torsional spring*. Consider a wire or string which exerts a torque proportional to the angle through which its end has been twisted and in the opposite direction of that angular displacement:
$\tau_{SPRING} = -\ \kappa\ \Delta\theta$.
The potential energy term might then be

$PE = {}^1/_2\kappa\, [\Delta\theta]^2$.
What about the units? Well, κ is in Nm/radians = Nm (yet another quantity with the same dimension as energy!) and the PE_{ROT} is in $(Nm)rad^2$ or Nm, so this looks O.K. dimensionally.

A special example of an object translating and rotating is one which 'rolls without slipping.' In that case, there is a relationship between the angular velocity and the translational velocity of the centre of mass: $v_{CM} = R\omega$.

Consider this example:
A hoop of mass M and radius R rests at the top of an incline (height h). It's released and rolls down the incline. What is the hoop's speed when it arrives at the foot of the incline?
Let's try using conservation of mechanical energy. We start with potential energy Mgh and end with none. We start with no KE and end with a combination of translational and rotational KE:
$Mgh = {}^1/_2Mv_f^2 + {}^1/_2I\omega_f^2$.
For a hoop, $I = MR^2$, so
$Mgh = {}^1/_2Mv_f^2 + {}^1/_2MR^2\omega_f^2$.
If the hoop rolls without slipping, we can make use of the relationship $v_{CM} = R\omega$ to obtain:
$Mgh = {}^1/_2Mv_f^2 + {}^1/_2MR^2(v_f/R)^2$.
Now some interesting developments. First, the mass drops out, so our answer is independent of the mass of the hoop. Also, R drops out, so the result is independent of the size of the hoop.
$gh = {}^1/_2v_f^2 + {}^1/_2v_f^2 = v_f^2$.
$v_f = [gh]^{1/2}$.
Compare this to the result when an object simply slides without friction down such an incline:
$v_f = [2gh]^{1/2}$.
In this case, the speed is lower because some of the potential energy had to go into rotational KE, leaving less for the translational KE, and thereby resulting in a lower final speed.

But if it had been a smooth incline, the final speed of the hoop would have been $v_f = [2gh]^{1/2}$. Why the difference?

Hint

Hang on, doesn't that preclude the use of conservation of mechanical energy?
Answer

The Race:
Consider a hoop, a disc, and a solid sphere (each with mass M and radius R) at the top of an incline of height h. If they are released from rest at the same time, which will arrive first at the foot of the incline?
We could argue that the one with the highest final velocity will also have the highest average velocity, and so arrive first. Review the solution above:
$Mgh = {}^1/_2Mv_f^2 + {}^1/_2I\omega_f^2$.
Since we really don't want to do the problem three times, let's let the moment of inertia be γMR^2, where γ = 1 for the hoop, ${}^1/_2$ for the disc, and ${}^2/_5$ for the solid sphere. If the objects roll without slipping, we can also use $v_{CM} = R\omega$.
$Mgh = {}^1/_2Mv_f^2 + {}^1/_2[\gamma MR^2](v_f/R)^2$
As expected, the Rs and Ms cancel, leaving
$gh = {}^1/_2v_f^2 + {}^1/_2\gamma v_f^2 = {}^1/_2[1+\gamma]v_f^2$
$v_f = [2gh/(1+\gamma)]^{1/2}$.
So, the object with the highest moment (the hoop) will be the slowest and arrive last, while the object with

the lowest moment (the sphere) will arrive first. This should make sense, if one thinks of the fraction of the original potential energy each object puts into its rotational energy.

Interestingly, any sphere will beat any disc, which in turn will beat any hoop.

Angular Momentum

Again as an analogy, we might suspect that there is such a thing as *angular momentum* (**L**), and we might guess that it is defined as $I\boldsymbol{\omega}$ (analogous to $m\mathbf{v}$). Let's see:
Starting from the rotational form of the Second Law,
$\tau_{EXT} = I\alpha$,
substitute the definition of angular acceleration (and assume that I is constant!) to get
$\tau_{EXT} = I[\Delta\omega/\Delta t] = \Delta[I\omega]/\Delta t$
$\tau_{EXT}\ \Delta t = \Delta[I\omega] = \Delta\mathbf{L}$
The left hand side of the preceding relationship is the rotational equivalent of impulse, and we can see that, in the absence of any external rotational impulses, the total amount of angular momentum is constant, or conserved.
We can derive analogous relations for the final angular velocities for totally inelastic 'collisions' and for totally elastic 'collisions' by substituting moments of inertial for masses and angular velocities for linear velocities, although there are some restrictions on when these will be valid (Is should be constant, for example!).

Two demonstrations in class:
Rotating student with barbells. By pulling the barbells in towards his body, he reduces the moment of inertia. If there are no external torques, the angular velocity correspondingly increases. This is the same effect used by figure skaters.
Student with bicycle wheel. A non-rotating student holds a wheel that is rotating so as to have (say) one unit of angular momentum, pointing upward (call this +1). Inverting the wheel causes the student to begin rotating. In the absence of external torques, the total angular momentum must remain +1. Inverting the wheel changes its angular momentum to -1, and the student then acquires angular momentum +2, so that the sum remains +1. How does the student magically acquire just the right amount of angular momentum? Inverting the wheel required that the student apply a torque, and so, by the third law, a torque equal in magnitude but opposite in direction was applied by the wheel on the student.

Two notes on angular momentum:

First, for linear momentum, we expected that the masses of objects could not change, so that any changes in momentum **p** were due to changes in velocity. For angular momentum, we see that a change in angular momentum can be effected by changing either ω or I or both.
Secondly, and more interestingly, we remember the constant, droning repetition that all three of the pictures we developed in linear motion (force and acceleration, work and kinetic energy, and impulse and momentum) were not only equally valid, but derivable from each other. We might expect the same from the three pictures developed for rotational motion, namely torque and angular acceleration, work and rotational kinetic energy, rotational 'impulse' and angular momentum. In the classical world we are studying this semestre this is so, but in the real world, we find the suggestion that angular momentum is somewhat more fundamental as a concept than the other two. In your chemistry courses, you may have come across the notion that angular momentum is *quantized*, that is, that only certain numerical values are allowed; this can be true of energies also, but the values allowed depend on the exact system.

A Note on Rolling Objects

Often, a question arises about the effect of friction on rolling objects. Consider a ball rolling on a flat horizontal surface. It has translational kinetic energy $^1/_2 m v_{CM}^2$ and rotational kinetic energy $^1/_2 I\omega^2$. If it rolling without slipping, then there is a relationship between v_{CM} and ω, namely that $v_{CM} = \omega R$. The friction, if any, will be static, but due to the synchonization of the two types of motion, there is no tendency to slip, and the frictional force, which is only as large as it needs to be, will be zero. But what about an object on an inclined plane (length l)? We discussed in class that a rolling object will have a lower velocity at the bottom of the plane than will a sliding object, due to the fact that the same amount of initial potential energy is converted into both translational and rotational energy, instead of into only translational kinetic energy as for the sliding object. In this case, it is the static frictional force which causes the object to begin to rotate in the first place. Let's consider the work done by F_f:

In terms of translation, the frictional force is directed opposite to the displacement, so the work done is $F_f\, l \cos 180^o = -F_f\, l$.

In terms of rotation, however, the torque supplied by the the force will be $F_f R$ and that is in the same direction as the angular displacement, θ. The angular displacement will be l/R, so that the work done by the torque will be

$W = \tau * \Delta\theta = + F_f R * l/R = + F_f l$.

So the total work done by the friction is zero. The result though is that some energy is 'stolen' from the translational motion and 'given' to the rotational motion.

Mastery Question

Consider a bowling ball that is released with initial translational velocity v_o but which is not initially rotating. Calculate the velocity of the ball and how far down the alley it is when it begins to roll without slipping.

Click here for a Solution.

Return to Notes Directory
Continue to Next Section of Notes

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